REACTOR ANALYSIS

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REACTOR ANALYSIS

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PREFACE

The purpose of the present work is to provide a complete and consistent mathematical development of what is commonly known as reactor analysis, that is, the mathematical study of the nuclear behavior of reactors based on certain approximate physical models. The subject of reactor analysis differs somewhat from that of reactor physics in both viewpoint and content. Whereas reactor analysis deals primarily with the mathematical tools for treating the physical behavior of reactors, reactor physics places much more emphasis on the physical aspects themselves of these systems. The tone of this book is therefore much closer to that of advanced treatments in engineering analysis rather than to that of books on physics.

The formal level of the presentation is directed primarily toward the first- and second-year graduate student in engineering science although it is expected that students in physics will also find it useful. Considerable pains have been taken to provide a textbook which could also be used in a first course on reactor analysis. The introductory sections of each of the principal chapters have been organized and written with this thought in mind. In these sections the treatment is initiated with the aid of elementary mathematical models and emphasis is placed on a discussion of the principal physical concepts to be developed. The more sophisticated mathematical considerations and the development of the broader theory are left in each case to later sections. Thus it is expected that this work will serve as an elementary text which can also be used in an intermediate course by simply including the complete treatment.

The material presented in this book was developed from a course organized and presented by the authors over a period of five years at the Oak Ridge School of Reactor Technology at the Oak Ridge National Laboratory. The development of the subject matter in the ORSORT course constitutes one-third of this book.

It is presumed that the reader who desires a complete understanding of the contents of this book has had at least a course in advanced calculus and preferably a general course also in partial differential equations and boundary-value problems, or a first course in the methods of mathematical physics. It is also assumed that he is acquainted with the fundamental concepts involved in modern physics and has been introduced to

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the use of analytical methods in the solution of engineering problems. For those who desire only an introductory knowledge of the subject and therefore limit their study to the elementary sections, the usual undergraduate course in differential equations will suffice.

In this treatment the subject matter of reactor analysis has been developed with the aid of various mathematical models. The models selected for this purpose are those which have proved to be useful in describing the various neutron phenomena peculiar to nuclear reactors. Emphasis is placed upon detailed presentations of each method in order that the reader becomes sufficiently well equipped to treat new and different situations. In nearly every instance the mathematical treatment has been extended to include the derivation of working formulas, and these are usually followed by numerical examples which display the computational techniques which may be used in application. In a few instances only a formal presentation is supplied, and in these cases the intent is merely to exhibit the principal physical ideas involved.

The authors have attempted to present a discussion of all the principal topics of reactor analysis, with an entire chapter devoted to each. In many instances several analytical methods are presented in order to provide as wide a treatment as possible. These include Chap. 2 on probability concepts; Chap. 3 on the neutron flux; Chap. 4 on slowing down; Chap. 5 on diffusion theory; Chap. 6 on the Fermi age model; Chap. 7 on transport theory; Chap. 8 on reflected reactors; Chap. 9 on reactor kinetics; and Chap. 10 on heterogeneity. It is important to mention that the remaining chapters represent in main part extensions and applications of these general topics.

The material in Chaps. 1, 2, and 3 and in the first section or so of Chaps. 4, 5, 6, 8, 9, and 10 constitutes a comprehensive first course in reactor analysis. A complete coverage of this text would constitute a second or intermediate course.

The authors wish to express their appreciation for the assistance and encouragement given by their friends and colleagues. To L. Nelson they are especially indebted for his penetrating criticism and gentle tolerance. To R. R. Coveyou, L. Dresner, R. K. Osborn, and H. Schweinler they are grateful for many suggestions and hours of stimulating discussion; to R. A. Charpie, W. K. Ergen, E. Guth, G. Leibfried, L. W. Nordheim, A. Simon, A. M. Weinberg, and T. A. Welton for reviews and comments; and to H. Honek, D. H. Platus, and D. L. Platus for help with the numerical examples.

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CHAPTER 1

INTRODUCTION

1.1 Nuclear Chain Reactions

An understanding of the properties and behavior of nuclear chain reactors is achieved through a study of the neutron population which supports the chain. Information about the neutron population is conveniently expressed in terms of the neutron-density-distribution function.

The detailed features of the chain reaction are determined by the various nuclear processes which can occur between the free neutrons and the materials of the reactor system. As in chemical chain reactions, the rates of the reactions involved in the chain are directly dependent upon the density of the chain carrier, in the present case the neutrons. Thus in order to determine the various properties of a reactor, such as the power-production rate and the radiation-shielding requirements, it is necessary to obtain the fission reaction rate throughout the system and, therefore, the neutron-density distribution. In fact, all the basic nuclear and engineering features of a reactor may be traced back ultimately to a knowledge of these distribution functions.

The subject of reactor analysis is the study of the analytical methods and models used to obtain neutron-density-distribution functions. Since these functions are intimately related to various neutron-induced nuclear reactions, a knowledge of at least the basic concepts of nuclear physics is essential to a thorough understanding of reactor analysis.

The first section of this chapter is a brief discussion of those aspects of nuclear reactions which are of principal interest to reactor physics. This presentation assumes that the reader is equipped with an introductory course in nuclear physics. The second section is an outline of the basic nuclear components of reactors and of the various types of reactors, and the last section is a summary of the principal problems of reactor physics and the analytical methods of attack. It is intended that the last section be used primarily for purposes of review and to aid the reader in orienting the various topics with regard to the over-all structure and scope of the subject.

a. Fission Reaction. In introducing the general subject of chain reactions it will be helpful to begin with a review of some elementary but basic notions about nuclear reactions, in particular the fission reaction.

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This information is included here primarily as a convenient reference for explaining the physical features of neutron phenomena relevant to reactors.

The bulk of the neutrons participating in the chain reaction within a reactor possess kinetic energies which range from thermal energy[1] (hundredths of an electron volt[2]) up to fission neutron energies (several million electron volts). Even though this range extends over some eight orders of magnitude, it is nevertheless possible (and convenient) to describe nearly all the important neutron-induced reactions within a reactor by means of a single conceptual model, namely, the compound-nucleus idea of Bohr.[3] This model is especially useful in studying the fission process.

The formation of the compound nucleus constitutes the first step in the reaction between the neutron and a nucleus. It may be represented symbolically by

$$X^A + n^1 \rightarrow \{X^{A+1}\}^* \tag{1.1}$$

where X^A denotes some nucleus of mass number A which has captured (absorbed) a neutron n^1. The symbol $\{\quad\}^*$ indicates that the resultant compound nuclear structure X^{A+1} is in an excited state. The excitation energy of this nucleus is the combined kinetic and binding energies of the captured neutron (in the compound nucleus). If the captured neutron had exactly zero velocity relative to the nucleus, then the excitation energy would be precisely the binding energy E_b. This point is easily demonstrated with the aid of the inverse to the complete reaction implied in (1.1). The complete reaction is accomplished when the compound nucleus achieves one of several possible stable states by simply ejecting the excess energy E_γ in the form of electromagnetic radiation (gamma rays); thus,

$$X^A + n^1 \rightarrow \{X^{A+1}\}^* \rightarrow X^{A+1} + E_\gamma \tag{1.2}$$

Now consider the situation in which the nucleus X^{A+1} in the unexcited (ground) state acquires an energy E_b just large enough to separate a neutron. The appropriate reaction would be

$$X^{A+1} + E_b \rightarrow \{X^{A+1}\}^* \rightarrow X^A + n^1 \tag{1.3}$$

This reaction is called the photoelectric liberation of a neutron. Now, by the theorem of detailed balance,[4,5] this reaction is just the

[1] Energy comparable to the thermal motion of the nuclei in the medium supporting the neutron population.

[2] One electron volt (ev) = 1.6023×10^{-12} erg = 1.6023×10^{-19} watt sec.

[3] N. Bohr, *Nature*, **137**, 344 (1936); J. M. Blatt and V. F. Weisskopf, "Theoretical Nuclear Physics," pp. 340–342, John Wiley & Sons, Inc., New York, 1952.

[4] Blatt and Weisskopf, *op. cit.*, pp. 601–602.

[5] E. Fermi, "Nuclear Physics," rev. ed., pp. 145–146, course notes compiled by Jay Orear, A. H. Rosenfeld, and R. A. Schluter, University of Chicago Press, Chicago, 1950.

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inverse of (1.2); a comparison of Eqs. (1.2) and (1.3) therefore reveals that

$$E_\gamma = E_b \tag{1.4}$$

The binding-energy concept is essential to an understanding of the nuclear-fission process. A short summary of the idea is therefore in order. By definition, the *total binding energy* E_b^{tot} of a nucleus is given by the difference between the mass of the nucleus and the sum of the masses of its constituent nucleons (protons and neutrons). If M is the nuclear mass, A the mass number (number of nucleons), N_n the number of neutrons, m_n and m_p the masses[1] of the neutron and proton, respectively, and c the velocity of light, then

$$\begin{aligned} E_b^{\text{tot}} &\equiv c^2[m_n N_n + m_p(A - N_n) - M] = c^2[N_n(m_n - m_p) + m_p A - M] \\ &\simeq c^2(Am_n - M) \end{aligned} \tag{1.5}$$

where the approximation follows from the fact that $m_n \simeq m_p$. In order to obtain the *average binding energy per nucleon* $\bar{E}_b$, we divide through by A:

$$\bar{E}_b \equiv \frac{E_b^{\text{tot}}}{A} \simeq c^2\left(m_n - \frac{M}{A}\right) \tag{1.6}$$

This is the average energy required per nucleon to separate the nucleus into its constituent particles. The value of $\bar{E}_b$ varies from about 1 to 9 Mev over the entire mass scale.[2] In the mass range of interest to reactor physics ($A \gtrsim 70$), $\bar{E}_b$ decreases monotonically from 8.6 Mev at $A = 70$ to 7.5 Mev at $A = 238$. It is this variation which determines two fundamental features of the fission reaction. It is shown later that this indicates (1) that there is a positive energy release if a nucleus with $A > 85$ is caused to disintegrate and (2) that nuclei in this range are theoretically unstable with regard to the fission process.

Consider first the question of the energy released when a nucleus disintegrates. As an example, let us take the case of the neutron-induced fission of the nucleus X^A which results in the formation of two fragments Y^{A_1} and Z^{A_2} of masses M_1 and M_2, respectively. On the basis of the compound-nucleus concept, this reaction may be written

$$n^1 + X^A \rightarrow \{X^{A+1}\}^* \rightarrow Y^{A_1} + Z^{A_2} \tag{1.7}$$

In general, the binding energies of the fragments will differ from the binding energy of the original nucleus; that is, the combined masses of the fragments will not equal the mass of the fissioned nucleus X^{A+1}. The difference appears as an energy release E_f which may be determined

[1] These are: $m_n = 1.00893$ amu (atomic mass units) and $m_p = 1.00758$ amu, where 1 amu $= 1.66 \times 10^{-24}$ g.

[2] See, for example, C. F. Bonilla, "Nuclear Engineering," Fig. 3-5, p. 63, McGraw-Hill Book Company, Inc., New York, 1957.

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from the Einstein mass-energy relation. In the present case

$$E_f \equiv c^2[M_c - (M_1 + M_2)] \tag{1.8}$$

where the subscript c refers to the compound nucleus. This expression may be written in terms of the various binding energies if we use the relation (1.6); thus, in general, for nucleus i

$$M_i \simeq A_i\left[m_n - \frac{\bar{E}_b^{(i)}}{c^2}\right] \tag{1.9}$$

The substitution of this equation into (1.8) yields

$$E_f \simeq (A + 1)[\bar{E}_b^{(2)} - \bar{E}_b^{(c)}] + A_1[\bar{E}_b^{(1)} - \bar{E}_b^{(2)}] \tag{1.10}$$

where we have used $A_c \equiv A + 1 = A_1 + A_2$. The variation of this function with mass number has the general shape shown in Fig. 1.1.

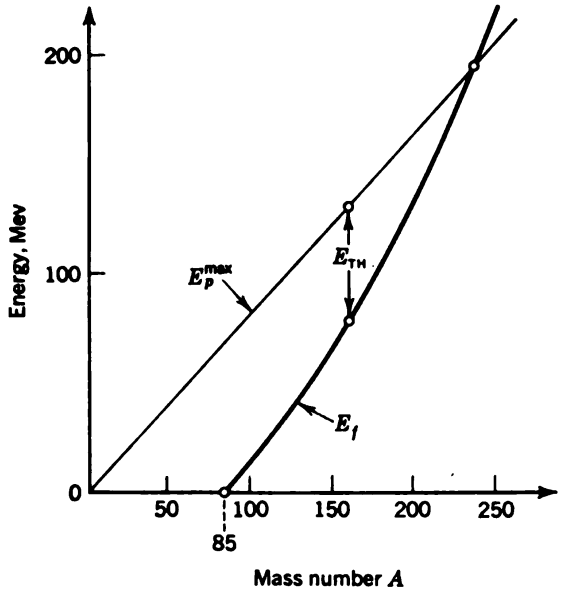

FIG. 1.1 Fission energy and electrostatic repulsion energy of a nucleus as a function of the mass number.

Precisely at what value of A, E_f becomes positive depends, of course, upon the mode of disintegration. In the case of symmetric division ($A_1 \equiv A_2$), the critical value of A is about 85. For mass numbers substantially greater than 85, there are many modes of disintegration which result in a positive energy release.

The principal observation to be drawn from the fission-energy relation (1.10) is that, in the case of the heavier nuclei, the fission fragments represent a lower energy state than the original nucleus. This would imply that the heavier nuclei are inherently unstable toward fission and could conceivably undergo spontaneous disintegration. Experience shows, however, that spontaneous fission does not occur at anything

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like the rate one might be led to expect on the basis of the fission energetics alone. The explanation rests with the fact that a nucleus cannot disintegrate until it has first acquired a certain activation or threshold energy. Thus there exists an energy barrier between the state of the whole nucleus and the fragmented state. If one were to take the potential energy of the fragments at infinite separation to be zero, then the merged state of the fragments (forming the original nucleus) would have energy E_f. The variation of the potential-energy curve between these limits would exhibit a maximum, as shown in Fig. 1.2. If the maximum value of the potential energy is $E_p^{\max}$, then $E_p^{\max} - E_f \equiv E_{\text{TH}}$ would be the threshold energy for fission.

A nucleus can acquire the necessary excitation energy to overcome this barrier by absorbing either a nuclear particle or electromagnetic

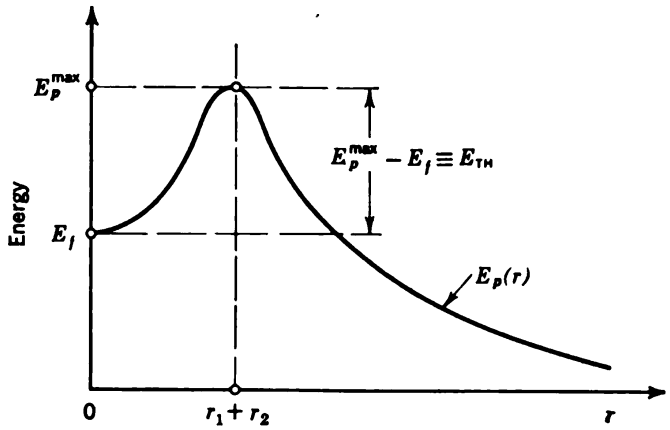

FIG. 1.2 Potential energy of fission fragments as a function of separation.

radiation. Of the two possibilities, the former leads more easily to fission since the absorption of a particle makes available not only its kinetic energy but its binding energy as well. This point was previously noted in connection with the neutron-capture reaction. If the energy acquired in this way is sufficiently large, the nuclear structure experiences increasingly violent oscillations which eventually rupture it, forming the various fragments.

An estimate of the fission threshold can be obtained from the energy required to distort the nucleus into an extreme shape which results in complete separation into fragments. It has been shown[1,2] that this calculation can be based on the *liquid-drop model* of the nucleus. The two principal contributions to the distortion energy of the nucleus are the "surface-tension" effect from the nuclear forces between the constituent

[1] N. Bohr and J. A. Wheeler, *Phys. Rev.*, **56**, 426 (1939); J. Frenkel, *J. Phys.* (*U.S.S.R.*), **1**, 125 (1939).

[2] An elementary treatment is given by D. Halliday, "Introductory Nuclear Physics," pp. 417–421, John Wiley & Sons, Inc., New York, 1950.

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nucleons and the electrostatic repulsion due to the charge on the protons. When the nucleus is set into oscillation, any departure from its original shape results in an increase in its potential energy because of the "surface tension." Such distortions tend, however, to separate the proton population and thereby create centers of electrostatic repulsion; these forces decrease the potential energy of the system and increase the distortions further. If the distortional oscillations lead to a "dumbbell-like" nuclear configuration, the electrostatic repulsive forces may eventually overcome the nuclear attractive forces and the nucleus will divide.

When the separation results in two major fragments, the threshold energy for fission is given by the potential energy of the fragments at the instant of separation. If one assumes that separation yields two spherical collections of nucleons with Z_i protons and of radius r_i, then the maximum value of the potential energy $E_p^{\max}$ due to the electrostatic field is proportional to $Z_1Z_2(r_1 + r_2)^{-1}$. When the separation r exceeds $r_1 + r_2$, the coulomb forces predominate, and when $r < r_1 + r_2$, nuclear forces predominate. A representative curve of the potential energy E_p of this system is shown in Fig. 1.2.

The variation of the potential energy $E_p^{\max}$ with mass number has the general shape shown in Fig. 1.1. It is seen that for all $A \lesssim 250$ the threshold energy $E_p^{\max} - E_f > 0$; energy must be added in order to produce fission in all nuclei. The figure reveals, however, that the threshold energy becomes progressively smaller with increasing mass number. As the mass number approaches 250, there is a rapid increase in the probability for spontaneous fission through the mechanism of barrier penetration.[1] When $A \gtrsim 250$, the probability is essentially 1, and the nucleons do not remain together long enough to be described as a nuclear structure. It is not surprising, therefore, that nuclei with such large A do not exist in nature.

b. Nuclear Fuels. Nuclei with mass numbers in the range $230 < A < 240$ have fission thresholds of some several Mev. Thus in these cases fission can be brought about by the absorption of radiation or neutrons of only a few Mev kinetic energy. There are a few nuclei that can even be caused to fission by thermal (very slow) neutrons. The specific energy requirements for a particular nucleus depend strongly upon the excitation energy which the captured particle can impart to it. It was shown previously that, in the case of the neutron-induced reaction, the binding energy of the neutron represented a large part (if not all) of this excitation energy. However, even though the mass numbers of the more easily fissionable nuclei differ but little, the binding energies vary by as much as 50 per cent; hence, the variation in the fissionability of the various nuclei. This relatively large variation is due primarily to the influence of the even-odd term in the nuclear-mass formula. If $M(A,Z)$ is the mass of a

[1] Blatt and Weisskopf, *op. cit.*, p. 567.

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nucleus containing A nucleons of which Z are protons, then, in atomic mass units,[1]

$$M(A,Z) = 0.99395A - 0.00084Z + 0.0141A^{\frac{1}{3}} + 0.000627\frac{Z^2}{A^{\frac{1}{3}}} + 0.083\frac{(A/2 - Z)^2}{A} + \delta \quad (1.11)$$

The term of interest to the present discussion is the quantity δ, the so-called even-odd term. It is defined

$$\begin{aligned} \delta &\equiv 0 && \text{when } A \text{ is odd} \\ &\equiv -\frac{0.036}{A^{\frac{1}{3}}} && \text{when } N_n \text{ is even, } Z \text{ is even} \\ &\equiv +\frac{0.036}{A^{\frac{1}{3}}} && \text{when } N_n \text{ is odd, } Z \text{ is odd} \end{aligned} \quad (1.12)$$

These relations indicate the dependence of the mass of a nucleus, and therefore its binding energy, upon the number and type of nucleons it contains. An accurate computation of the binding energy of a neutron E_b in a compound nucleus is obtained from the equation

$$E_b = c^2(M + m_n - M_c) \quad (1.13)$$

which may be compared to the approximation (1.6).

Equations (1.11) through (1.13) reveal that neutron absorptions which result in compound structures with even numbers of protons and neutrons acquire the largest excitation energies since the δ term is negative for these nuclei. Compound nuclei with an odd number of nucleons acquire the next largest excitation energies, and odd-odd nuclei, the least. It is on this basis that the isotopes U^{233}, U^{235}, and Pu^{239} can be made to fission by neutron captures of any energy, whereas Th^{232} and U^{238} will fission only with very fast neutrons. In the case of the first three nuclei, a neutron capture leads to an even-even compound structure, and the excitation energy due to the neutron binding energy alone (~6.8 Mev) is equal to the fission threshold. Thus these nuclei can fission by thermal (very slow) neutron capture, as well as by captures of fast neutrons. It is this characteristic which makes these nuclei especially important as nuclear fuels. As will be discussed later, these nuclei fission with such relative ease in the thermal-energy range that it is well worth while to provide means to moderate (slow down) fission neutrons to thermal energies so that this characteristic may be fully exploited. In fact, the problem of neutron moderation is a principal consideration in reactor analysis.

In the case of Th^{232} and U^{238}, the compound nucleus has an even-odd collection of nucleons and the binding energy for this state (~5.3 Mev)

[1] Bonilla, *op. cit.*, pp. 69–72.

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is less than the threshold energy (~7.1 Mev). Evidently, neutron-induced fission is not possible with these nuclei when the neutron kinetic energy is less than 1.8 Mev. This limited fissionability makes these isotopes much less attractive as the primary fuel component for a steady chain reaction, although their presence in a chain supported by any of the "thermal fuels" can lead to significant augmentation.

The energy released from the fission reaction of any of the isotopes mentioned above may be computed from the equation for E_f (1.8). This expression, which applies for the two-fragment divisions, will suffice for most situations of interest since a three-fragment (or more complex) division of the compound nucleus has a very low probability. A rough estimate of the fission energies generally available from such nuclei may be obtained from a sample computation for U^{235}. The resulting value will be representative since the various mass numbers differ but little, and the principal dependence on binding energy occurs in the $\bar{E}_b^{(2)} - \bar{E}_b^{(c)}$ term [cf. Eq. (1.10)]. Consider, therefore, the reaction

$$n^1 + U^{235} \rightarrow \{U^{236}\}^* \rightarrow Kr^{94} + Ba^{142} \tag{1.14}$$

The average binding energy per nucleon $\bar{E}_b$ for a nucleus of mass number 236 is about 7.5 Mev, and for nuclei of mass numbers 94 and 142, 8.6 and 8.3 Mev, respectively. The approximate relation (1.10) yields for E_f:

$$E_f = (236)(8.6 - 7.5) + (142)(8.3 - 8.6) = 217 \text{ Mev}$$

Thus, roughly 200 Mev is released in a typical fission reaction involving a heavy nucleus.

The energy from fission appears principally as the kinetic energy of the fission fragments. As these fragments speed outward from the point of reaction, they encounter the various nuclei of the surrounding environment. Such encounters are relatively frequent since the fragments are usually highly ionized and therefore experience strong coulomb interactions with the electron clouds of these nuclei. These interactions are primarily scattering collisions, and each collision results in the transfer of some of the kinetic energy of the fragment to the struck nucleus. A series of such collisions eventually slow the fragments to thermal equilibrium with the environment. Approximately 85 per cent of the fission energy is liberated in this way and must be removed by a suitable cooling system. The remaining 15 per cent appears either as radiation or as the kinetic energy of neutrinos[1] and neutrons evaporated from the fragments or released at the instant of fission. A detailed breakdown of the energy distribution for U^{235} is given in Table 1.1.

The reaction of Eq. (1.14) gives one mode of division of the U^{235} nucleus as a consequence of fission. Of course, this is not the only one

[1] Halliday, *op. cit.*, p. 94.

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possible and, in fact, any division consistent with the conservation of mass and energy can occur. The fragments which appear cover the entire mass scale; however, certain divisions are favored, depending on the target nucleus and the energy of the captured neutron. The mass distribution of the various fragments resulting from the fission of U^{235}, U^{238}, and Pu^{239} is shown in Fig. 1.3. For a given nucleus, the shape of the yield curve is highly sensitive to the energy of the incident neutron. The figure shows clearly the characteristic grouping of the fission fragments into two distinct regions of the mass scale for the case of a thermal neutron-induced fission. It is seen that such reactions most frequently produce two fragments, one of mass number around 95 and the other of mass number 140. Although other subdivisions can occur, their likelihood is very improbable. Of these alternative schemes, the division of the compound nucleus into two fragments of equal mass occurs in about 0.01 per cent of the time.

TABLE 1.1 DISTRIBUTION OF FISSION ENERGY FOR U^{235}

Form	*Energy, Mev*
Kinetic energy of fission fragments	165 ± 15
Prompt gamma rays	5
Kinetic energy of fission neutrons	5
Fission product decay:	
Gamma	6
Beta	5
Neutrinos	11
Total energy per fission	197 ± 15

The fragments formed in fission are generally very rich in neutrons. This is a consequence of the fact that among the stable nuclei[1] the ratio N_n/Z increases with Z. Thus the fissionable nuclei with $Z \sim 90$ will possess far more neutrons than are required in the nuclear structures of the stable fragments.[2] Nearly all these excess neutrons[3] are released at the instant of fission. These are the *prompt* neutrons. The remaining small fraction of neutrons to be released are evaporated off from the fragments at various time intervals after fission; these constitute the *delayed* neutron groups. In the event that the emission of a neutron or two by a primary fragment does not leave the nucleus in a stable configuration, further nuclear readjustment may then occur through the emission of β radiation (electrons).

For the purpose of studying the neutron economy in a reactor, it will be convenient to rewrite Eq. (1.7) so as to include the neutrons which are emitted by the primary fragments. For the present, we can omit

[1] Bonilla, *op. cit.*, Fig. 3-6, p. 68.
[2] Halliday, *op. cit.*, pp. 408–412.
[3] 99.245 per cent in the case of U^{235}.

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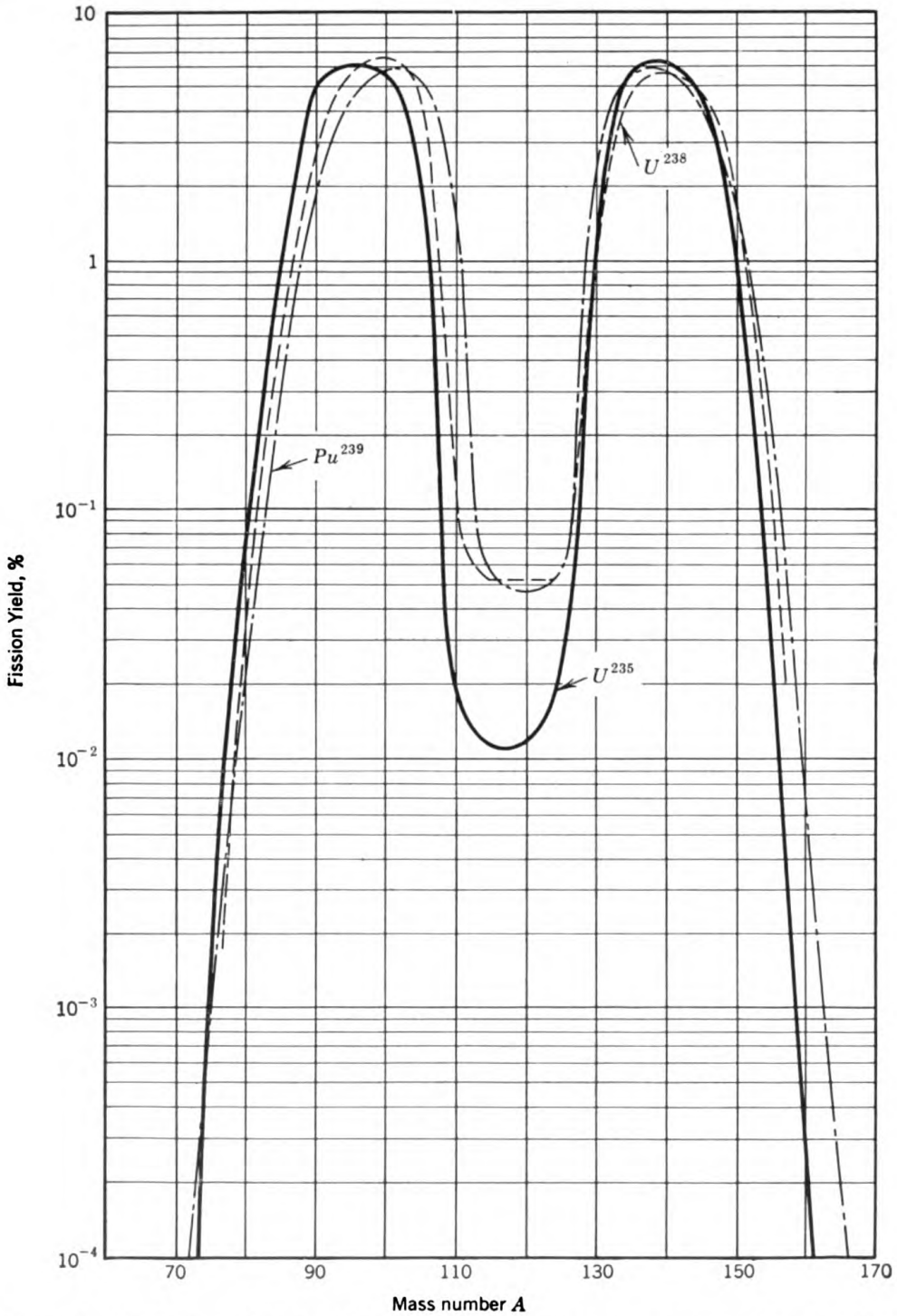

FIG. 1.3 Yields from fission of U^{235}, U^{238}, and Pu^{239} versus mass number A. [*Reprinted from C. DuBois Coryell and N. Sugarman (eds.), "Radiochemical Studies: The Fission Products," Part 3, National Nuclear Energy Series, Div.* IV, *vol.* IX, *McGraw-Hill Book Company, Inc., New York,* 1951; U^{238} *from Paper* 219 *by Steinberg and Freedman, and* U^{235} *and* Pu^{239} *from Appendix B.*]

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the distinction between the neutrons that appear instantly and those which appear at later stages in the decay processes of the fragments. If we denote by ν the total number of neutrons which appear from fission, then it is clear that the nuclei Y and Z will contain fewer neutrons by exactly this amount; therefore, we now write reaction (1.7) in the representative form

$$n^1 + X^A \rightarrow \{X^{A+1}\}^* \rightarrow Y^{A_1} + Z^{A-A_1-\nu+1} + \nu n^1 \tag{1.15}$$

For the case of U^{235} considered in Eq. (1.14), we might have, for example, the following division:

$$n^1 + U^{235} \rightarrow \{U^{236}\}^* \rightarrow Kr^{93} + Ba^{140} + 3n^1 \tag{1.16}$$

We have assumed here that three neutrons are emitted by the fragments.

The number of neutrons emitted from fission is dependent upon the energy of the incident neutron. In first approximation, it is permissible to neglect this dependence and to assume that the *average number* of neutrons produced per fission is a function only of the target nucleus. It should be recognized that this is not a true picture and that, in fact,

TABLE 1.2 NEUTRON YIELDS DUE TO THERMAL FISSION IN NUCLEAR FUELS

Fissionable nucleus	ν, neutrons per fission	η, neutrons per absorption
U^{233}	2.54 ± 0.04	2.31 ± 0.03
U^{235}	2.46 ± 0.03	2.08 ± 0.02
Pu^{239}	2.88 ± 0.04	2.03 ± 0.03

the average number of neutrons increases with the energy of the incident particle. For many reactor calculations, however, this dependence is neglected and some average number is selected for the entire energy range.

The actual value of ν for a given fissionable material can be obtained from experiment. In such measurements, even if the incident neutrons were all of some fixed energy, the number of neutrons produced by fission reactions would be a statistical quantity and would vary from one fission to the next. As a matter of fact, the actual number of neutrons released by any one reaction is of little interest in reactor physics, and one would certainly prefer an average value which could be assigned to a large number of such reactions. Measurements of this type have been made, and the presently acceptable values of ν (the average number of neutrons per fission) for the three principal nuclear fuels are listed in Table 1.2.

c. Nuclear Reactions and Neutron Economy. So far in our discussion of the chain-reacting system, we have focused attention on the fission reaction. Fission is but one of several neutron-induced nuclear reactions

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that can occur in a fissionable material; however, essentially all the remaining neutron-capture reactions are nonproductive, by which we mean that they do not lead to significant energy releases and additional neutrons. It is the competition between these nonproductive processes and fission which in part determines the relative merits of a specific nuclear species as a "fuel material." The other essential factor is the number of neutrons produced per fission (ν).

The two competing nonproductive neutron-capture processes are (1) radiative capture and (2) inelastic scattering. The radiative-capture reaction can be symbolized by the equation

$$X^A + n^1 \rightarrow \{X^{A+1}\}^* \rightarrow X^{A+1} + E_\gamma \tag{1.2}$$

The symbol E_γ here denotes energy liberation in the form of gamma rays (electromagnetic radiation of wavelength in the order of 10^{-10} cm) having about 1 Mev energy. In this process, a neutron is captured by the fissionable material and forms a compound state. However, instead of releasing its excess energy by fissioning, it does so by the emission of some radiation immediately after the capture. This emission may remove all the excitation energy of the compound nucleus and thereby leave it in a stable state. If this initial emission leaves the nucleus with some residual energy, further emissions must eventually follow until a stable configuration is achieved. For the case of U^{235}, the radiative-capture reaction could be written

$$n^1 + U^{235} \rightarrow \{U^{236}\}^* \rightarrow U^{236} + E_\gamma \tag{1.17}$$

The symbol E_γ represents all the excess energy of the compound state (in this case, the excited U^{236} nucleus) which must be released in order to achieve the stable state U^{236}.

The second nonproductive process, inelastic scattering, does not physically remove neutrons from the system, it merely degrades (as a rule) the kinetic energy of the neutron population. For the nuclei of primary interest to reactor technology (heavy nuclei), the inelastic-scattering process is conveniently described with the aid of the compound-nucleus model. In these reactions the nucleus "captures" a neutron at one energy and releases it at another; thus,

$$n^1 + X^A \rightarrow \{X^{A+1}\}^* \rightarrow \{X^A\}^* + n^1 \tag{1.18}$$

A compound nucleus is not always formed, however; some inelastic-scattering reactions are properly symbolized by

$$n^1 + X^A \rightarrow \{X^A\}^* + n^1 \tag{1.19}$$

But, in either case, the essential feature of the inelastic-scattering reaction is that the total kinetic energy of the original particles differs from that of the product particles. If the struck nucleus X^A was before

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collision in a stable state (the ground state), then after collision it will occupy an excited state. The energy of excitation, which appears as an increase in the internal energy of the nucleus, is precisely the energy loss suffered by the incident neutron. Thus, the sum of the kinetic energies of the emitted neutron and the struck nucleus (after collision) will differ from the total kinetic energy of the system before collision by exactly the amount of the excitation energy of the residual nucleus.

A third nonproductive neutron-nucleus reaction which is of some consequence is elastic scattering. Although reactions of this type are of great importance where nonfuel components of a reactor system are concerned, they do not represent truly competitive processes in the case of fuels. When an elastic scattering occurs, the incident neutron leaves the region of the target nucleus with a different kinetic energy[1] from that which it had before the collision. However, the total kinetic energy of the particles involved before and after collision is conserved. From a phenomenological viewpoint, neutron-nucleus interactions of this type have the characteristics of both a "capture" type of reaction and a "deflection" type. The so-called potential scattering component is a deflection-like process and does not involve the formation of a compound nuclear structure. Resonance scattering, on the other hand, may be interpreted by means of the compound-nucleus concept;[2] thus in terms of Eq. (1.19) it can be described as an inelastic-scattering reaction which leaves the struck nucleus with no residual energy. Whereas this process is limited to relatively narrow ranges of incident neutron energies, the potential-type scattering extends over the entire energy spectrum. Of primary importance to the present discussion is the fact that the nuclear fuels are generally of large nuclear mass, and elastic collisions with neutrons produce little change in the kinetic energy of the neutron. Thus, elastic-scattering collisions with fuel nuclei effect little or no change in the neutron population, either by altering the neutron energies or by serving as neutron removers.

Inelastic scattering and radiative capture are the two processes which compete with the fission reaction. The extent of this competition is determined by the relative magnitudes of the rate at which nonproductive reactions occur as compared to the rate at which fissions occur. If the nonproductive reactions for a particular fissionable material constitute only a small fraction of all neutron-nucleus reactions, then the possibilities of this substance as a nuclear fuel are greatly enhanced. The deciding factor, however, is the number of neutrons released by fission. For a given substance, the likelihood of fission taking place may be very

[1] In the laboratory system of coordinates. The kinetic energies in the center-of-mass system are conserved.

[2] See, for example, Halliday, *op. cit.*, pp. 396–402; also, Blatt and Weisskopf, *op. cit.*, chap. VIII, especially pp. 325–329.

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high, but for the purposes of a chain-reacting system, this is of little value if on the average the number of neutrons released per fission is negligibly small. Thus it is the combined characteristics, reaction rates of non-productive processes and number of neutrons per fission, which determine the merits of a fissionable material as a nuclear fuel.

The focus of attention in reactor analysis is upon the fate of the neutrons produced by the fission reactions, rather than upon the fission process proper with its resultant fragments and electromagnetic radiation. For most situations in reactor analysis, the nuclear-fission process may be adequately represented by picturing a given neutron as disappearing at a point and being immediately replaced by ν neutrons whose kinetic energies and directions of motion are very different from those of the incident neutron. The principal objective is to trace the life histories of these neutrons, of the neutrons which they in turn may produce, and of all the subsequent generations.

A detailed study of the life history of the neutrons in a chain-reacting system depends upon a knowledge of the materials and geometry of the assembly which supports the chain. A complete description of a reactor complex is determined by the purpose and application of the device; thus, the configuration and composition will vary from system to system, and an all-inclusive description is not meaningful. Nevertheless, there are certain primary components of every reactor which are common to all. These are:

1. The *core:* The region (usually central) which contains the nuclear fuel, along with various structural materials, moderator, and coolant.
2. The *reflector:* A shell region surrounding the core which usually consists of moderator-type material, supporting structure, and possibly coolant materials.
3. The *shield:* An outer shell which serves as a stopping medium for the biologically dangerous radiations from the core and the reflector.

The core and the reflector region (if any) constitute the principal nuclear components of the reactor. From the viewpoint of the reactor analyst, the main interest rests with the distribution of the neutrons throughout these two regions. Neutrons which pass through the outer boundary of the reflector are effectively lost to the system. Attention is therefore focused on the neutron population within the confines of the core and the reflector.

The core region of the reactor is the source of the neutrons which support the chain reaction. A large fraction of the neutrons produced by fission is lost to the chain process, either by passing through the boundaries of the reactor and thereby escaping from the system, or by removal through radiative capture. Unless the reactor configuration is unusually small, the fraction of the neutron population which is lost by escape is relatively small and the bulk of the losses is due to absorption

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(capture) reactions which do not produce fission. We have already noted that radiative capture in the fuel is a competitive process with fission and, therefore, serves as a neutron-removal process (sink) in the chain-reacting system. This phenomenon is by no means confined to the fuel nuclei. Radiative capture of neutrons by nonfuel components of the reactor is a significant loss factor as well. As in the case of the fuels, the nonproductive capture process in these materials can be described by means of an equation of the type (1.17).

It was mentioned previously that the core region may contain some moderator-type material. (This will not be the case, however, in all reactors; thus in the so-called fast reactors, the absence of moderator materials is greatly desired.) The primary function of a moderator is to degrade the neutron energies through the mechanism of scattering collisions. We observed previously that scattering collisions, both elastic and inelastic, cause the incident neutron to transfer part of its kinetic energy to the struck nucleus. Whether or not collisions of this type tend to improve the conditions for maintaining a chain reaction depends upon the type and function of the reactor.

The primary purpose of the reflector is to return some of the neutrons which escape from the core. Generally speaking, suitable reflector materials are also good moderators; thus, intentionally or not, the reflector region serves both to degrade the neutron energies and to reduce the escapes from the system.

In undertaking a first crude attempt to understand the various physical features of a chain-reacting system, it will be helpful to picture the life history of a typical neutron as consisting of the following stages: (1) The neutron is born from fission somewhere in the core. (2) It will then wander about through the core and possibly pass back and forth into the reflector several times during its lifetime. (3) Eventually, it will disappear by one of two processes, absorption by a nucleus in either the core or the reflector, or by escaping from the reflector.

Only if the neutron is absorbed by a fuel nucleus is there a chance that additional neutrons will be produced.[1] If there is to be a stable neutron chain reaction, the ν neutrons produced by a given fission must excite, on the average, one further fission, thus replacing these ν neutrons and thereby perpetuating the chain.

This simple model may be used to describe in rather broad terms the requirements on a stable chain reaction. If n denotes the total number of free neutrons available at any time instant in the reactor, then it is clear that if the neutron population is to be stable the number n must be maintained for all subsequent time. A certain fraction of these, however,

[1] This is not strictly true since there are such nuclear reactions as the $(n,2n)$ reaction in beryllium (capture of neutron producing two neutrons) which can serve as secondary neutron sources.

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will escape from the system, and the remainder will be captured by various nuclei. Thus absorption or escape is the only eventuality open to a neutron. If we define

$f_a \equiv$ fraction of neutrons in the reactor which are eventually absorbed (by either fuel or nonfuel nuclei)
$f_e \equiv$ fraction of neutrons in the reactor which eventually escape
$f_f \equiv$ fraction of neutrons in the reactor which cause fission in the fuel nuclei
$f_\gamma \equiv$ fraction of neutrons which experience radiative capture by fuel nuclei
$f_o \equiv$ fraction of neutrons which experience radiative capture by nonfuel nuclei

then of the available n neutrons, nf_a are absorbed and nf_e escape; therefore,

$$1 = f_a + f_e \tag{1.20}$$

where, by definition,

$$f_a \equiv f_f + f_\gamma + f_o \tag{1.21}$$

Of the nf_a neutrons absorbed, nf_f cause fission, and from these fissions, νnf_f neutrons are emitted. If the population n is to remain at steady state, we must require that

$$n = nf_a + nf_e = \nu nf_f \tag{1.22}$$

and it follows that

$$\frac{\nu f_f}{f_a + f_e} = 1 \tag{1.23}$$

This relation may be rewritten with the aid of Eq. (1.21); thus,

$$\frac{\nu f_f}{f_f + f_\gamma + f_o + f_e} = 1 \tag{1.24}$$

For convenience, we define the ratio

$$\alpha \equiv \frac{f_\gamma}{f_f} \tag{1.25}$$

This quantity is strictly a function of the nuclear characteristics of the fuel. Equation (1.24) may be written in terms of α by making a direct substitution from (1.25). The result is

$$\nu = 1 + \alpha + \left(\frac{f_o + f_e}{f_f}\right) \tag{1.26}\star$$

The importance of this relation rests with the fact that for a given fuel, ν, the average number of neutrons per fission, must be *at least as large as* $1 + \alpha$ if the chain reaction is to be self-sustaining.[1] Thus, the number

[1] Without any external sources, i.e., no sources of neutrons other than fission reactions.

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of neutrons per fission must exceed unity by precisely the amount which accounts for the nonproductive captures in the fuel (represented by the a term) and the captures in the nonfuel components and the escapes from the reactor [the last term on the right in (1.26)]. It should be noted that the fraction of neutrons which escapes from the reactor is dependent on the size of the system. At best, $f_e = 0$, a situation which is possible only in an infinite geometry. Clearly, even for an infinite system, some losses (radiative capture) still occur.

Much of the attention in reactor analysis is devoted to the calculation of the fractions f_a, f_e, etc., defined above. A knowledge of these quantities forms the basis for determining the neutron-density distributions in the reactor. We will later derive suitable procedures for computing these quantities, not only as a function of the spatial coordinates, but also in terms of other independent variables of interest, such as the neutron energies.

Some mention must be made of the frequent use of the phrase "on the average." As in the kinetic theory of gases, we are dealing in the problems of reactor physics with very large numbers of particles which are moving sufficiently rapidly so that the elapsed time between collisions is very short on the laboratory scale. For example, the entire lifetime of a neutron in a reactor from birth in a fission process to final death in absorption or escape may be of the order of 10^{-5} to 10^{-3} sec. The properties of the assembly which are usually measured are bulk properties which depend on the occurrence of huge numbers of nuclear collisions; thus, the actual fate of a given neutron is of little interest. What we really seek in every case are the various possible life histories a neutron may have and the probability of each such life history; then we may take an average over these probabilities, to find some macroscopic property of the system which is of interest. Inasmuch as these problems involve large numbers of particles and time intervals long in comparison with their average free flight time (time between collisions), the deviations in the behavior of the assembly from the average expected behavior will be very small. So long as these conditions hold, we may replace the actual assembly by a model in which all particles behave according to some average pattern. This approach is used over and over again in reactor analysis; consequently, the phrase "on the average" is heavily worked and becomes such a burden that it is often omitted as understood.

1.2 Nuclear Reactors

a. Classification of Nuclear Reactors. From the viewpoint of the reactor physicist, nuclear reactors may be classified according to two criteria. These are: (1) the geometric arrangement of the nuclear fuel and moderator materials, and (2) the average kinetic energy of the neu-

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trons which cause the bulk of the fission reactions. In terms of the first criterion, nuclear reactors are separated into two categories, namely: (*a*) homogeneous reactors, and (*b*) heterogeneous reactors.

The principal nuclear constituents in the core of a homogeneous reactor are intimately mixed. The fuel-bearing substance will appear, in general, as some molecular compound, dissolved in a suitable solvent to form a solution, suspended in a fluid carrier to form a slurry, or fused at some appropriate temperature (with or without a carrier) to form a homogeneous fluid. The purpose and operating conditions of the reactor complex determine which of these possibilities is the most attractive for a given engineering application.

Heterogeneous reactors are characterized by the geometric separation of fuel and moderator materials. In systems of this type, the fuel may appear in the form of metal (or ceramic) plates, rods, or lumps which are distributed throughout the moderator according to some prescribed lattice configuration. Heterogeneous reactors necessarily contain moderator material, in contrast to homogeneous systems which may not. However, the mere physical separation of the fuel and moderator materials does not establish a reactor as belonging to the heterogeneous category. This point is perhaps better illustrated by the extreme example of a reactor core which consists of "tissue-thin" foils of fuel material closely packed in a medium of moderator. In so far as the neutron population is affected, this configuration is equivalent to a homogeneous mixture of fuel and moderator. The neutrons do not "see" the geometric discontinuities in passing from fuel to moderator to fuel, etc., because of the "fineness" of the physical structure. This degree of fineness is determined jointly by the various dimensions of the system and by considerations of the average energy of the neutron population and the material composition.

The basis for the choice between a heterogeneous and a homogeneous reactor is determined in large part by the available fuel enrichments, heterogeneous arrangements being most attractive (if not the only possibility) for low-enrichment fuel materials. The now classic example of this application was the choice of a heterogeneous configuration for the first nuclear reactors. The famous University of Chicago reactor (CP-1) designed and operated by Enrico Fermi's group, the X-10 graphite reactor at the Oak Ridge National Laboratory, and the Hanford reactors were all heterogeneous systems. In each case the selection of the heterogeneous configuration was dictated by the fact that the then-available quantities of enriched U^{235} (the only nuclear fuel found in nature) were insufficient to charge a reactor, and it was necessary to employ natural uranium which contains only 0.714 per cent U^{235}. Homogeneous systems utilizing mixtures of various moderators and natural uranium, however, cannot support self-sustaining chain reactions because of the high neu-

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tron-absorption properties of these materials. The only exception is heavy water (D_2O). The other good moderators, such as graphite, Be, BeO, and H_2O, require enriched fuels, or if natural uranium is to be used, it must be arranged in a heterogeneous configuration. (The "lumping" of the fuel gives it an advantage over the moderator in competing for the available neutrons and thereby improves the possibilities for sustaining the chain reaction.) In the early days of reactor technology, only graphite and water were easily available, and of the two, graphite had the lesser affinity for neutrons and was therefore selected for the earliest designs.

The energy spectrum of the neutron population in a reactor is another useful criterion for classifying reactors. This description, when coupled with the specification of the geometric arrangement of fuel and moderator, gives a reasonably complete picture of the reactor physics of the system. Now it was noted in the preceding section that fission neutrons are born into the system with energies of a few Mev, but are subsequently slowed to thermal speeds by a succession of elastic- and inelastic-scattering collisions with the surrounding nuclei. Thus neutrons are distributed over the entire energy scale from thermal to several Mev. If one were to plot the density of neutrons as a function of energy, the resulting distribution would reveal that in each system the bulk of the neutron population occupies a relatively narrow band. It is convenient, therefore, to speak in terms of an "average energy" of the neutron population.

It was also mentioned previously that the fission-reaction rate, as well as the rates of the other neutron-induced reactions, is a direct function of the neutron energy. Thus, if the neutron-energy spectrum is known, the total fission rate in the reactor can be computed as a function of neutron energy, and an "average energy of fission" can be derived from this function. It is customary to classify reactor types in terms of this average fission energy. For this purpose the entire energy scale is divided into three domains. If the average fission energy is greater than 100 Kev, the system is called a *fast reactor;* if the average fission energy lies between 100 Kev and thermal energy, the reactor is designated *intermediate;* and if the bulk of the fissions are caused by thermal neutrons, the reactor is designated *thermal.* In some areas of reactor technology, the expression *epithermal* has become popular for describing reactors in which the average fission energy is in the electron-volt range.

Figure 1.4 shows typical energy spectrums for the neutron population in the three principal reactor types. The ordinate in these curves gives the relative density of neutrons as a function of their kinetic energy. The intent here is to present some illustrative sketches which will give some idea of the nature of the distribution function. Other than indicating the general shape of these curves, the only feature of interest is the relative "sharpness" of the spectrum in the thermal reactors.

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The classification of the neutron-energy spectrum of a given reactor is determined principally by the neutron-moderating materials which it contains. If the nuclear masses of the nonfuel components of the reactor are relatively low, then the neutron spectrum will correspond to that of a thermal reactor (cf. Fig. 1.4*c*); if they are large, a "fast" spectrum will result (cf. Fig. 1.4*a*). The spectrums of intermediate reactors may be due to a number of nuclear characteristics, the presence of nuclear masses of moderate magnitudes being one cause.

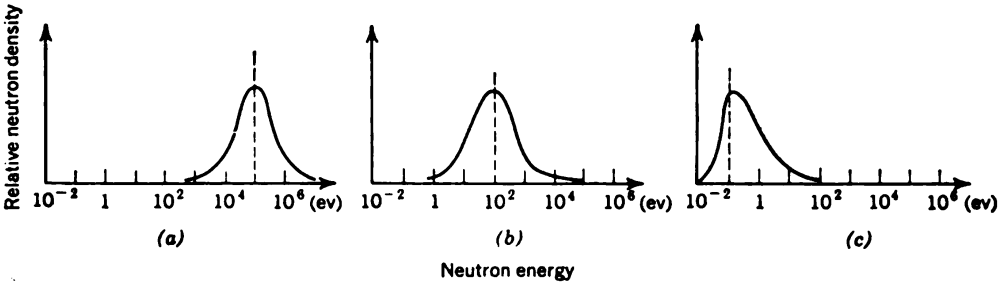

FIG. 1.4 Neutron-energy spectrum in (*a*) fast, (*b*) intermediate, and (*c*) thermal reactors.

b. Applications of Nuclear Reactors. Since the successful operation of the first reactor by Fermi's group at Chicago in 1942, nuclear reactors have been utilized for a number of important applications. These can be grouped under the following three headings: (1) power production, (2) research, and (3) breeding and converting.

The first of these applications, power production, may be further classified according to three specific functions: central-station power, package power, and mobile power. Central-station-power applications of nuclear reactors refers to the production of power for large municipal or industrial areas. Package-power applications, however, refer to power production for limited facilities. In this classification, one might include power plants for areas in unusual climatic conditions, advanced bases, or small, isolated establishments. The application of nuclear reactors to power mobile units can conceivably include any device which is designed for terrestrial operation or, for that matter, space craft as well. The possibilities in this area of application are as yet hardly known.

The application of nuclear reactors for research purposes includes the production of high-intensity neutron and radiation fields and the production of radioisotopes. The use of radiation and neutron fields for experimental research in physical and biological sciences is well established, and the construction of nuclear reactors for use in university and research laboratories is already under way. The production of radioisotopes has, likewise, become an important application of nuclear reactors, no doubt stimulated by the considerable success of radiotracer techniques.

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The third, and the earliest, application of nuclear reactors was for the production of fissionable materials. The first of these devices were the Hanford reactors which were built for the production of plutonium by the conversion of the U^{238} in natural uranium.[1] Nuclear reactors which produce fissionable material are generally classified as *breeders* or *converters*. It has become the custom to distinguish between these two reactor types by designating as breeders those reactors which produce fissionable material of the same kind as the actual nuclear fuel which they consume. Converters, on the other hand, are reactors that produce fissionable material different from the reactor fuel which maintains the chain reaction. The best-known example of a converter is the Hanford type reactor which produces Pu^{239} from U^{238}, using the U^{235} in the natural uranium as the fuel. The important fact in any case is that some fissionable material is produced from the *fertile material* added to the reactor. The U^{238} in the Hanford reactors is such a material, as is Th^{232} (which yields U^{233}). In either of these fertile materials, the nuclear process involved is the capture of neutrons by the nucleus and the subsequent formation of fissionable nuclei. Although breeders and converters provide only a portion of the fissionable material required in present reactor technology, they are, nevertheless, an important source. Another application of these reactors which is being given increasing attention in recent times is the combined production of fissionable material and power. The idea of a power source which is capable of producing its own fuel is, of course, highly attractive from the economic standpoint. Although many such devices have been proposed, their practicability is not yet entirely established. Many questions in both nuclear physics and economics must be answered before such applications can become engineering realities.

In Sec. 1.2*a*, we discussed the classification of reactors according to their nuclear characteristics. There are important correlations between these characteristics and the operational purpose of the reactor. This interdependence is due in part to engineering considerations and in part to the nuclear properties of the reactor constituents. A simple illustration of this point is found in the case of the thermal reactors. Great pains are taken in the design of these systems to obtain efficient moderator materials to slow the fast fission neutrons to thermal energy. The reason a thermal spectrum is desired is that the fission characteristics of the principal fuels are superior in the thermal-energy range. As will be shown later, this results in a minimum requirement in fuel mass. This consideration would be important in a design situation which is limited by the scarcity of nuclear fuels. We have here another reason for the selection of the thermal-reactor type for the earliest designs.

[1] The radiative capture of a neutron in U^{238} yields Pu^{239} [cf. Eq. (1.17)].

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1.3 The General Reactor Problem

a. Description of Neutron Population. A complete understanding of the nuclear behavior of a reactor requires a detailed description of its neutron population. In its most general form, such a description must give the distribution of the neutrons in space, energy, direction of motion, and time. A study of the various nuclear features of a reactor usually begins with a general statement consisting of a detailed neutron count which is summed up in an integrodifferential equation of the Boltzmann type. These equations are conveniently written in terms of the neutron-density function, and the statement of the equation gives a balance relation between the various nuclear processes which can affect the neutron population.

Generally speaking, the dependence of the neutron density upon all seven independent variables implied above is customarily indicated in the initial, very broad statement of the problem. Very seldom is such generality required in actual practice. In most situations, attention is confined to certain specific aspects of the neutron problem, and less precise descriptions involving only a few variables will often suffice. For example, the time-dependent behavior of the neutron population is of interest only in regard to the nuclear stability and control of the reactor, and in many cases an analysis of the steady-state problem is entirely adequate. Likewise, the directional dependence of the neutron density (i.e., the specification of the direction of motion of the neutrons being studied) is of little interest except in regions of the reactor system close to discontinuities in the material composition (boundaries, etc.).

Only when a complete description including all seven variables is required is it necessary to solve the Boltzmann equation in all its generalities. Frequently, simplifying assumptions and limiting conditions can be imposed which reduce the integrodifferential equation to more tractable form. Thus much of the subject of reactor analysis is devoted to the development and the application of simplified analytical models which define, within the limits of engineering needs, the nuclear characteristics of the reactor complex.

b. The Problems of Reactor Physics. It is convenient to list the various types of nuclear-reactor problems under the following general categories: (1) critical-mass calculations, (2) neutron-density and fission-energy distributions, (3) control effectiveness, (4) reactor stability, (5) fuel burnup and production of reactor poisons, (6) nuclear accidents and special problems. The computation of the *critical mass*[1] for a given reactor system is one of the first tasks with which the reactor analyst is confronted. It is, likewise, one of the easier calculations to perform.

[1] The nuclear fuel mass required to sustain a chain reaction in a reactor configuration of specified composition.

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Frequently coupled with this problem is the determination of the optimum nuclear configuration which yields a minimum fuel mass. Reasonable estimates[1] for preliminary studies can be made with relatively little effort, and many crude analytical models are available for this purpose. Accurate estimates require more elegant methods or the use of *critical experiments*. Although precise mass figures per se are only infrequently required in modern practice, this information is usually available in every reactor study as the by-product of solutions to more essential problems involving neutron-density distributions. As a practical matter, relatively large discrepancies in mass estimates can be readily accommodated with the increased availability of high-enrichment fuel samples.

Neutron-density and fission-energy distributions in a reactor can also be obtained with sufficient accuracy for many needs by the use of relatively simple models. Again, more exact information requires elaborate mathematical analysis which is usually carried out with the aid of fast computing machines or simulators. Occasionally associated with these problems is the determination of the fuel-loading pattern which will yield a spatially uniform production of power throughout the reactor. Except in the case of relatively simple core geometries, nonuniform fuel-loading problems can be treated only by numerical methods. The distribution of neutron density and power production are closely related and of paramount interest to the reactor analyst. A knowledge of these distributions determines the design of the cooling system and provides a detailed count of the neutron population. Once the density of neutrons is known at every point in a reactor, the reaction rates of all the important neutron-nucleus interactions are easily computed. Such information yields the rate of fuel consumption, the production of radiation, the rate of fuel breeding and conversion, and the flow of neutrons out from the reactor surfaces. This last factor determines, in part, the shielding requirements and the intensity of neutrons beam which may be available for research purposes.

The problems of *control effectiveness* deal with those considerations which enter into the design and the operation of the nuclear elements which control the neutron population. The mechanical configuration of these elements may take many forms; the determining factor is usually design and operating convenience. The customary device is a rod or some other cylindrical unit which contains a material that is a strong neutron absorber. By some appropriate motion of the unit, this material is brought into a region of the reactor which contains a high density of neutrons. By either withdrawing or inserting this neutron *sink*, the neutron balance can be quickly altered and the population caused to rise or fall, thereby accelerating the chain reaction or shutting it off. Other control devices that have also been successful include the

[1] In the order of 10 to 15 per cent accuracy.

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use of *burnout poisons* and *movable reflectors.* The use of burnout poisons for control would be better included with the group of reactor-physics problems dealing with long-time nonstationary processes, such as fuel burnup. Control devices which utilize movable reflectors rely upon the control of the neutron population through changes in the rate of neutron loss by escapes from the reactor. The problem of estimating accurately the effects of such devices on the neutron-density distribution is difficult; however, if the effect of the device is not "too large" in the nuclear sense, perturbation methods may be useful for such calculations. Otherwise, multiregion analyses may be required which in some instances involve the solution of the Boltzmann equation. Here again critical experiments are helpful in checking the analysis.

Although most reactors contain some sort of external control device, at least to meet emergencies and make planned shutdowns, such devices are not required for all types of reactors. Many reactors in operation today contain some degree of inherent nuclear stability. Reactor designs which incorporate liquid coolants are frequently of this type. In such reactors, any disturbance (or control command for that matter) which momentarily changes the operating condition of the system (e.g., opening the throttle on the turbine of a power reactor) will cause some fluctuations in the neutron population and power production, but these quickly decay and the system returns to steady-state operation. The prediction of the thermal response of the reactor to these disturbances belongs to a related class of problems. All these involve questions of dynamic response and stability, and in the majority of the cases of engineering interest they are nonlinear in character. Although some insight into the physical features of a given situation can be gained from linearized approximations to the problem, the nonlinear solutions are ultimately required for a complete understanding. Much of this work is carried out with the aid of simulators and high-speed computing machines.

Another class of time-dependent problems of concern to the reactor physicist are questions on fuel burnup, poison production and burnup, breeding ratio, and the like. These problems differ from those on reactor stability in that they involve time scales measured in hours (or years) in contrast to stability problems which are concerned with fractions of a second. Reactor-analysis problems, such as the determination of critical mass and neutron-density distributions, are based on the steady-state operating condition of the reactor. The day-to-day operation of the reactor at steady state involves, however, long-time changes in the fuel concentration. Except in the case of circulating-fuel reactors, the fuel is introduced into the reactor according to some predetermined cycle. As the fuel is consumed,[1] some gradual adjustments can be made by means

[1] The absorption of a neutron by a fuel nucleus "destroys" the nucleus whether it fissions or not.

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of the controls (shim rods) which will introduce additional fuel into the reactor and thereby maintain the chain reaction.

The rate of fuel consumption and the amount of control adjustment available for this purpose are among the problems which confront the analyst. Closely related to the fuel-burnup problem is the production of fission fragments which are strong neutron sinks. These nuclei are called *poisons* and are serious competitors for the neutrons in the chain reaction. The production of these nuclear species and their concentrations at all times during the operating life of the reactor are determining factors in the design of the control elements and the fuel-loading requirements. Another set of problems which are similar in nature to those of fuel burnup and poison production is the production of fissionable material from fertile nuclei. The time scales involved are comparable, and the same analytical methods may be applied in each case.

In addition to the class of conventional reactor-analysis problems mentioned above, one frequently encounters special problems which arise from emergency or unusual operating conditions of the reactor. It is convenient to include also in this set the analytical problems dealing with special loading configurations designed to accommodate experiments. Such situations as the placing of test specimens and measuring devices in or near the reactor proper require special attention.

c. Methods of Attack and Analytical Models. In the present discussion we summarize the principal analytical models and tools which have proved useful for treating reactor problems. It is believed that this information will serve best as an outline to locate and orient the various ideas and methods within the over-all structure of the subject.

The general field of problems described above, except in some special areas, may be treated by the well-known methods and analytical models of mathematical physics. It has already been noted that the most general description of the neutron population usually starts with a neutron-balance relation of the Boltzmann type. The Boltzmann equation was developed in connection with the study of nonuniform gas mixtures, and the application to the neutron problem represents a considerable simplification of the general gas problem. (Whereas in gas problems all the particles are in motion, in reactor problems only the neutrons are in motion.[1]) The fundamental equation of reactor physics, then, is already a familiar one from the kinetic theory. Further, many of the most useful neutron models obtained from approximations to the Boltzmann equation reduce to familiar forms, such as the heat-conduction, Helmholtz, and telegraphist's equations. These simplifications result from the elimination of various independent variables in the

[1] This is not strictly true because the nuclei themselves are in motion, but in many problems the neutron speeds are very great, and nuclear motion can be neglected in comparison.

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general formulation and are usually introduced by virtue of certain limitations imposed on the physical situation.

Perhaps the most elementary, and limited, model in reactor analysis is the so-called *infinite medium*. The essential feature of this model is, of course, that the geometric confines of the real problem are ignored and attention is focused on other aspects of the situation. The infinite-medium model is especially useful for treating reactor problems which deal with phenomena well inside the physical boundaries of the system. Also, chain reactions in systems that are large in a nuclear sense can be handled in this way. An especially important problem in reactor physics, the neutron-energy spectrum, may be studied by means of this model. The question to be answered here is the determination of the density distribution of neutrons as a function of kinetic energy from the instant a neutron appears as a fission particle down through the slowing-down process until it is captured.

A second model which has great usefulness in treating reactor problems is the *one-velocity* model. We noted in Sec. 1.2 that the energy distribution (spectrum) of the neutrons in the chain has a sharply peaked form (cf. Fig. 1.4). As a first approximation, one can treat the neutrons in the reactor as though they all possess the same average energy defined by the shape of the spectrum. As crude as the approximation may seem at first, it has proved to be extremely effective for treating a large variety of reactor problems. It is especially useful for studying thermal reactors. By eliminating the general energy dependence through the assignment of a single energy (or the corresponding one velocity) to all the neutrons in the chain, the Boltzmann equation for the multiplying medium[1] can be reduced ultimately to the Helmholtz (stationary-wave) equation; this approximation is frequently called the *neutron-diffusion equation*. The application of this model limits the description of the neutron population to its spatial dependence.

A third analytical model which combines the most important features of the two already described is the continuous slowing-down diffusion model, better known as the *Fermi age* model. Fermi age theory offers a first-order approximation to the Boltzmann equation. In this model, the neutron population is described in terms of two independent variables, energy and position. The directional dependence of the neutron density is eliminated by assuming that the velocity vectors of neutrons in regions far from boundaries are nearly uniformly distributed, i.e., isotropic. The application of this assumption reduces the Boltzmann equation to a partial differential equation of the heat-conduction type. Fermi age theory is most effective for analyzing reactors which contain a moderator of relatively large mass (anything heavier than deuterium). It is most useful for obtaining fuel-mass calculations and neutron-density distributions.

As we have already mentioned, reactors are usually designed to

[1] A system containing fissionable material.

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include a reflector. Thus, the simplest configurations, at best, require the solution of a two-region (core and reflector) problem in order to obtain the neutron-density distributions. As a rule, engineering requirements complicate the reactor system considerably, and the analyst is frequently confronted with three- and four-region boundary-value-type problems. Both the elementary one-velocity diffusion model and the Fermi age theory may be applied to these systems. Diffusion theory is the most easily applied of the two and can be extremely useful in handling multiregion problems, provided that the one-velocity approximation gives an adequate description of the neutron-energy spectrum. If diffusion theory is inadequate, and the energy distribution must be taken into account, the Fermi age model may still be used; however, the analysis of multiregion systems which include the neutron-energy spectrum usually requires the application of numerical methods. The so-called *multigroup* methods have been developed for this purpose. Many of these methods are based on the continuous slowing-down model of Fermi age theory. One very useful model which has been devised for treating multiregion reactors is the *two-group model.* Various forms of this model have been used for analyzing reactors of both the thermal and fast type. The essential feature of the model is the description of the neutron population by means of two energy groups. The bulk of the neutrons which cause most of the fissions is treated as one group, and neutrons of all other energies are treated as a second group. The particular merit of this method is that reasonably good estimates of the nuclear properties of the reactor can be obtained by hand calculation. More involved multigroup (three or more) methods must be handled on large computing machines.

Another type of nonhomogeneity encountered in reactor analysis is that associated with heterogeneous reactors. The multiregion configurations mentioned above are on a relatively large scale, and each region in itself (e.g., the core, the reflector) is a homogeneous medium. In a heterogeneous reactor, the core consists of a lattice of many small cells, each of which contains both moderator and fuel. The moderator and the fuel exist as separate regions; however, the scale of the nonhomogeneity is much smaller than in the general multiregion problem. An analysis of this type of reactor requires a detailed study of the nuclear characteristics of the individual cells. In particular, the analyst must determine the space and the energy distribution of neutrons in these cells. From this information, he can readily predict the over-all nuclear characteristics of the reactor. For many purposes, simple diffusion theory and the Fermi age model will give adequate descriptions of these distributions in the cells. More accurate analyses require the application of semiempirical models or higher-order approximations to the Boltzmann equation. *Exponential experiments* are also used in this connection.

All the models which have been mentioned can be altered to include

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the time dependence of the neutron-density distributions. Thus, each of these models can be applied equally well to nonstationary reactor-physics problems as to steady-state problems. However, the differential equations which include the time dependence can be easily solved for only a few simple reactor geometries. More complex configurations (which involve two or more regions), and systems which include energy dependence as well, are more difficult to treat. Again, one must resort to numerical or graphical methods for analyzing these systems. Time-dependent problems which include the coupling between the reactor-power output and the neutron population will, in general, involve a nonseparable relationship between the time and the space variables. However, in many instances, the important features of the problem can be determined by using simple physical models which do separate out the space and the time variables. Such solutions will not yield an accurate description of the neutron physics, but they can give valuable insight into the basic characteristics of the system in question.

The analytical treatment of reactor-physics problems is derived in main part from the fundamental Boltzmann equation. These models are approximations of this equation to various degrees of accuracy. Which model is selected for a particular reactor problem is determined by the accuracy required and the nature of the phenomenon to be analyzed. When the problems deal with the bulk, or integral, neutron characteristics of the reactor, relatively crude models can usually be applied. Nuclear-fuel-mass calculations fall in this category. On the other hand, more detailed studies require more elegant methods. This is a consequence of the relatively small dimensions which are involved in these problems. In these situations, many of the simplifying assumptions about the neutron-density distribution and the composition of the media (such as isotropy) are no longer valid, and more precise descriptions must be used. Problems which involve regions close to material boundaries and heterogeneous cell calculations are problems of this type. Accurate descriptions of the physical situation in these problems can be given only by the Boltzmann equation. This is not to say, however, that the Boltzmann equation must be solved in all such cases. Again, cruder models have been devised which can be applied even to these problems, and previous experience can always reduce appreciably the work required to obtain adequate solutions. One of the purposes of the present work is to explain some of these approximations which have proved to be useful and to demonstrate their application to typical situations.

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CHAPTER 2

PROBABILITY CONCEPTS AND NUCLEAR CROSS SECTIONS

2.1 Neutron-Nucleus Interactions

a. Neutron Life Histories. Detailed analyses of the problems of reactor physics require in each instance a description of the average behavior of the neutrons in the reactor. The average behavior is obtained through an understanding of the probabilities of the various neutron-nucleus interactions which can occur. From a knowledge of these probabilities can be obtained a complete description of the neutron-density distributions and of the reaction rates of all neutron-induced processes throughout the reactor. The nuclear reaction rates determine the competition between the fuel and other nuclear components of the reactor for the free neutrons and thereby establish the characteristics of the chain reaction. If the fuel can compete favorably, the chain will be self-supporting and the neutron population either will be maintained or will increase. If the competition is unfavorable, the neutron population will eventually die off and all nuclear reactions come to a halt.

As an example of a typical life history of a neutron, consider the trajectory of a neutron born from fission in a reactor which consists of a single medium (i.e., no reflector). On the average, the neutron will be ejected from the point of fission with a relatively high kinetic energy (about 2 Mev) and with a random direction. It will move outward from this point in a straight-line path until it encounters a nucleus or passes through the boundaries of the system. Throughout reactor analysis, we consider the region outside the medium of interest as being void of material so that there is no chance for back-scattering of the neutrons.[1] Thus, a neutron which passes through the boundaries of the reactor is lost for good. If, on the other hand, the neutron encounters a nucleus, by which we mean if the neutron passes sufficiently close to a nucleus that the nuclear forces interact,[2] then a reaction will occur which will either remove the neutron or alter its energy and direction.

[1] In reality, this will seldom be the case. In most situations, the reactor will be surrounded by the atmosphere, and some neutrons will always be scattered back to the reactor by collisions with air molecules. But, these events are negligibly few in number and can be ignored in so far as their effects on the chain reaction are concerned.

[2] The range of nuclear forces is in the order of 10^{-13} cm.

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For the present, we will not concern ourselves with the details of the various collisions which can occur. It will suffice to distinguish between scatterings and captures. If the neutron is scattered on its first collision, it will lose some energy and will move on in a new direction. Again, there is a possibility that this new direction will allow the neutron to reach the boundary of the reactor without further collisions and so escape from the system. On the other hand, if a collision occurs, there is once

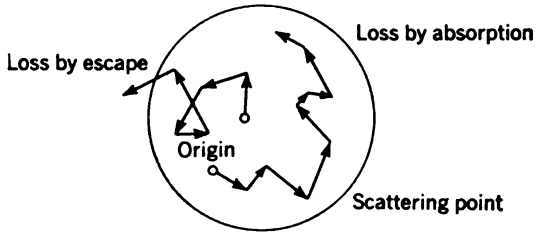

FIG. 2.1 Spatial trajectory of a neutron.

more a probability that the neutron will be absorbed or that it will be scattered, with subsequent loss of energy and change in direction. If this second collision occurs and it too is a scattering, then once again there is a possibility that the neutron will be lost by escape or absorption. This scattering process will continue until the neutron is eventually lost to the system. For any single neutron, the spatial trajectory will be some complicated series of broken straight-line paths which extend from its point of birth to the point at which it is eventually absorbed or leaves the boundary of the reactor (refer to Fig. 2.1).

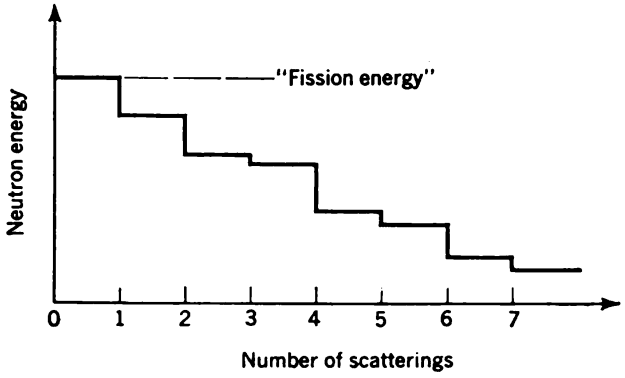

FIG. 2.2 Neutron trajectory in energy-space.

The trajectory of the neutron in terms of its kinetic energy is determined from its speed over each straight-line portion of its path. Thus, the trajectory in *energy-space* is a staircase function, as illustrated in Fig. 2.2. Note that each scattering collision causes a jump change in the neutron energy. As we have already mentioned, the history of any one neutron is of little importance to the over-all neutron problem. We are concerned only with the average behavior of the neutrons, and this can be

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described in terms of the probabilities of the various nuclear events and the probability for escape at each instant during the life of a neutron. One of the goals of reactor analysis, then, is the calculation of the probability of a neutron-nucleus interaction of every possible type.

b. Types of Interactions. Nuclear collisions may be classified under two main categories, absorptions and scatterings. It will be convenient for the purposes of reactor analysis to define absorptions in the following way:

Absorption: Includes all neutron-nucleus reactions whose end products do *not* include a neutron and the original nucleus. The following processes are specifically included in the absorption interaction:

1. *Capture* of a neutron: all neutron absorptions which do not produce fission, for example, radiative capture (n,γ) and charged-particle producing reactions such as (n,α) and (n,p), etc.
2. *Fission*

The various scattering interactions are included under the following general heading:

Scattering: All neutron-nucleus reactions which produce a neutron and a nucleus that differ from the reactants only in their energy content (kinetic and/or internal). Scattering reactions are of two types:

1. *Elastic scattering:* The products of the reaction are a neutron and the struck nucleus in its original internal-energy state.
2. *Inelastic scattering:* The products are a neutron and the struck nucleus in a higher internal-energy state.

For the present discussion, all the neutron-nucleus reactions defined above will be included under the general heading *collision.* Thus, when we talk about the probability of a nuclear collision, we imply any one of the four principal reactions: capture, fission, elastic scattering, or inelastic scattering.

2.2 Probability of Collision

a. Average Behavior. The calculation of the number of neutron-nucleus collisions that occur per unit volume per unit time at each space point of a medium supporting a neutron gas is based on a knowledge of the various interaction probabilities between the neutrons and the nuclei of the medium. These probabilities depend upon the varieties and densities of the nuclei present, as well as upon the relative speed of the neutrons.[1]

A typical cubic centimeter of a reactor may contain 10^{22} nuclei and 10^{7} neutrons. In any calculation involving such enormous numbers

[1] The nuclei are generally in motion, and the relative speed of the neutrons is given by the magnitude of the vector difference between the velocity of the neutron and that of the nucleus.

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of particles, it is neither possible nor desirable to follow the detailed behavior of each particle and to obtain in this way the exact number of collisions at each space point and instant of time. Instead, it is entirely sufficient to concentrate attention upon the *average behavior* of the particles and to calculate the *expected* number of collisions. As usual in statistical problems, the large number of events considered guarantees that deviations from the expected behavior will be slight for the individual particle. The exact details of the life history of a neutron wandering through a medium are not important, since attention is confined to the average behavior. This average behavior is obtained by consideration of the functions describing the probabilities of the various nuclear collisions[1] that the neutron experiences. The present section is devoted to the construction of these probability functions.

b. A Physical Approach. As an initial approach to the basic probabilities for neutron-nucleus interactions, consider the following situation: A number of neutrons n_0, all of a given speed, are incident normally upon a unit area of surface of a uniform semi-infinite medium (see Fig. 2.3). Each neutron will penetrate some distance into the medium before suffering its first collision with a nucleus. For the present, the type of nuclear interaction which occurs will be left unspecified. The problem is to find the distribution in space of these first collisions. Let the number of neutrons surviving as far as distance x without colliding with a nucleus be defined as $n(x)$. Now we assume that the *fraction* of the neutrons which suffer their first collision in a small interval dx is a constant, independent of the position of the interval. If we denote the number of neutrons making their first collision in dx by $dn(x)$, then the mathematical statement of our assumption may be written

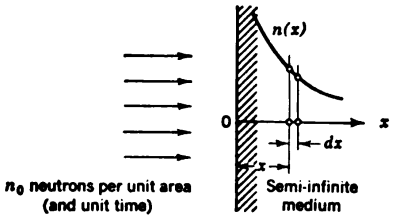

FIG. 2.3 Collimated beam of neutrons incident upon the face of a semi-infinite medium.

$$-\frac{1}{dx}\frac{dn(x)}{n(x)} = \Sigma \tag{2.1}$$

where Σ is a constant appropriate for the given medium, the particular neutron-nucleus interaction, and the neutron speed.

The solution to Eq. (2.1) is easily seen to be

$$n(x) = n_0 e^{-\Sigma x} \tag{2.2}$$

under the boundary condition

$$n(0) = n_0 \tag{2.3}$$

[1] The frequency of neutron-neutron collisions is some six orders of magnitude less than neutron-nucleus collisions and, therefore, entirely ignored in reactor calculations.

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The physical content of Eq. (2.2), that the number of uncollided neutrons falls off exponentially with distance, may be subjected to experimental verification directly, and the fact that the form of (2.2) is found to fit the data very well is evidence for the validity of the assumption expressed by Eq. (2.1).

The result (2.2) may also be stated as the product of two factors; thus

$$\begin{bmatrix}\text{The number of}\\ \text{neutrons}\\ \text{surviving to}\\ \text{distance } x\end{bmatrix} = \begin{bmatrix}\text{the number}\\ \text{of neutrons}\\ \text{starting}\\ \text{at } x = 0\end{bmatrix}\begin{bmatrix}\text{the probability that}\\ \text{a neutron will } \textit{not}\\ \text{suffer a collision}\\ \text{in traveling distance } x\end{bmatrix} \tag{2.4}$$

If this equation is compared to (2.2), we arrive at one of the basic probability functions for neutron-nucleus interactions:

$$w(x) \equiv \begin{matrix}\text{probability that a neutron}\\ \text{have an uncollided flight of}\\ \text{length } x\end{matrix} \equiv e^{-\Sigma x} \tag{2.5}$$

The related probability that a neutron *will* suffer a first collision *before* reaching distance x is called the *distribution function for the first collision* and is given by

$$\begin{aligned} F(x) &\equiv \begin{matrix}\text{probability that a neutron } \textit{will} \text{ have suffered its}\\ \text{first collision before distance } x\end{matrix}\\ &= 1 - w(x) = 1 - e^{-\Sigma x} \end{aligned} \tag{2.6}$$

Further, *the frequency function for first collision*, i.e., the probability that the neutron have its first collision in unit distance about x, may be obtained from the distribution function; thus

$$\begin{aligned} f(x) &= \begin{matrix}\text{probability that a neutron have its first collision}\\ \text{in unit distance about } x\end{matrix}\\ &\equiv \frac{d}{dx}F(x) = \Sigma e^{-\Sigma x} \end{aligned} \tag{2.7}$$

In the sections which immediately follow (*c* and *d*) these functions are again derived in a more complete treatment which exhibits more clearly the underlying hypothesis (the general form of the distribution and frequency functions is shown in Fig. 2.4). The physical content of these probability functions is discussed in Sec. 2.3.

c. The "As Good As New" Hypothesis. There is an alternative formulation of the problem of calculating the probability of collision which may be obtained by asking the following question: If a neutron is observed moving through a medium at some speed v, what is the probability that it suffers a nuclear collision before it has traveled a distance x? This question is stated in very general terms. No detailed description is given of the physical condition, composition, or geometric form of the

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medium, nor of the type of collision which is to occur. Although each of these factors is pertinent and together they determine a particular situation, the general question is still meaningful and the approach generally applicable. We will discover shortly that the answer to our question may be written in terms of certain parameters which are related only to the physical properties of the medium and the speed of the neutron. Thus, if one asks what is the probability that a 100-ev neutron will be absorbed by a U^{235} nucleus within a distance of 5 cm, the answer can be expressed in terms of the density and certain nuclear characteristics of the U^{235} nuclei in the medium. The specific question is concerned with a particular physical situation and involves detailed information. The general problem inquires into the analytical description of the phenomenon and examines the conclusions predicted by the model.

The essential feature of the analytical model we select for calculating the collision probabilities is the "as good as new" hypothesis which is defined in the following way: If a neutron moving through a medium is known to have traveled a distance x along its path without suffering a collision, then it is *as good as new*. Thus, if a neutron is as yet uncollided at position x, the probability that it will travel an additional distance y along its path before it encounters a nucleus is now *measured from the position* x. That is, the neutron is as good as new and can be considered to have originated at point x. The mathematical statement of this physical principle may be written in terms of the probability function[1] $P(\xi \leq x)$, which we define as

$$P(\xi \leq x) \equiv F(x) = \begin{array}{l}\text{probability that a neutron make its first} \\ \text{collision before reaching a given point } x\end{array} \tag{2.8}$$

where ξ is a random variable and $F(x)$ is the corresponding distribution function. For this calculation, we require also the conditional probability function. The conditional probability is defined as

$$P(x < \xi \leq x + y | \xi > x) = \begin{array}{l}\text{probability that given the collision} \\ \text{occurs after } x(\xi > x), \text{ it occurs} \\ \text{between } x \text{ and } x + y\end{array} \tag{2.9}$$

The "as good as new" hypothesis can be stated with the aid of these functions. Thus, if the neutron is known to have survived to x, then the probability that the first collision occurs before it moves an additional distance y [i.e., Eq. (2.9)] is identically the probability that a neutron first observed at $x = 0$ makes a collision before it reaches $x = y$ [i.e., Eq. (2.8)]. Or,

$$P(x < \xi \leq x + y | \xi > x) = P(\xi \leq y) \tag{2.10}\star$$

[1] H. Cramer, "Mathematical Methods of Statistics," chaps. 13, 14, and 15, Princeton University Press, Princeton, N.J., 1951.

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This equation is the mathematical statement of the "as good as new" hypothesis and is the basis from which we construct the collision probability functions.

d. Distribution Function for Collision. The distribution function for collision based on the relation (2.10) may be derived by applying the properties of a conditional probability, namely,

$$P(x < \xi \leq x + y | \xi > x) = \frac{P(x < \xi \leq x + y, \xi > x)}{P(\xi > x)} \tag{2.11}$$

The function $P(x < \xi \leq x + y, \xi > x)$ is the joint probability that the collision occurs in the interval $(x, x + y)$ and that the collision point is beyond x. If we substitute this relation into (2.10), we obtain

$$P(\xi \leq y)\, P(\xi > x) = P(x < \xi \leq x + y, \xi > x) = P(x < \xi \leq x + y) \tag{2.12}$$

The last equality follows from the fact that $P(x < \xi \leq x + y, \xi > x)$ implies $P(x < \xi \leq x + y)$. It is convenient to write this equation in terms of the corresponding distribution functions. We apply the definition (2.8) and write[1]

$$F(y)[1 - F(x)] = F(x + y) - F(x) \tag{2.13}$$

This relation is in the form of a difference equation. If we divide through by y, we obtain

$$\frac{1}{y} F(y)[1 - F(x)] = \frac{1}{y} [F(x + y) - F(x)] \tag{2.14}$$

If we suppose that, for each x, $F(x)$ has a derivative, the right-hand side of (2.14) tends to $F'(x)$ as y tends to zero; the left-hand side must therefore also converge. Thus we find

$$\Sigma[1 - F(x)] = F'(x) \tag{2.15}$$

where the constant Σ is given by

$$\lim_{y \to 0} \frac{F(y)}{y} \equiv F'(0) \equiv \Sigma \tag{2.16}$$

As noted, this limit must exist, and since it does exist, $F(0) = 0$, hence the identification (2.16). The solution to Eq. (2.15) is

$$F(x) = 1 - e^{-\Sigma x}$$

which agrees with our previous result (2.6).

As we have already noted, the function $F(x)$ is properly called the distribution function for the first collision of a neutron along its path. The derivative of this function, which is called the frequency function,

[1] *Ibid.*, p. 166.

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is usually of somewhat greater physical interest. This function has been given the symbol $f(x)$, and a formal definition was made in (2.7).

We summarize the first collision distribution function and frequency function below:

$$F(x) = 1 - e^{-\Sigma x} \qquad f(x) = \Sigma e^{-\Sigma x} \tag{2.17}$$

Note also that

$$F(x) = \int_{-\infty}^{x} f(w)\, dw = \int_{0}^{x} f(w)\, dw \tag{2.18}$$

The general form of these functions is shown in Fig. 2.4 where, for convenience, the parameter Σ is taken as unity. The origin of the x axis is the point at which the uncollided neutron is first observed. The functions $F(x)$ and $f(x)$ are identically zero for all $x < 0$. The distribution function $F(x)$ approaches unity in the limit as x becomes arbitrarily large, i.e., for very large x, the probability that the neutron has made a collision is nearly a certainty. Note, also, that the probability that the neutron makes its first collision per unit path of travel is greatest near its point of origin. Likewise, for any position $x_0 < x_1$

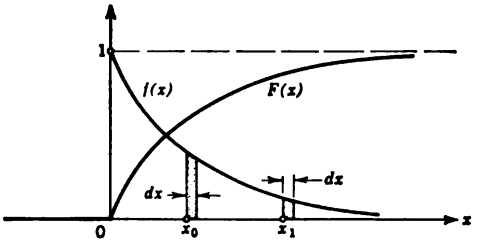

FIG. 2.4 Distribution and frequency functions for first collision.

$$f(x_0)\, dx > f(x_1)\, dx \tag{2.19}$$

It is easy to show that $f(x)$ is indeed the probability of collision per unit path of travel around x by means of the following computation: The probability that the first collision per unit path traveled occurs around x is given by

$$\lim_{\Delta x \to 0} \frac{P(x < \xi \leq x + \Delta x)}{\Delta x}$$

But, $$P(x < \xi \leq x + \Delta x) \equiv F(x + \Delta x) - F(x)$$

Therefore,

$$\lim_{\Delta x \to 0} \frac{P(x < \xi \leq x + \Delta x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} \equiv f(x) \tag{2.20}$$

2.3 Macroscopic Cross Section

a. Expected Number of Collisions. The parameter Σ introduced in Eq. (2.1) is intimately related to the nuclear properties of the struck nucleus and the relative speed of the neutron. In addition, it has a physical significance associated with the concept of a *mean free path.* This relationship is easily demonstrated by computing the average distance to the point of the first collision, i.e., the first moment of the fre-

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quency function $f(x)$. In the usual way, we write[1] [using (2.17)]

$$\bar{x} = \int_{-\infty}^{\infty} x f(x)\, dx = \Sigma \int_0^{\infty} x e^{-\Sigma x}\, dx = \frac{1}{\Sigma} \equiv \lambda \qquad (2.21)\star$$

Thus the parameter $\Sigma \equiv 1/\lambda$, the reciprocal of the average distance to collision (mean free path); or, saying it in another way

$$\Sigma \equiv \text{average number of collisions per unit path of travel} \qquad (2.22)\star$$

A physical argument which displays the basis for this definition of Σ may be obtained from the simple experiment discussed in Sec. 2.2*b*. According to Eq. (2.1)

$$-\frac{1}{dx}\frac{dn}{n} = \Sigma \equiv \text{fraction of first collisions in } dx \text{ about } x$$

where $n(x)$ is the density of uncollided neutrons at x. Or, rewriting

$$\frac{dn}{dx} = -n(x)\Sigma \qquad (2.23)$$

Since the derivative at the left of this relation gives the change in the neutron density $n(x)$ over the distance dx, it must correspond also to the number of collisions which occur in dx. Thus a verbal statement of (2.23) might be as follows:

$$\begin{bmatrix}\text{Number of}\\ \text{collisions in}\\ \text{traversing } dx\end{bmatrix} = \begin{bmatrix}\text{number of}\\ \text{uncollided}\\ \text{neutrons at } x\end{bmatrix}\begin{bmatrix}\text{number of collisions}\\ \text{per neutron per}\\ \text{unit path}\end{bmatrix} \qquad (2.24)$$

If this equation is compared to (2.23), we immediately identify Σ as the quantity defined in (2.22).

b. Radioactive Decay. By introducing appropriate changes in terminology, the analysis given above for the collision processes applies directly to the problem of radioactive decay. This is possible since the physical principle which served as a starting point for the analysis of the collision problem has an exact analogue in radioactive decay; namely, if a nucleus has not suffered radioactive decay by a given time t, it is "as good as new." For this calculation, we define Λ, the *decay constant*, as the number of decays per unit time. This quantity is analogous to the parameter Σ. On this basis all the analysis from Eqs. (2.8) through (2.18) applies directly. If we call the frequency function for decay $g(t)$, then

$$\begin{aligned} g(t)\, dt &= \Lambda e^{-\Lambda t}\, dt \\ &= \text{probability of decay between time } t \text{ and } t + dt \end{aligned} \qquad (2.25)$$

The distribution function $G(t)$ is obtained by integrating $g(t)$.

[1] *Ibid.*, p. 174.

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$$G(t) = \int_{-\infty}^{t} g(\theta)\, d\theta = 1 - e^{-\Lambda t}$$
$$= \text{probability that nucleus decays before time } t \qquad (2.26)$$

If $N(0)$ is the number of undecayed nuclei present at $t = 0$ and $N(t)$ the number at t, then

$$N(t) = N(0)[1 - G(t)] = N(0)e^{-\Lambda t} \qquad (2.27)\star$$

We have used here the function

$$1 - G(t) = \begin{matrix}\text{probability that nucleus has } \textit{not} \\ \text{decayed by time } t\end{matrix} \qquad (2.28)$$

The decay per unit time is given by the derivative $N'(t)$,

$$\left|\frac{dN}{dt}\right| = \Lambda N(t)$$

Thus, the fraction of the nuclei present at time t which decay per unit time is

$$\frac{1}{N}\left|\frac{dN}{dt}\right| = \Lambda \qquad (2.29)$$

which is in accord with our previous definition of Λ.

A quantity frequently used in describing a radioactive decay process, and which is related to the decay constant Λ, is the *half-life* of the nucleus. The half-life is defined as the time it takes a given concentration of a radioactive (unstable) nucleus to diminish to half its initial value by decay. If we call the half-life t_0, then the number of undecayed nuclei present at time t_0 is

$$N(t_0) = N(0)e^{-\Lambda t_0} \qquad (2.30)$$

As before, $N(0)$ denotes the number of nuclei present at $t = 0$; thus, t_0 is the time instant (measured from $t = 0$) at which the number of nuclei present is exactly half the initial amount. We require, therefore, that

$$N(t_0) = \tfrac{1}{2}N(0) \qquad (2.31)$$

The half-life t_0 is obtained by using this relation in Eq. (2.30). The result is

$$t_0 = \frac{\ln 2}{\Lambda} \qquad (2.32)\star$$

The half-life, decay scheme, and energy of radiation (or of the emitted particle) for some radioisotopes of interest to reactor technology are listed in Table 2.1. The various known decay schemes for these nuclei are given in the third column. The symbol α denotes radioactive decay of the unstable isotope by the emission of an α particle; β^-, emission of a negative electron; γ, emission of electromagnetic radiation; K, orbital

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TABLE 2.1 RADIOACTIVE ISOTOPES

Isotope	Half-life	Decay scheme	Energy, Mev	Isotope	Half-life	Decay scheme	Energy, Mev
$_5B^{12}$	0.022 sec	β^-	12	$_{48}Cd^{115}$	43 days	β^-	1.85
						γ	0.50
$_6C^{14}$	5100 years	β^-	0.15	$_{52}Te^{135}$	<2 min	β^-	
$_7N^{16}$	7.35 sec	β^-	3.5, 10	$_{53}I^{135}$	6.7 hr	β^-	1.40, 1.00, 0.47
		γ	6.2, 6.7			γ	1.6
$_8O^{19}$	28 sec	β^-	2.9, 4.5	$_{54}Xe^{135}$	9.2 hr	$(e^-)\gamma$	0.247
		γ	1.6			β^-	0.93
$_{11}Na^{24}$	14.8 hr	β^-	1.39	$_{60}Nd^{149}$	1.7 hr	β^-	1.6
		γ	1.38, 2.76				
$_{13}Al^{28}$	2.30 min	β^-	3.01	$_{61}Pm^{149}$	47 hr	β^-	1.1
		γ	1.8			γ	0.25
$_{19}K^{40}$	1.8×10^9 years	β^-	1.9	$_{90}Th^{232}$	1.39×10^{10} years	α	3.98
		γ	1.54				
$_{27}Co^{60}$	10.7 min	β^-	1.35	$_{92}U^{235}$	8.91×10^8 years	α	4.56, 4.40
		γ	0.056, 1.5			γ	0.162
$_{30}Zn^{65}$	250 days	$K(\beta^+)$	0.32	$_{94}Pu^{239}$	24,110 years	α	5.15
		$(e^-)\gamma$	1.11			$(e^-)\gamma$	0.42, 0.20

electron capture; β^+, positive electron emission; and e^-, internal conversion of a gamma ray.[1]

The preceding analysis of the radioactive-decay process is entirely general and applies to all the decay schemes mentioned above. Some of the isotopes listed in Table 2.1 may decay by the emission of more than one type of particle. For example, Al^{28} decays by either β^- or γ emission. The half-lives of nuclei that possess multiple decay schemes are based on the combined probabilities of the various possible events. Thus, in the case of Al^{28}, the 2.30-min half-life means that a given sample of this material will be reduced to one-half its initial concentration in 2.30 min by *either* β^- emission *or* γ emission.

Although it was not specifically stated, the decay process selected for the above derivation implied a single decay scheme. The analysis is more general than this, however, and can be changed to account for the possibility of additional modes of decay. To continue our example, consider the case of Al^{28}. Let $G_\beta(t)$ denote the probability that an Al^{28} nucleus will decay before time t by β^- emission and $G_\gamma(t)$ the probability that it will decay before time t by γ emission. Then [cf. Eq. (2.28)]

$$[1 - G_\beta(t)][1 - G_\gamma(t)] = \begin{array}{l}\text{probability that nucleus has } \textit{not} \text{ decayed} \\ \text{before time } t \text{ by } \textit{either } \beta^- \textit{ or } \gamma \text{ emission}\end{array} \tag{2.33}$$

[1] A detailed description of these processes can be found in any of the standard texts on nuclear physics. An introductory treatment is given by F. Bitter, "Nuclear Physics," chap. 4, Addison-Wesley Publishing Company, Reading, Mass., 1950.

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This statement follows from the assumption that the two events γ and β^- emission are *independent*, and, therefore, the *joint probability* is given by the product of the individual probabilities. If Λ_β and Λ_γ denote the decay constants for these two processes, then by (2.26)

$$[1 - G_\beta(t)][1 - G_\gamma(t)] = e^{-(\Lambda_\beta + \Lambda_\gamma)t} \tag{2.34}$$

The number of Al^{28} nuclei which survive to time t (i.e., have not decayed by either γ or β^- emission) is the product of this joint probability and the initial population $N_{Al}(0)$; thus

$$N_{Al}(t) = N_{Al}(0)e^{-(\Lambda_\beta + \Lambda_\gamma)t} \tag{2.35}$$

It is easy to show that the corresponding half-life of the joint events is

$$t_0^{(Al)} = \frac{\ln 2}{\Lambda_\beta + \Lambda_\gamma} \tag{2.36}$$

This general approach can also be applied to nuclei which decay by three or more modes.

2.4 Macroscopic Cross Sections and Physical Properties

a. Microscopic Cross Section. As presented in Eq. (2.22) the parameter Σ is the expected number of neutron-nucleus collisions per unit length of path traveled. In this section we shall attempt to relate this parameter, from this definition, to the basic properties of the medium through which the neutron is passing. As a first, admittedly crude, model of the physical situation, introduced primarily for its simplicity and to give some feel for the physical situation, consider the nuclei as spheres of radius R. Then, if the neutron passes within a distance R of the center of the nucleus, a collision is sure to result; otherwise, there is no collision. Thus in traveling a unit distance along its path the neutron will collide with any nucleus whose center falls within a distance R measured perpendicularly from the path line. If N is defined as the number of nuclei per unit volume in the medium, then along unit length of path a total of $\pi R^2 N$ nuclei fall within the required range for collision. Thus on the basis of this simple model

$$\Sigma = (\pi R^2)N \tag{2.37}$$

Thus Σ is directly proportional to the nuclear density and to the target area presented by a given nucleus.

Of course, the picture of nuclei as hard "billiard balls" is not really allowable; a more valid calculation might proceed as follows:

Define a probability:

$$h(r)\,dr = \begin{array}{l}\text{probability per unit time that a neutron found}\\ \text{between radial distances } r \text{ and } r + dr \text{ from a given}\\ \text{nucleus will suffer a collision with that nucleus}\end{array} \tag{2.38}$$

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Then, as the neutron moves an infinitesimal distance ds, it spends a time ds/v at a distance r from all nuclei in the spherical shell r to $r + dr$ whose center is at the neutron (see Fig. 2.5). Thus in this formulation the parameter Σ is given by

$$\Sigma = \int_0^\infty \frac{(4\pi r^2\, dr) N\, h(r)\, \dfrac{ds}{v}}{ds} = N \int_0^\infty \frac{4\pi r^2}{v}\, h(r)\, dr \equiv N\sigma \quad (2.39)\star$$

In the last step in (2.39) the integral of the weighted probability function over all space is designated by the symbol σ. Equation (2.37) may also be written in this form with a particular choice for the function $h(r)$.

Thus, quite generally, the expected number of collisions per unit distance traveled Σ is proportional to the nuclear density through a proportionality constant which has the units of length squared and plays the role of an effective target area per nucleus. This proportionality constant will be designated by the symbol σ and referred to as the *microscopic cross section* for neutron-nucleus collision throughout this work. Correspondingly, the parameter Σ will be referred to as the *macroscopic cross section*, although its dimensions are, as shown by Eq. (2.21), those of reciprocal length. The microscopic cross sections for nuclei are conveniently measured in *barns*, with 1 barn $= 10^{-24}$ cm^2; from the result of Eq. (2.37) a nucleus with a σ of 1 barn would be expected to have a radius of the order of 6×10^{-13} cm, which agrees roughly with various methods of estimating nuclear radii. However, the hard-sphere model is not really valid, as may be seen from the fact that for a given nucleus the value of σ for interaction with a neutron of speed v may be a few barns, while the value of σ for the same nucleus, but for interaction with a neutron of speed v', only slightly different from v, may be several thousand barns.

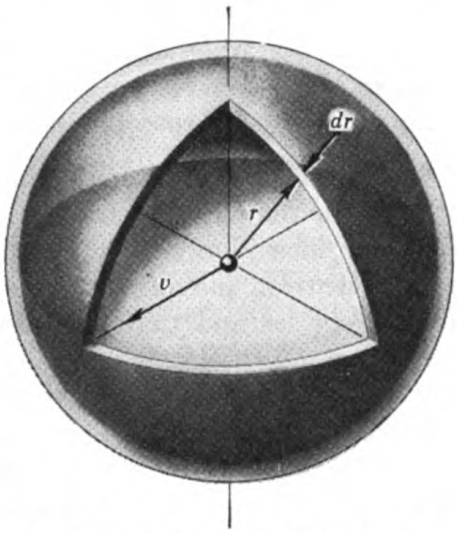

FIG. 2.5 Spherical shell dr about r.

b. Cross Sections for Neutron Reactions. In our general treatment of the collision problem (Sec. 2.2), we were not concerned with the physical process which took place between the neutron and the nucleus at the instant of collision. Instead, we confined our attention to determining the probability that a collision occurs. Thus our initial examination of the collision problem has not been limited to any particular type of reaction, and the results obtained are generally applicable to interactions between many kinds of elementary particles. For the problems of reactor

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physics, the interaction phenomena of primary importance are the neutron-nucleus reactions. We should like then to consider the application of the results of Sec. 2.2 to the description of the various types of neutron-induced reactions.

It has become the practice in the field of reactor technology to represent the various types of neutron reactions by a fixed set of nomenclature. It will be convenient for our present needs to introduce this set here. A particular type of reaction will be described by the cross-section symbols given above, distinguished by an appropriate subscript. The most commonly used quantities are:

$$\begin{aligned}
\Sigma_a &= N\sigma_a = \text{macroscopic absorption cross section} \\
\Sigma_\gamma &= N\sigma_\gamma = \text{macroscopic radiative-capture cross section} \\
\Sigma_f &= N\sigma_f = \text{macroscopic fission cross section (by convention, absorption includes radiative capture and fission when dealing with nuclear fuels)} \\
\Sigma_s &= N\sigma_s = \text{macroscopic elastic-scattering cross section} \\
\Sigma_{in} &= N\sigma_{in} = \text{macroscopic inelastic-scattering cross section}
\end{aligned} \tag{2.40}\star$$

The application of the probability concept to the description of these processes is the purpose of the present discussion. We will demonstrate the basic idea involved by considering an example. Suppose that we have a neutron moving through a medium which consists of a single nuclear species and that this medium has only absorption and elastic-scattering cross sections; i.e., the only possible neutron-nucleus reactions are capture and elastic scattering. It is meaningful then to ask the following question: What is the probability that the neutron will travel a distance x before it is either absorbed *or* scattered? We can answer this question by means of a calculation much like the one we used for the radioactive-decay problem of Al^{28}.

The two processes elastic scattering and absorption are independent, and their respective distribution functions are

$$F_s(x) = 1 - e^{-\Sigma_s x} \qquad F_a(x) = 1 - e^{-\Sigma_a x} \tag{2.41}$$

These functions give the probability that the particular event denoted by the subscript occurs before the neutron reaches x. Thus, the probability that the neutron remains uncollided by the time it reaches x is [cf. Eqs. (2.28) and (2.33)]

$$[1 - F_s(x)][1 - F_a(x)] \equiv 1 - F_o(x) = e^{-(\Sigma_s+\Sigma_a)x} \tag{2.42}$$

We have defined the product at the left as some function $1 - F_o(x)$. This notation is used for convenience. Note that

$$\begin{aligned}
F_o(x) &\equiv 1 - e^{-(\Sigma_s+\Sigma_a)x} \\
&\equiv \text{probability that neutron is either absorbed } \textit{or} \text{ elastically scattered before it reaches } x
\end{aligned} \tag{2.43}$$

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The corresponding frequency function $f_o(x)$ is

$$f_o(x) = \Sigma_o e^{-\Sigma_o x} \tag{2.44}$$

with $\Sigma_o \equiv \Sigma_a + \Sigma_s$. We conclude from these results that, in a situation which allows two types of neutron reactions, the joint probability that either of the events occur may be described by means of a combined cross section. This cross section, called Σ_o *in the present instant*, is simply the sum of the cross sections of the two reactions. This idea is easily extended to situations which involve more than two types of collisions. In general, if a nucleus has neutron-reaction cross sections for n different types of collisions, then the sum Σ_t

$$\Sigma_t = \Sigma_1 + \Sigma_2 + \cdots + \Sigma_n \tag{2.45}\star$$

gives the total number of collisions of all n types which the neutron experiences per unit path of travel. Any partial sum, e.g., $\Sigma_1 + \Sigma_2$, gives the number of collisions of types 1 and 2 per unit path of travel. Macroscopic cross sections, then, are additive. This operation, however, offers some conceptual difficulties in connection with the absorption process. The idea of a single neutron being absorbed more than once appears to be in contradiction to the definition of the absorption process. And, indeed it is. But, this is not what Eq. (2.45) implies. In dealing with the absorption process (and with any of the others, for that matter), it is more meaningful to talk about the mean free path to absorption λ_a, which is the reciprocal of Σ_a [cf. Eq. (2.21)]. A physical interpretation of (2.45) is better made, then, in terms of the corresponding mean free paths. Thus, we write

$$\lambda_t = \frac{1}{\Sigma_t} = \frac{1}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \cdots + \frac{1}{\lambda_n}} \tag{2.46}$$

The total mean free path is the average distance along its path that a neutron travels before any one of the n events occurs. Note that, if the mean free path to any one of these individual events approaches zero (i.e., the probability of the event occurring increases to unity), the total mean free path also goes to zero.

The functions $F_o(x)$ and $f_o(x)$ derived above may be used to describe the neutron population and distribution of collisions in the example considered. If $n(0)$ neutrons leave the origin of the x axis, moving in the positive x direction, then the number of neutrons which reach x without collision is [cf. Eq. (2.2)]

$$n(x) = n(0)e^{-\Sigma_o x} \tag{2.47}$$

which follows directly from the relation (2.42). We can also compute the total number of first scattering *and* absorption collisions $A_o(x)$ that occur

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in unit distance around x by applying (2.44). Thus

$$A_o(x) = n(0)\Sigma_o e^{-\Sigma_o x} \tag{2.48}$$

The number of the individual events which occur at x is

$$\begin{aligned} &\textit{Scatterings:} && A_s(x) = n(0)\Sigma_s e^{-\Sigma_o x} \\ &\textit{Absorptions:} && A_a(x) = n(0)\Sigma_a e^{-\Sigma_o x} \end{aligned} \tag{2.49}$$

Note that the neutron population and the density of events all decrease exponentially from the origin.

It has become the practice in dealing with combined events to define a suitable set of symbols for some of the important reactions. The standard practice is to call the *total cross section* Σ_t of a given material the sum of the cross sections of all the possible types. Thus

$$\Sigma_t = \Sigma_a + \Sigma_s + \Sigma_{in} \tag{2.50}\star$$

In situations wherein the inelastic-scattering process is unimportant, the usual definition is to write $\Sigma_t = \Sigma_a + \Sigma_s$. As previously remarked [Eqs. (2.40)], the absorption cross section is given by

$$\Sigma_a = \Sigma_\gamma + \Sigma_f \tag{2.51}\star$$

In the case of nonfuel materials ($\Sigma_f = 0$), the absorption cross section is just the radiative-capture cross section.

c. Measurement of Cross Sections. We found in Sec. 2.4*a* that the macroscopic cross section Σ could be described in terms of the nuclear density N and the microscopic cross section σ. Thus, given the density of material and either cross section, the other is immediately known. However, as yet we have given no explanation of how the number σ (or Σ) is to be obtained. With our present imperfect knowledge of nuclear dynamics, it is not possible to calculate from first principles neutron cross sections with better accuracy than roughly a factor of 2. Another approach to the calculation of cross sections is the phenomenological one. From a knowledge of experimentally determined values of σ, one attempts to compute σ for other reactions in the same nucleus. Such calculations are generally of greater accuracy than those based on first principles. Thus when high accuracy is required, we must rely principally upon experimental measurements.

Unfortunately, this approach is not free of major obstacles either. Except in the case of the total cross section [Eq. (2.50)], cross-section measurements are difficult to perform and require extensive instrumentation for precision. This is especially true in situations wherein one attempts to measure the cross sections for the individual processes listed in Eqs. (2.40). However, considerable effort has gone into cross-section-measurement programs, and a large body of data is gradually being accumulated.

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Although the basic problems of reactor physics are not generally concerned with the analysis of cross-section experiments, there are some special areas of reactor analysis which involve the same techniques and analytical models. The analysis of radiation experiments in research reactors and criticality studies are examples of such problems. There is one cross-section experiment, however, which is extremely simple to study from the analytical point of view and which will serve very well to demonstrate some of the ideas presented in this section. This problem has the additional merit that it gives some further insight into the physical concept of a cross section. The experiment in question is the measurement of the total cross section of a given material.

The essential features of this experiment are as follows: A collimated beam of monoenergetic neutrons is allowed to strike a slab of the material of interest. If $n(0)$ is the number of neutrons crossing a unit area of the slab surface per unit time at $x = 0$ (see Fig. 2.6), the number of uncollided neutrons crossing a unit area per unit time at distance x inside the slab is [cf. Eq. (2.47)]

$$n(x) = n(0)e^{-\Sigma_t x}$$

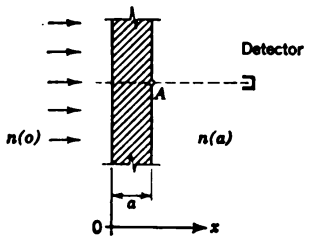

FIG. 2.6 Collimated neutron beam incident on a slab.

Suppose that a neutron detector is placed on the opposite side from the face exposed to the incident beam and that this detector registers a count only when a neutron reaches it by a path normal to the slab surface. Thus, neutrons that are scattered about inside the slab and eventually emerge at face $x = a$ traveling in some random direction are not counted. It is possible that some of these neutrons might still leave the surface with a trajectory normal to the surface and at a point directly opposite the detector (point A in Fig. 2.6); however, these are few in number, and most of them can be eliminated by using very thin slabs (i.e., by requiring that $a\Sigma_t \ll 1$). The number of neutrons which survive the passage through the slab without being scattered or absorbed is

$$n(a) = n(0)e^{-\Sigma_t a} \tag{2.52}$$

The numbers $n(0)$ and a are known, and $n(a)$ is measured by the detector. Thus, one can compute Σ_t

$$\Sigma_t = N\sigma_t = \frac{1}{a} \ln \frac{n(0)}{n(a)} \tag{2.53}$$

A knowledge of the nuclear density N of the slab material yields the microscopic total cross section σ_t.

Experiments of this type are most useful for measuring the total cross section of a given material. As already noted, cross-section meas-

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urements for specific neutron reactions are much more difficult to make and will not be discussed here. Extensive compilations of such data have been made, nevertheless, for many materials of interest to reactor technology and are now available in several standard sources.[1] These data for the various elements are customarily presented in the form of cross-section curves as a function of neutron energy. Also available in many sources are the thermal-neutron cross sections. The thermal-neutron cross sections refer to the cross section for neutron-nucleus interaction in a system wherein the kinetic energy of the neutron corresponds to the equilibrium temperature of the material. This reference temperature is customarily taken as 20°C (0.025 ev). The corresponding neutron

TABLE 2.2 THERMAL NEUTRON CROSS SECTIONS[a]

Material	Density, g/cm³	σ_a	σ_s	Material	Density	σ_a	σ_s	σ_f
H	.....	0.330	38.0	Mo	10.2	2.5	7	
D	.....	0.46 mb	7.0	Cd	8.65	2550	7	
Li	0.534	71	1.4	In	7.28	190	2.2	
Be	1.85	10 mb	7.0	Xe^{135}	.....	2.7×10^6		
B	.....	755	4.0	Sm^{149}	7.7	50,000		
C	1.60	3.20 mb	4.8	Th^{232}	11.3	7	12.6	
N	.....	1.88	10	U^{233}	18.7	585	.....	533
O	.....	<0.2 mb	4.2	U^{235}	18.7	687	10	580
Na	0.971	0.505	4.0	U^{238}	18.7	2.75	8.2	
Al	2.70	0.230	1.4	Pu^{239}	.....	1065	9.6	750
K	0.87	1.97	1.5	Pu^{241}	.....	.....	.....	1100
Fe	7.86	2.53	11.0	H_2O	1.0	0.66	108	
Co	8.9	37.8	7.0	D_2O	1.10	0.93 mb	14.3	
Ni	8.9	4.6	17.5	BeO	3.01	9 mb	10.0	

All cross sections are given in barns except where noted as millibarns (mb).

[a] Refers to thermal energy (20°C or 2,200 m/sec).

speed is 2,200 m/sec. The thermal-absorption, elastic-scattering, and fission cross sections for some common reactor materials are listed in Table 2.2. The density of the material at 20°C is also included. The millibarn notation (mb) used in the table is defined as one-thousandth of a barn.

d. Calculation of Macroscopic Cross Sections. In order to determine the various neutron-nucleus reaction rates, we require the appropriate probabilities for collision. Since these probabilities are related to the collision cross sections Σ (or the mean free path λ), the reaction rates may be computed from a knowledge of the microscopic cross sections σ

[1] See, for example, Donald J. Hughes and John A. Harvey, in U.S. Atomic Energy Commission, "Neutron Cross Sections," McGraw-Hill Book Company, Inc., New York, 1955.

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and the nuclear densities N. The densities are obtained from the relation:

$$N \equiv \frac{\text{nuclei}}{\text{vol}} = \left(\frac{\text{mass}}{\text{vol}}\right)\left(\frac{\text{mole}}{\text{mass}}\right)\left(\frac{\text{nuclei}}{\text{mole}}\right)$$

By the usual definitions:

ρ = density = mass/volume
$\mathfrak{A}$ = nuclei/mole = Avogadro's number (0.6023×10^{24})
A = mass/mole = atomic (or molecular) weight

We use these definitions in the expression for N and obtain

$$N = \frac{\rho \mathfrak{A}}{A} \qquad (2.54)\star$$

The macroscopic cross section may be written in terms of these quantities

$$\Sigma = \frac{\sigma \rho \mathfrak{A}}{A} \qquad (2.55)$$

Thus, Σ is immediately determined once the density of the material ρ is specified.

The magnitudes of the numbers involved in these relations may be demonstrated by means of a sample calculation. As an example, consider BeO. The pertinent data for computing the macroscopic cross sections are: $\rho = 3.01$ g/cm^3, $A = 25$, $\mathfrak{A} = 0.6023 \times 10^{24}$, $\sigma_s = 10$ barns, $\sigma_a = 9$ mb. These yield $N = 7.26 \times 10^{22}$ nuclei/cm^3, $\Sigma_s = 0.726$ cm^{-1}, and $\Sigma_a = 0.000654$ cm^{-1}.

e. Mixtures of Nuclear Species. The usual reactor configuration consists of various homogeneous regions composed of mixtures of different nuclear species. Thus, the general problem is to compute the reaction rates for media which contain more than one type of nucleus. This calculation is easily carried out when the media are homogeneous. The methods derived in the preceding portions of this section for mononuclear media apply directly to the uniform mixture.

Suppose that the medium through which the neutron is moving consists of a mixture of nuclear species such that

$$N_i \equiv \text{number of nuclei of the } i\text{th species present per unit volume of mixture} \qquad i = 1, 2, \ldots, k \qquad (2.56)$$

The frequency function for collision is still of the form given in Eq. (2.17), but the cross section Σ is now to be computed from the properties and densities of all nuclei present. Consider k fictitious media, each of which consists solely of one of the nuclear species of the final mixture. Let the nuclear density of the medium composed of the ith nuclear type be the same as the density of that type *in the mixture*. Then, for each of these

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media, we can define a distribution function $F_i(x)$ such that

$$F_i(x) \equiv \begin{matrix}\text{probability that a neutron collides with a nucleus of type} \\ i \text{ before reaching } x\end{matrix}$$
$$= 1 - e^{-\Sigma^{(i)}x} \tag{2.57}$$

The parameter $\Sigma^{(i)}$ is to be computed from the properties of this ith medium, using the known $\sigma^{(i)}$ and the density N_i from Eq. (2.54). Now imagine these various media to be combined to give the resultant mixture. Furthermore, let us assume that collisions with the various nuclear species are independent events, that is, the presence of other nuclear species in no manner affects the interaction between a given nuclear type and a neutron. Then the probability in the *mixture* that a neutron reach distance x without suffering a collision with a nucleus of type i is given by $1 - F_i(x)$. If a neutron is to survive a distance x without collision in the mixture, it must avoid collision with each nuclear type. Thus, the joint probability is some function $1 - F_m(x)$, where

$$1 - F_m(x) \equiv \text{probability of survival to } x \text{ in the mixture}$$
$$= [1 - F_1(x)][1 - F_2(x)] \cdots [1 - F_k(x)]$$
$$= \prod_{i=1}^{k} e^{-\Sigma^{(i)}x} = e^{-\Sigma^{(m)}x} \tag{2.58}$$

with
$$\Sigma^{(m)} = \Sigma^{(1)} + \Sigma^{(2)} + \cdots + \Sigma^{(k)} = \sum_{i=1}^{k} N_i\sigma^{(i)} \tag{2.59}$$

The last equality follows from our usual definition of σ. The frequency function $f_m(x)$ for collision in the medium may be computed from the distribution function $F_m(x)$ by means of the result (2.58).

$$f_m(x) = F_m'(x) = \Sigma^{(m)}e^{-\Sigma^{(m)}x} \tag{2.60}$$

The mean free path to first collision in the mixture is

$$\lambda_m = \frac{1}{\Sigma^{(m)}} = \frac{1}{N_1\sigma^{(1)} + N_2\sigma^{(2)} + \cdots + N_k\sigma^{(k)}} \tag{2.61}$$

f. Cross Sections of Compounds and Nuclei in Bound States. The calculation of the cross sections for molecules may be obtained from a knowledge of the microscopic cross sections of the individual atoms in the structure. The procedure is a straightforward one for capture-type reactions; the capture cross section of the compound is given by the sum of the cross sections for all the atoms present. Scattering collisions, however, are much more complicated phenomena. A scattering reaction between a neutron and a nucleus bound to a molecular structure differs greatly from the reaction between the two corresponding particles in the free state. The chemical bonds between the struck nucleus and the rest

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of the molecule give it an increased rigidity; consequently, the neutron responds as though it had collided with a much larger mass than that of the target nucleus.

The nature of the collision is determined in main part by the speed of the neutron. At neutron energies in excess of 1 ev, the target nucleus responds as a free particle because at these energies the chemical binding forces are negligible in comparison to the collision forces, and so the bonds are disrupted. At lower neutron energies, internal vibrational states of the molecule can be excited, and the neutron will experience *inelastic* collision with the molecule. However, there exists some neutron speed below which the lowest vibrational mode of the molecule cannot be excited, and the target nucleus responds to the collision with effectively the mass of the molecule. In addition to these considerations, the scattering from molecules is further complicated by interference effects due to the presence of multiple scattering centers. Some attempts have been made to evaluate these various factors, and we will consider them later in connection with the study of the scattering process proper.

PROBLEMS

2.1 Monoenergetic neutrons are being emitted isotropically from a point source in an infinite medium at a rate q_0 neutrons/sec. If the nuclei of the medium have an absorption cross section σ_a for neutrons of the given energy, a negligible scattering cross section, and a nuclear density N, find the number of neutron absorptions occurring per cubic centimeter per second at each point of the infinite medium.

2.2 Suppose that in the same medium as in Prob. 2.1 neutrons are emitted perpendicularly from a line source at a rate q_1 neutrons/sec per centimeter length of line. Find the number of absorptions per cubic centimeter per second at each point in the infinite medium.

2.3 Suppose that the neutron source in the infinite medium of Prob. 2.1 is a perpendicular-emitting plane with a strength q_2 neutrons/cm^2 of source area per second. Find the number of absorptions per cubic centimeter per second for all points in the medium.

2.4 Suppose that the neutron source in the infinite medium of Prob. 2.1 is a plane which emits neutrons such that from each side of a small area $d\mathbf{S}$, $q_3\, d\mathbf{S}$ neutrons are emitted per second with an angular distribution given by

$$q(\theta)\, d\theta\, d\mathbf{S} = 2q_3\, d\mathbf{S} \sin\theta \cos\theta\, d\theta$$

where θ is measured from the normal to the plane (θ having the range $0 \to \pi/2$ for each side). Show that the number of absorptions per cubic centimeter per second for all points in the medium is given by

$$\text{Absorptions} = 2q_3\Sigma \left(e^{-\Sigma|x|} - \Sigma|x| \int_{\Sigma|x|}^{\infty} \frac{e^{-z}}{z}\, dz \right)$$

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CHAPTER 3

MULTIPLICATION CONSTANT AND NEUTRON FLUX

3.1 Collision Density

The importance of the nuclear reaction rates (the number of neutron-nucleus collisions per unit volume per unit time) to the problems of reactor physics was discussed in the preceding chapters. The purpose of the present chapter is to establish the relationship between the various reaction rates and the neutron population and the nuclear characteristics of the reactor materials. The ideas developed in this connection will be used to examine, in greater detail than was done in Chap. 1, the requirements for a self-sustaining chain reaction.

The expected number of collisions per unit volume per unit time can be expressed in terms of the neutron density and the appropriate cross section for the type of reaction in question. In order to introduce the necessary concepts for this calculation, let us examine the following simplified situation. Consider a medium which is supporting a neutron gas of a specific kinetic energy. Thus, all the neutrons in this system have the same speed, but their directions of motion are randomly distributed. We should like to compute the average number of neutron-nucleus collisions which occur in time dt in the element of volume $d\mathbf{r}$ around some space point[1] $\mathbf{r}$ in this medium. This quantity may be expressed in terms of the following factors:

$$\begin{bmatrix}\text{Expected number of collisions in time } dt \text{ in volume } d\mathbf{r}\end{bmatrix} = \begin{bmatrix}\text{expected number of collisions per neutron per unit distance traveled}\end{bmatrix}\begin{bmatrix}\text{distance traveled by neutron in time } dt\end{bmatrix}\begin{bmatrix}\text{number of neutrons in volume } d\mathbf{r}\end{bmatrix} \tag{3.1}$$

But, by our previous definition [cf. Eq. (2.22)]

$$\Sigma(v) = \text{expected number of collisions by neutrons of speed } v \text{ per unit path of travel} \tag{3.2}$$

[1] The symbol $\mathbf{r}$ denotes a set of three coordinates to locate a point in space.

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If we define, also,

$$\begin{aligned} n(\mathbf{r},v,t) &= \text{number of neutrons of speed } v \text{ per unit volume around } \mathbf{r} \text{ at time } t \\ F(\mathbf{r},v,t) &= \text{expected number of collisions by neutrons of speed } v \text{ per unit volume around } \mathbf{r} \text{ and per unit time at time } t \end{aligned} \tag{3.3}$$

then, (3.1) may be written

$$F(\mathbf{r},v,t)\, d\mathbf{r}\, dt = \Sigma(v)\, v\, dt\, n(\mathbf{r},v,t)\, d\mathbf{r}$$

or

$$F(\mathbf{r},v,t) = \Sigma(v)\, v\, n(\mathbf{r},v,t) \tag{3.4}$$

The quantity $F(\mathbf{r},v,t)$ is called the *collision density* for the nuclear reaction described by the cross section $\Sigma(v)$. No subscript has been assigned to Σ so as to keep the derivation general. Equation (3.4) shows clearly the dependence of the reaction rate (or collision density) on the neutron density $n(\mathbf{r},v,t)$. This relationship was stated previously in the introduction to Chap. 1.

In dealing with specific types of reactions, we add the appropriate subscripts to the cross section according to the notation of (2.40). Thus, for example,

$$\begin{aligned} \Sigma_t(v)\, v\, n(\mathbf{r},v,t) &\equiv \text{total number of collisions of all types which occur between nuclei and neutrons of speed } v \text{ per unit volume around } \mathbf{r} \text{ per unit time at } t \\ \Sigma_a(v)\, v\, n(\mathbf{r},v,t) &\equiv \text{number of absorptions which occur between nuclei and neutrons of speed } v \text{ per unit volume around } \mathbf{r} \text{ per unit time at } t \end{aligned} \tag{3.5}$$

and so on. It is important to remember that all these expressions for the various collision densities are statistical quantities and should be taken to give the *expected* number of collisions. As with all statistical results, the actual number of collisions will vary around the expected number with a variation which will depend upon the neutron and nuclear densities.

The macroscopic cross sections $\Sigma(v)$ may be written in terms of the nuclear densities in the usual way [cf. Eq. (2.39)]. In general, these densities will be spatially variant functions and so should be properly written in the form $N(\mathbf{r})$. The resulting equation for the collision density is then

$$F(\mathbf{r},v,t) = \sigma(v)\, N(\mathbf{r})\, v\, n(\mathbf{r},v,t) \tag{3.6}$$

Note that we have retained the specification of the neutron speed in defining the microscopic cross section $\sigma(v)$. This is in keeping with our recognition of the fact that the neutron cross sections are in general a function of the neutron speed (cf. Sec. 2.4*a*).

3.2 Neutron Balance

a. Infinite-medium Model. The actual situation in an operating reactor may be described as a continuous cycle of neutron production

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and destruction. The principal nuclear events which constitute this cycle are the fission, scattering, and nonproductive capture reactions which take place throughout the reactor. At virtually every point of the reactor core, high-energy neutrons are continually appearing from fission. Likewise, neutrons of all energies are being continually removed from the general population at *all* points of the reactor system because of capture reactions. Neutrons are also continually lost by the net flow outward through the boundaries of the system. In the time interval between birth and death, a neutron wanders about the reactor and experiences many scattering collisions which cause it to deposit by decrements its initial kinetic energy from fission at the points of collision.

If we were to examine a small volume of the reactor, we would find that the "microscopic" picture would be much the same as the over-all picture of the reactor system, with neutrons of all energies continually streaming in and out of the little volume. Thus, at any given time instant, there would be present in this volume neutrons of all energies, corresponding to different stages in their life history. Some of these neutrons would subsequently be absorbed in the volume element. Others might be scattered about for a while before they finally pass out through the boundaries. The neutrons which escape would be leaving with energies less than that with which they entered.[1] If the volume element we selected were in the core, the situation would be further complicated by the continual appearance of fission neutrons. As in the case of neutrons which arrive in this volume by passing through its boundaries (transport), the fission neutrons might disappear by absorption while still in the volume, or they might be scattered about before leaving. In any event, we realize that the physical situation in any part of the reactor is extremely complex.

For the present, we will not attempt a detailed description of all the processes that occur in an actual system, but instead will limit our study of the reactor system to a considerably simplified model. The model we select is admittedly an unrealistic one; however, it will serve well to display some of the basic features of all reactors. The reactor model we propose has the following characteristics:

(3.7)

(1) The reactor medium contains fissionable material and is infinite in extent and homogeneous and isotropic in composition
(2) All fission neutrons appear at the same speed v
(3) Neutrons *do not* lose speed (energy) in scattering collisions
(4) The reactor material has cross sections for only absorption and elastic-scattering reactions

[1] These remarks do not apply to thermal neutrons, as will be explained later, but only to *fast neutrons*.

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The first of these assumptions is imposed in order to remove the spatial dependence of the neutron density. In an infinite homogeneous medium, no neutrons can be lost by leakage (escape), and all regions of this medium are identical. Thus, there are no special space points in this system (all points are alike), and consequently there can be no net change in the neutron population in going from one point to another. We recognize then that the neutron-density function n will not include spatial coordinates in its argument [cf. Eq. (3.3)]. Assumption (2) further restricts the system to neutrons of a fixed speed. This is known as the *one-velocity* model. Thus, in our definition of the neutron density n, we can omit the speed variable v, it being understood that all the neutrons in this medium have the same speed. Although we specified a similar situation in the discussion of Sec. 3.1, it was convenient to retain the independent variable v in that instance in order to emphasize the functional dependence of the quantities involved. Assumption (3) is introduced so as to simplify the problem and allow attention to be focused on other aspects of the physics. The energy changes resulting from scattering collisions will be discussed in detail in Chap. 4. Assumption (4) is again a simplification, but it does not impose any great limitation on the usefulness of the results. As in the case of the neutron density n, the cross sections will refer to neutrons of speed v, but the symbols will not include the argument v for the reason given above.

b. Neutron Density in Infinite Medium. Even though the system described above is a highly fictitious one, we can nevertheless acquire from it a considerable insight into the nature of a chain-reacting complex. We approach this objective by asking the fundamental question of all reactor problems: What is the distribution of neutrons in the system? Because of the nature of the limitations we have imposed, this distribution will be expressed in terms of the only remaining variable, the time t. Thus any balance condition we write for the neutron population must be in terms of its time dependence. On the basis of the assumptions (3.7), all the materials are uniformly distributed throughout the space. Furthermore, the only sources of neutrons are the fission reactions, and the only sinks, the absorption reactions. No neutrons can be lost by escape through the boundaries of this reactor. The neutron-balance condition for this system must be expressed then in terms of the time rate of change of these competing processes. The verbal statement of this condition is as follows:

$$\begin{bmatrix}\text{Number of}\\ \text{neutrons gained}\\ \text{per unit volume}\\ \text{per unit time}\end{bmatrix} = \begin{bmatrix}\text{number of}\\ \text{neutrons produced}\\ \text{from fission per}\\ \text{unit volume per}\\ \text{unit time}\end{bmatrix} - \begin{bmatrix}\text{number of}\\ \text{neutrons lost by}\\ \text{absorption per}\\ \text{unit volume per}\\ \text{unit time}\end{bmatrix}$$

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This statement may be written in terms of the concepts developed in Sec. 3.1. If $n(t)$ denotes the number of neutrons per unit volume at time t, then by introducing the appropriate collision densities for absorption and fission [see Eq. (3.5)], we have

$$\frac{d}{dt}n(t) = \nu\Sigma_f v n(t) - \Sigma_a v n(t) \tag{3.8}$$

where $\Sigma_f \equiv N_F\sigma_f$ = macroscopic-fission cross section of fuel
$\Sigma_a \equiv \Sigma_a^{(F)} + \Sigma_a^{(O)}$ = total macroscopic-absorption cross section of reactor material (3.9)

and $\nu \equiv$ neutrons per fission
N_F = fuel nuclei per unit volume
N_O = nonfuel nuclei per unit volume
$\Sigma_a^{(F)}$ = macroscopic-absorption cross section of fuel
$\Sigma_a^{(O)}$ = macroscopic-absorption cross section of all nonfuel materials (3.10)

In general, if there are M such materials

$$\begin{aligned}\Sigma_a^{(O)} &= \Sigma_a^{(1)} + \Sigma_a^{(2)} + \cdots + \Sigma_a^{(M)} \\ &= N_1\sigma_a^{(1)} + N_2\sigma_a^{(2)} + \cdots + N_M\sigma_a^{(M)}\end{aligned} \tag{3.11}$$

which follows from Eq. (2.59). The symbols N_1, N_2, . . . are the nuclear densities of the nonfuel components and the $\sigma_a^{(1)}$, $\sigma_a^{(2)}$, . . . their respective microscopic-absorption cross sections.

Two points are to be noted from Eq. (3.8): (1) No mention is made of loss or gain by migration across the boundaries of the unit volume. In our idealized model, no such losses or gains occur on the average since each neutron lost by wandering out is immediately replaced by a neutron gained from wandering in [cf. assumption (1), Eq. (3.7)]. (2) No mention is made of the scattering cross section Σ_s; these reactions have no effect on the neutron population in the present model. Both of these facts result from the assumption of an infinite medium and the consequent uniform neutron density. In the more general case of a finite reactor, the scattering cross section *will* affect the neutron balance, and migration (or transport) losses and gains must be considered.

Suppose now we ask for the physical condition which must hold if a neutron density, once set up in this infinite medium, is to persist for all time, i.e., we ask that

$$\frac{d}{dt}n(t) = 0 \tag{3.12}$$

It follows from Eq. (3.8) that

$$\frac{\nu\Sigma_f}{\Sigma_a} = 1 \tag{3.13}\star$$

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(so long as we have not selected $n = 0$). Equation (3.13) is one of a class of conditions called the *criticality condition;* a reactor which satisfies such a condition is said to be *critical.*

If condition (3.13) is not satisfied for the infinite reactor, then (3.12) does not apply, and the neutron density must change in time. As an example, suppose that in some way a uniform density n_0 is established in the infinite medium and maintained until time $t = 0$. At $t = 0$ all *nonfission* sources are removed; then for all $t > 0$, Eq. (3.8) applies. The solution of this equation gives the neutron density $n(t)$ for all subsequent time. We write (3.8) in the form

$$\frac{d}{dt} n(t) = (\nu\Sigma_f - \Sigma_a)vn(t) \quad (3.14)$$

The solution to this equation is

$$n(t) = n_0 e^{(\nu\Sigma_f - \Sigma_a)vt} \quad (3.15)\star$$

where we have applied the initial condition $n(0) = n_0$. Note that if $\nu\Sigma_f \equiv \Sigma_a$, $n(t) \equiv n_0$ for all $t > 0$; this is the condition we examined above. But, if $\nu\Sigma_f > \Sigma_a$, the neutron population rises exponentially with time, and if $\nu\Sigma_f < \Sigma_a$, it decays exponentially. These three possible situations are shown graphically in Fig. 3.1.

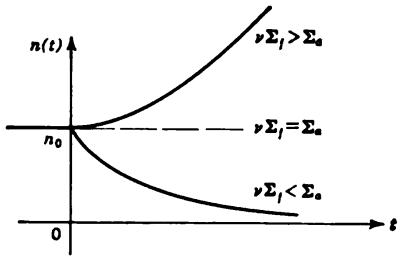

FIG. 3.1 Time dependence of neutron density in infinite medium.

3.3 Infinite-medium Criticality Problem

a. Neutron Chain and Multiplication Constant. The criterion for a self-sustaining chain reaction in a system containing fissionable material is that, on the average, the number of neutrons produced per unit volume per unit time be equal to the number of neutrons lost per unit volume per unit time [cf. Eqs. (3.12) and (3.13)]. From the viewpoint of the behavior of the neutrons, the cycle may be imagined to be as follows: n_0 neutrons are produced per unit volume per unit time. If these n_0 neutrons are produced as the result of fission reactions, they will, in general, have kinetic energies of about 2 Mev. Each of these n_0 neutrons moves through the system and makes collisions with the various species of nuclei present; the results of these collisions will be: (1) change in direction of motion of the neutron, (2) loss in kinetic energy of the neutron, (3) disappearance of the neutron by absorption in the nucleus, and (4) loss of the neutron by passage from a final scattering collision across the boundaries of the system.

Each of the original n_0 neutrons will have a different history, but, on the average, a certain fraction of them will disappear as a result of processes (3) and (4). Only those absorbed in nuclei have a chance to produce additional neutrons; a certain fraction of the absorbing nuclei

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will be those of the moderator, the coolant, and the structural materials, all of which have cross sections for nonproductive capture of neutrons. Only the fraction of neutrons absorbed in the fissionable fuel can be productive. Furthermore, not all captures in fissionable materials result in fission and, hence, more neutrons, because each fissionable element has also a cross section for radiative capture. In any event, a certain fraction of the n_0 will cause fission, and these fissions in turn will yield, say, n_1 neutrons. If the reactor system is self-sustaining, the n_1 neutrons of the second generation will be identical in number with the n_0 neutrons of the first generation; thus, the population is maintained.

The ratio of the neutron population in two successive generations, such as n_0 and n_1, is a useful number for describing the characteristics of the chain reaction. This ratio is called the *multiplication constant* of the reactor and is formally defined as

$$k \equiv \frac{n_1}{n_0} = \begin{array}{l}\text{ratio of number of neutrons of any one generation to} \\ \text{number in preceding generation}\end{array} \qquad (3.16)\star$$

As noted above, the system is at steady state when $k = 1$. In this case, each generation reproduces itself, and the neutron population remains fixed. If $k > 1$, the subsequent generations of neutrons increase in population; if $k < 1$, the subsequent generations decrease in population. The case $k = 1$ corresponds exactly to the condition given by Eq. (3.13), i.e., the criticality condition. This is easily demonstrated for the infinite-medium reactor model defined by the assumptions (3.7).

In our treatment of the infinite homogeneous reactor model, we argued that the neutron density was a spatially invariant function. Thus, if we determine the neutron-balance conditions in any one volume element of this infinite system, these conditions will apply to all such elements throughout the space. Let us write, then, the quantities involved in our definition of the multiplication constant in terms of the nuclear events which occur in a unit volume of the reactor. It is convenient here to write Eq. (3.16) in the form

$$k = \frac{\text{number of neutrons produced per unit volume per unit time}}{\text{number of neutrons introduced per unit volume per unit time}} \qquad (3.17)$$

According to the definition of the collision density, the neutron production per unit volume per unit time is given by $\nu\Sigma_f nv$. These neutrons are the offspring of the previous generation, of which there must have been exactly $\Sigma_a nv$ produced per unit volume per unit time. That $\Sigma_a nv$ is indeed the number of neutrons introduced follows from the fact that all neutrons must eventually be absorbed somewhere. Although very few of the neutrons introduced into the unit volume are actually absorbed there, every one of the neutrons which leave this volume element to be

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absorbed elsewhere is replaced by one which arrives from another region of the reactor. It follows, then, that

$$k = \frac{\nu\Sigma_f nv}{\Sigma_a nv} = \frac{\nu\Sigma_f}{\Sigma_a} \tag{3.18}\star$$

Thus, the multiplication constant for the infinite-medium reactor is given by the ratio $\nu\Sigma_f/\Sigma_a$. When $k = 1$, we have a critical system, and the neutron population maintains itself. Equation (3.18) is, therefore, consistent with (3.13).

Equation (3.18) is frequently written in an alternative form which better shows the essential features of the reactor. Let us write the total absorption cross section Σ_a according to the definition (3.9). Then

$$k = \frac{\nu\Sigma_f}{\Sigma_a^{(F)} + \Sigma_a^{(O)}}$$

If we multiply numerator and denominator by the absorption cross section of the fuel $\Sigma_a^{(F)}$, we obtain

$$k = \frac{\nu\Sigma_f}{\Sigma_a^{(F)}} \frac{\Sigma_a^{(F)}}{\Sigma_a^{(F)} + \Sigma_a^{(O)}} = \frac{\nu N_F \sigma_f}{N_F \sigma_a^{(F)}} \frac{\Sigma_a^{(F)}}{\Sigma_a^{(F)} + \Sigma_a^{(O)}} = \left(\frac{\nu\sigma_f}{\sigma_a}\right)_F \frac{\Sigma_a^{(F)}}{\Sigma_a^{(F)} + \Sigma_a^{(O)}} \tag{3.19}$$

The ratio $(\nu\sigma_f/\sigma_a)_F$, which depends only on the nuclear properties of the fissionable material, is usually denoted by the symbol η and is to some extent a measure of the value of a fuel in a reactor. Of course, the value of η varies with the relative speed (energy) of the neutron, but for many reactors there is one neutron speed which is of predominant importance (cf. Sec. 1.2*a*), and the value of η at that speed is an appropriate criterion for the value of a fuel. The quantity η may also be defined in terms of the ratio α defined previously [cf. Eq. (1.25)]. In terms of the fuel cross sections,

$$\alpha = \frac{\sigma_\gamma^{(F)}}{\sigma_f} \tag{3.20}$$

and it follows that

$$\eta = \frac{\nu\sigma_f}{\sigma_f + \sigma_\gamma^{(F)}} = \frac{\nu}{1 + \alpha} \tag{3.21}\star$$

The quantity $\Sigma_a^{(F)}/[\Sigma_a^{(F)} + \Sigma_a^{(O)}]$ is usually denoted by the symbol f and is called the *thermal utilization;*[1] it is dependent upon the concentration of the fuel and is a measure of the competition for neutrons between the fuel nuclei and the nonfuel nuclei in the reactor. Note that if $N_F \to 0$, $f \to 0$; likewise, if $N_O \to 0$, $f \to 1$. If this notation is used in (3.19), the multiplication constant may be written

$$k = \eta f \tag{3.22}\star$$

where

$$f \equiv \frac{\Sigma_a^{(F)}}{\Sigma_a^{(F)} + \Sigma_a^{(O)}} \tag{3.23}\star$$

[1] This expression was first used in conjunction with "thermal reactors"; thus, the cross sections would refer directly to the property of thermal neutrons.

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It was remarked in Sec. 3.2*b* that the scattering cross section did not play a role in the analysis of the infinite-medium model, and indeed the result (3.22) is independent of Σ_s. This is entirely a consequence of the assumptions of an infinite medium and no change of speed in scattering.

For more general finite multivelocity reactors, other expressions for the multiplication constant will be obtained; these must take into account loss by escape from the system and absorption and production of neutrons over the entire range of energies.

b. Critical Fuel Concentration. The criticality equation[1] (3.13) gives the condition which must be satisfied if an infinite homogeneous reactor is to operate at steady state, i.e., be self-sustaining. This equation is a condition on the material composition of the reactor. If the nonfuel components of the reactor (moderator, structure, coolant, etc.) are specified, Eq. (3.13) defines the *critical fuel concentration* for that system. We will show later that critical equations in general also involve the geometric proportions of the reactor. For the infinite-medium problem, however, the factor[2] in the critical equation which depends on the reactor dimensions is identically 1, and the resulting expression depends only on quantities involving the composition.

The infinite-medium one-velocity model has some useful applications in reactor analysis. One application is the estimation (in first approximation) of the critical fuel concentration of large, highly thermal reactor systems. A sample calculation using this model is presented in Sec. 3.4. In the present section, we derive the basic relations for determining the critical concentration. For simplicity in notation, we assume that the reactor core contains only nuclear fuel and moderator and that these components are intimately and homogeneously mixed. If the fuel and moderator are not homogeneously mixed, and this will be the case in many reactor designs, one can still introduce a suitable homogenization. The usual procedure is to assume that the various reactor materials, which in reality appear as distinct and separate regions, are mixed uniformly and spread throughout the entire reactor volume. Thus, if the region of interest is the core, then one assumes that the constituents are spread uniformly through the core volume. Consequently, each of the constituent materials of the core appears at a reduced density. The *density-reduction factor* is given by the ratio of the actual volume of the substance (as it is to be found in the reactor) to the total volume of the core. The total volume of the core is, therefore, maintained in the homogenized version of the system. A similar procedure may be applied to the other regions of the reactor, such as the reflector. These will be discussed later.

[1] See also Eq. (3.22) with $k = 1$.

[2] The neutron nonleakage probability.

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We compute the critical fuel concentration N_F from Eq. (3.22). If the symbol M denotes the moderator components, then the appropriate form of (3.22) for a homogenized fuel-moderator system is

$$1 = \eta f = \frac{\eta N_F \sigma_a^{(F)}}{N_F \sigma_a^{(F)} + N_M \sigma_a^{(M)}} \tag{3.24}$$

The solution for N_F is

$$N_F = \frac{N_M \sigma_a^{(M)}}{\sigma_a^{(F)}(\eta - 1)} \tag{3.25}$$

This equation gives one relation between the two unknowns N_M and N_F. A second relation may be obtained by requiring that the total volume of the system be conserved. This condition may be written

$$1 = \sum_i v_i \tag{3.26}$$

where the *volume fractions* are given by

$$v_i \equiv \frac{V_i}{V} \tag{3.27}$$

and V_i = actual volume of component i as it is to appear in reactor

V = total volume of system under consideration (3.28)

Equation (3.26) may be stated in terms of the nuclear concentrations. For this purpose, we use (2.54) in the form

$$N_i = \frac{P_i \mathfrak{A}}{A_i} \tag{3.29}\star$$

In this expression, P_i is the density of component i after it has been dispersed uniformly through the region V. According to our previous definition, this is simply the *normal density* ρ_i of material i times its volume fraction; thus,

$$P_i = \rho_i v_i \tag{3.30}\star$$

The normal density ρ_i is the actual density of material i at the reactor temperature and pressure of interest. It follows that

$$v_i = \frac{N_i A_i}{\rho_i \mathfrak{A}} \tag{3.31}$$

The substitution of this relation into (3.26) yields

$$1 = \frac{1}{\mathfrak{A}} \left(\frac{N_F A_F}{\rho_F} + \frac{N_M A_M}{\rho_M} \right) \tag{3.32}$$

This is the second equation between N_M and N_F. The simultaneous.

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solution with (3.25) gives

$$N_F = \frac{\rho_F \mathfrak{A}}{A_F[(\eta - 1)m/sp + 1]}$$
$$N_M = \frac{\rho_M \mathfrak{A}}{A_M}\left[\frac{(\eta - 1)m/sp}{(\eta - 1)m/sp + 1}\right] \tag{3.33}$$

where

$$m \equiv \frac{A_M}{A_F} \qquad s \equiv \frac{\sigma_a^{(M)}}{\sigma_a^{(F)}} \qquad p \equiv \frac{\rho_M}{\rho_F} \tag{3.34}$$

3.4 Application to Safety Storage Calculation of a Chemical-processing Plant

In the present section, we carry out a sample calculation for the safety storage requirements in a chemical-processing plant. The calculation is based on the infinite-medium one-velocity model.

Our consideration of nuclear reactors, thus far, has included only the principal features of the reactor proper. The reactor is, of course, only one element of an extensive engineering complex. The operation and the maintenance of this device require many additional component systems.

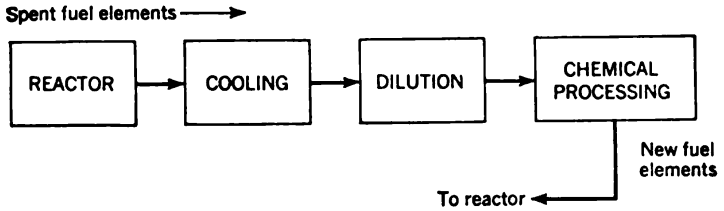

FIG. 3.2 Chemical processing cycle.

Precisely what supporting components are required is determined by the type and the design purpose of the reactor. Reactors designed for industrial or military application are primarily used for the production of power or fissionable materials. These applications are frequently made on a large scale; consequently, the supporting facilities are also large. Under such circumstances, the preparation and supply of the reactor fuel elements can become a mammoth undertaking. This is particularly true in the case of reactors which utilize fissionable materials in solid form. Such large quantities of solid fuel elements can be handled only in large chemical-processing plants.

When dealing with large amounts of fissionable material, a problem which inevitably arises is the criticality consideration of the system. The problem is important to the chemical-processing cycle for two reasons: (1) the safe storage of the fissionable material at each stage of the process, and (2) the optimization of the batch size handled by the process. Both of these factors can enter into the design considerations of the process at any one of a number of intermediate stages. The principal stages in the chemical processing of solid fuel elements are shown in the

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block diagram of Fig. 3.2. The first stage is denoted *cooling* and customarily involves the storage of the spent fuel elements in a suitable vault or canal wherein the radioactivity of the fission fragments present in the elements is allowed to decay to tolerable limits. The second stage, here called *dilution*, is the preparation of the fuel elements for processing (e.g., dissolving by some suitable acid solution). The final stage involves

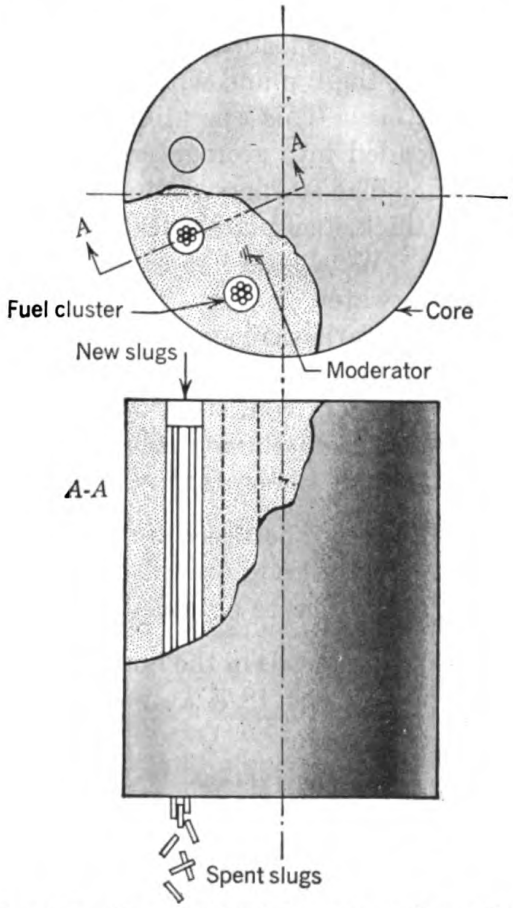

FIG. 3.3 Core and fuel-element configuration.

the reconcentration of the unburned fissionable material present in the element and the preparation of new elements.

At each stage of the process, the geometric configuration of the batch must be *safe*. We will illustrate the treatment of the criticality problem in this connection by carrying out a sample calculation for one stage of the process. For this calculation, we select the cooling stage and apply the one-velocity infinite-medium model to determine the criticality of the proposed cooling batch size for a representative reactor configuration.

Let us assume that the reactor in question is a heterogeneous system of slightly enriched natural uranium slugs imbedded in moderator. The slugs are cylindrical in shape and are loaded into the core through vertical

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channels so as to form clusters (see Fig. 3.3). The design details of these elements are given below:

Slug length	12 in.
Slug diameter	0.4 in.
Slugs per fuel-element cluster	500
Enrichment (% U^{235})	3
Total number of slugs	150,000

It is proposed that the reactor be loaded by adding new slugs (a complete vertical layer at a time) at the top and withdrawing spent slugs (a layer at a time) from the bottom. This operation would yield 25,000 slugs per layer, these to be loaded into storage crates in batches of 500 for cooling in a water canal. The storage crate is a rectangular box 9.2 by 8.8 by 12 in., with $\frac{1}{4}$-in.-thick steel-plate sides (no top or bottom). This crate can hold 506 slugs. We should like to know if the loaded crate is critical when immersed in water.

We compute the multiplication constant for this system by assuming that it is a homogenized infinite medium. Thus, we require the macroscopic cross sections of a system consisting of enriched slugs, steel, and water. The volume fractions of these constituents may be computed from the cross-sectional area of the crate:

$$\text{Cross-sectional area of crate occupied by 506 slugs} = 410 \text{ cm}^2$$
$$\text{Cross-sectional area of crate occupied by steel} = 56.4 \text{ cm}^2$$
$$\text{Cross-sectional area of crate occupied by water} = 55.7 \text{ cm}^2$$

The densities of the various materials in the homogenized system are computed from Eq. (3.30), using $\rho_F = 18.7$, $\rho_{\text{steel}} = 7.8$, and $\rho_{H_2O} = 1$ g/cm³ (cf. Table 2.2).

$$P_F = \rho_F v_F = (18.7)(\tfrac{410}{522}) = 14.7 \text{ g/cm}^3$$
$$P_{\text{steel}} = 0.843 \qquad P_{H_2O} = 0.107$$

For the microscopic cross sections (at 0.025 ev), we use

$$\sigma_a^{(28)} = 2.75 \text{ barns} \qquad \sigma_a^{(25)} = 687$$
$$\sigma_a^{(\text{steel})} = 2.53 \qquad \sigma_a^{(H_2O)} = 0.66$$

where the superscript (28) refers to U^{238} and (25) to U^{235}. We will use these cross sections directly in our calculations; this is not a correct procedure, but it will suffice for the present purposes.[1] In order to simplify the calculation of the macroscopic cross sections, we introduce the following conservative assumptions: (1) Spent fuel elements contain original concentration of U^{235}. (2) Fission fragments (which are strong neutron absorbers) in the spent slugs may be ignored.

[1] These data should be corrected for the energy distribution of thermal neutrons which are not all clustered about 0.025 ev, but are spread out over a wide range (refer to Chap. 4).

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For the 3 per cent enriched uranium slug, we obtain the absorption cross sections [cf. Eq. (3.29)]

$$\Sigma_a^{(25)} = N_F\sigma_a^{(25)} = \frac{P_F\mathcal{A}\sigma_a^{(25)}}{A_{25}} = \frac{(14.7)(0.6023)(687)(0.03)}{238} = 0.766 \text{ cm}^{-1}$$

$$\Sigma_a^{(28)} = \frac{(14.7)(0.6023)(2.75)(0.97)}{238} = 0.0992 \text{ cm}^{-1}$$

$$\Sigma_a^{(\text{steel})} = \frac{(0.843)(0.6023)(2.53)}{56} = 0.0229 \text{ cm}^{-1}$$

$$\Sigma_a^{(H_2O)} = \frac{(0.107)(0.6023)(0.66)}{18} = 0.00235 \text{ cm}^{-1}$$

These give for the thermal utilization f [cf. (3.23)]

$$f = \frac{\Sigma_a^{(25)}}{\Sigma_a^{(25)} + \Sigma_a^{(28)} + \Sigma_a^{(\text{steel})} + \Sigma_a^{(H_2O)}} = 0.860$$

And, the multiplication constant is, by (3.22),

$$k = \eta f = (2.08)(0.860) = 1.79$$

It is important to note that this result is based on a very elementary model of a chain-reacting system and should not be taken a priori as a representative value for the infinite-medium multiplication constant. Some of these limitations were mentioned in the discussion following Eq. (3.19). Better estimates for the infinite-medium multiplication constant than that given by Eq. (3.22) are developed in Chap. 4.

3.5 Neutron Flux

A statement of the neutron-balance condition involves the collision densities of the various possible neutron-nucleus reactions. The collision densities, in turn, may be described in terms of the neutron densities and the appropriate cross sections [cf. Eq. (3.5)]. Although the neutron density n is the fundamental quantity which describes the neutron population, it is frequently more convenient in reactor calculations to work with another function, called the *neutron flux*. The neutron flux ϕ is related to the neutron density through the definition (v = neutron speed)

$$\phi \equiv nv \tag{3.35}\star$$

If the neutron density is given as a function of space, speed, and time, then the flux is similarly specified; thus, if we use the definition (3.3), the flux is

$$\phi(\mathbf{r},v,t) = n(\mathbf{r},v,t)v \tag{3.36}$$

Note that the flux ϕ refers to neutrons of a particular speed v. More correctly, the function n, and therefore ϕ, must be given for an interval

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of speed dv. Thus, the formal definition of the density $n(\mathbf{r},v,t)$ is

$$n(\mathbf{r},v,t)\,dv\,d\mathbf{r} = \begin{array}{l}\text{number of neutrons in volume } d\mathbf{r} \text{ around space}\\ \text{point } \mathbf{r} \text{ at time } t \text{ which have speeds between } v \text{ and}\\ v + dv\end{array} \tag{3.37}$$

The corresponding definition of the neutron flux is

$$\phi(\mathbf{r},v,t)\,dv\,d\mathbf{r} = \begin{array}{l}\text{total } \textit{flux} \text{ of neutrons in volume } d\mathbf{r} \text{ around } \mathbf{r} \text{ at}\\ \text{time } t \text{ which have speeds between } v \text{ and } v + dv\end{array} \tag{3.38}$$

The quantities $n(\mathbf{r},v,t)$ and $\phi(\mathbf{r},v,t)$, therefore, are given as functions *per unit speed around* v and *per unit volume around* $\mathbf{r}$. By the expression "around v," we mean speeds which differ from v ever so slightly so that, if the deviation is dv, then as $dv \to 0$, the speeds in question approach v. It is customary to call ϕ the *total track length* per unit volume per unit time. The basis for this terminology is evident from the definition (3.35). We write this relation in an expanded form

$$\begin{aligned}\phi(\mathbf{r},v,t)\,dv\,d\mathbf{r} &= vn(\mathbf{r},v,t)\,dv\,d\mathbf{r}\\ &= \begin{bmatrix}\text{distance}\\ \text{traveled}\\ \text{per neutron}\\ \text{per unit}\\ \text{time}\end{bmatrix}\begin{bmatrix}\text{number of}\\ \text{neutrons}\\ \text{per unit}\\ \text{volume per}\\ \text{unit speed}\end{bmatrix}\begin{bmatrix}\text{speed}\\ \text{interval}\end{bmatrix}[\text{volume}]\end{aligned} \tag{3.39}$$

or

$$\phi(\mathbf{r},v,t)\,dv\,d\mathbf{r} \equiv \begin{array}{l}\text{total distance (track length) traveled by all}\\ \text{neutrons which have speeds in } dv \text{ around } v \text{ in}\\ \text{volume } d\mathbf{r} \text{ per unit time}\end{array} \tag{3.40}\star$$

This definition is frequently abbreviated to: *total track length per unit volume per unit speed.* Whenever this form is used, the complete statement (3.40) is implied.

It is sometimes more convenient to describe the neutron population in terms of the neutron kinetic energy instead of the speed. In this event, the neutron density and the flux are defined per unit of energy. In order to show the relationship between the speed and the energy definitions, let us consider the function

$$\mathrm{n}(\mathbf{r},E,t)\,dE\,d\mathbf{r} = \begin{array}{l}\text{number of neutrons in } d\mathbf{r} \text{ around } \mathbf{r} \text{ at time } t\\ \text{which have kinetic energies in } dE \text{ about } E\end{array} \tag{3.41}$$

This function gives the neutron density in the energy space E. It can be directly related to the density in the speed space [given by (3.37)] by selecting the speed interval dv which corresponds to the energy interval dE. Clearly, if we specify that dv is to correspond to dE, then the number of neutrons per unit volume around $\mathbf{r}$ must be the same whether we describe it in terms of the energy variable E or in terms of the speed

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variable v. Thus, if we use the function (3.37), then we require that (cf. Fig. 3.4)

$$\mathfrak{n}(\mathbf{r},E,t)\,dE = n(\mathbf{r},v,t)\,dv \tag{3.42}\star$$

or

$$\mathfrak{n}(\mathbf{r},E,t) = n(\mathbf{r},v,t)\,\frac{dv}{dE} \tag{3.43}$$

This statement tells us that if the neutron density is known in terms of either E or v the expression in terms of the other is obtained by using the transformation (3.43). The differential ratio[1] dv/dE is computed from the relation $E = \frac{1}{2}mv^2$, where m is the mass of the neutron. It follows that

$$\mathfrak{n}(E) = \frac{n(v)}{mv} = \frac{n[v(E)]}{\sqrt{2Em}} = \frac{n(\sqrt{2E/m})}{\sqrt{2Em}} \tag{3.44}$$

where we have omitted the variables $\mathbf{r}$ and t for the sake of clarity. The corresponding expression for the flux per unit energy range is obtained

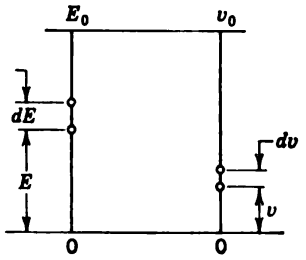

FIG. 3.4 Equivalent intervals in speed and energy spaces.

by a similar calculation. The fundamental relation between the expression for the flux in speed space and energy space is derived from (3.35) using (3.42)

$$v(E)\,\mathfrak{n}(\mathbf{r},E,t)\,dE = v\,n(\mathbf{r},v,t)\,dv \tag{3.45}$$

If we call $\varphi(\mathbf{r},E,t)$ the flux per unit energy range, where

$$\varphi(\mathbf{r},E,t) = v(E)\,\mathfrak{n}(\mathbf{r},E,t) \tag{3.46}$$

then

$$\varphi(\mathbf{r},E,t)\,dE = \phi(\mathbf{r},v,t)\,dv \tag{3.47}\star$$

This equation corresponds to the relation (3.42) and states that the total track length per unit volume per unit time around $\mathbf{r}$ at time t is conserved in going from speed space to energy space. If we introduce the Jacobian dv/dE, it is easily shown that (again omitting $\mathbf{r}$ and t)

$$\varphi(E) = \frac{\phi(\sqrt{2E/m})}{\sqrt{2Em}} = \frac{1}{m}\,n\left(\sqrt{\frac{2E}{m}}\right) \tag{3.48}$$

It has become customary to distinguish between the flux per unit speed

[1] The Jacobian for the transformation $v \to E$. See, for example, I. S. Sokolnikoff and E. S. Sokolnikoff, "Higher Mathematics for Engineers and Physicists," pp. 183–185, McGraw-Hill Book Company, Inc., New York, 1941.

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and the flux per unit energy solely by the argument following the flux symbol. Thus, if ϕ denotes flux and n denotes density, then

$$\begin{aligned}\phi(v) &= vn(v) = \text{total track length per unit volume of neutrons with speeds in unit interval around } v \\ \phi(E) &= v(E)\, n(E) = \text{total track length per unit volume of neutrons with energies in unit interval around } E\end{aligned} \tag{3.49}$$

The concepts introduced above may be extended to describe integral characteristics of the neutron population. The function $\phi(\mathbf{r},E,t)$, which gives a detailed description, can be used to construct a quantity which refers to the "bulk" properties of the system. Thus, if we wish to determine the total track length Φ of neutrons in a system of volume V which have energies in dE about E, we perform the following integration:

$$\Phi(E,t)\, dE = \int_V \phi(\mathbf{r},E,t)\, d\mathbf{r}\, dE \tag{3.50}$$

The total number $\mathfrak{N}$ of such neutrons is computed from the relation

$$\mathfrak{N}(E,t) = \frac{\Phi(E,t)}{v(E)} \tag{3.51}$$

Likewise we can compute the total track length $\mathbf{\Phi}$ of neutrons of all energies in volume V from the relation

$$\mathbf{\Phi}(t) = \int_E \Phi(E,t)\, dE = \int_E dE \int_V \phi(\mathbf{r},E,t)\, d\mathbf{r} \tag{3.52}$$

The application of these functions to the calculation of neutron populations in reactors will be considered later.

The flux concept is useful in describing the collision densities in reactor systems. The general expression for the collision density was developed in Sec. 3.1 with the result (in energy notation)

$$F(\mathbf{r},E,t) = \Sigma(E)\, v(E)\, n(\mathbf{r},E,t) \tag{3.53}$$

In terms of the neutron flux, this expression gives

$$\begin{aligned}F(\mathbf{r},E,t) &= \Sigma(E)\, \phi(\mathbf{r},E,t) \\ &= \text{total number of collisions per unit volume around } \mathbf{r} \text{ and per unit time at time } t \text{ due to neutrons in unit energy range around } E\end{aligned} \tag{3.54}*$$

Thus, for absorption collisions

$$\Sigma_a(E)\, \phi(\mathbf{r},E,t)\, dE = \text{total number of absorptions per unit volume around } \mathbf{r} \text{ and per unit time at time } t \text{ due to neutrons with energy between } E \text{ and } E + dE \tag{3.55}$$

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and so on. This relation should be compared to (3.5). The form (3.54) for the collision density is especially useful in writing neutron-balance conditions such as (3.8). In the present notation, this equation takes the form

$$\frac{1}{v}\frac{d}{dt}\phi(t) = \nu\Sigma_f\phi(t) - \Sigma_a\phi(t)$$

where we have applied the definition $\phi(t) = vn(t)$ from (3.35).

The neutron flux has a directly useful physical meaning when considering systems in which all the neutrons have the same energy and move in the same direction, i.e., a beam of monoenergetic neutrons. Suppose that there are n neutrons in each unit volume of the beam and that v is the speed of the neutrons. Now consider a unit area which is perpendicular to the beam and compute the number of neutrons passing through that area per unit time. Clearly, this is exactly the number of neutrons in a volume whose cross section is unit area and whose length is numerically the speed v, since all neutrons in this volume will be able to pass through the unit area in unit time (see Fig. 3.5). Then, in the

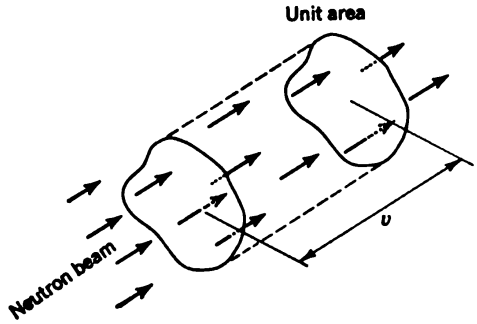

FIG. 3.5 Collimated beam of neutrons of speed v.

beam, the number of neutrons crossing unit area in unit time is $nv = \phi$, the neutron flux. This is true *only in the case of a monoenergetic beam.* In the more general case wherein neutrons are moving in all directions, the neutron flux *is not* the number of neutrons crossing a unit area in unit time. For example, in the case of a unit area placed in a gas of neutrons whose directions of motion are *isotropically* distributed, the number of neutrons crossing the unit area in unit time in any one direction is $\frac{1}{4}\phi$. This point will be demonstrated later.

We conclude the present chapter with a few remarks on the units of measure for neutron flux and collision density. It is customary to specify neutron densities in neutrons/cm^3; thus, if the neutron speed is given in cm/sec, the flux has the units neutrons/cm^2/sec. Likewise if, as is the custom, the neutron microscopic cross sections are given in square centimeters and the nuclear densities in nuclei/cm^3, then the macroscopic cross sections Σ are given in cm^{-1} and the collision densities in collisions/cm^3/sec. We can obtain some idea of the magnitudes of

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these quantities by considering some representative numbers. Thus, if we use the data of the sample computation in Sec. 2.4*d*,

$$N = 7.26 \times 10^{22} \text{ nuclei/cm}^3$$
$$\Sigma_s = 0.726 \text{ cm}^{-1}$$
$$\Sigma_a = 0.000654 \text{ cm}^{-1}$$

then for a flux of 10^{13} neutrons/cm^2/sec (a typical value), the rate a which neutrons are absorbed in BeO is

$$\Sigma_a \phi = 6.54 \times 10^9 \text{ absorptions/cm}^3\text{/sec}$$

These cross sections refer to neutrons of 0.025 ev energy or 2,200 m/sec speed. The density of these neutrons in the BeO is

$$n = \frac{\phi}{v} = 4.54 \times 10^7 \text{ neutrons/cm}^3$$

PROBLEMS

3.1 Consider an infinite reactor composed of a uniform mixture of beryllium and natural uranium. The atomic ratio of isotopes in natural uranium is $U^{238}:U^{235} = 993:7$. Assume that the one-velocity model applies, and use the microscopic cross sections and densities of Table 2.2.

a. Compute the atomic concentration of U^{235} required to make the reactor critical.

b. Show that, if it is assumed that the average density of the U-Be mixture can be approximated by the density of pure Be, then, *in this case*, the error in the U^{235} concentration is of the order of 1 per cent.

c. Find the mean number of scattering collisions made by a neutron during its lifetime.

3.2 It is proposed that a reactor of pure UF_6 be constructed, the fluorine to provide the moderation and the U^{235} in the uranium to serve as the nuclear fuel. A first estimate of the critical fuel concentration for such a system can be obtained from the one-velocity infinite-medium model.

a. If the uranium in the UF_6 is composed of U^{235} and U^{238}, determine the atomic ratio of U^{238} to U^{235} required to make the infinite medium critical. Use the data

$$\eta = 2.08 \ (U^{235}) \qquad \sigma_a = 9 \text{ mb for fluorine}$$

The uranium data are given in Table 2.2.

b. What is the infinite-medium multiplication constant for UF_6 with natural uranium (cf. Prob. 3.1)? NOTE: A much more realistic calculation would include the absorption of neutrons during the slowing-down process by the F and U^{238}. These nuclei have large Σ_a for high-energy neutrons, and the omission of this effect here gives an underestimate of the fuel requirements (cf. Chap. 4).

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CHAPTER 4

SLOWING-DOWN PROCESS IN THE INFINITE MEDIUM

4.1 Mechanism of Energy Loss by Scattering Collisions

a. Scattering Reactions. The importance of the neutron-energy spectrum in determining the nuclear features of a reactor was mentioned in Sec. 1.2. The spectrum itself is determined partly by the nuclear reactions which can occur and partly by the geometry and size of the system. In this chapter, attention is limited to the influence of the nuclear phenomena.

All three types of reactions enter into the determination of the spectrum. The fission reaction determines, of course, the average energy of the neutrons borne into the reactor. The scattering and absorption reactions together determine the detailed shape of the spectrum. The largest influence is due to the scattering interactions which establish the over-all shape of the spectrum. The effects of the absorptions are somewhat secondary, and in most reactor situations these tend primarily to distort the spectrum produced by the scatterings. Thus in the present treatment of the subject we begin with a study of the spectrum arising from scattering phenomena alone and then extend and modify this analysis to include the deviations which arise when one includes also the effects of the absorptions.

Both elastic- and inelastic-scattering reactions have a significant effect upon the neutron-energy distribution since each process is an important mechanism of energy degradation over some portion of the energy scale. The inelastic collisions are important when the neutron kinetic energy is in the Kev range or above. Below the Kev range, the elastic collisions are the primary mechanism. Before attempting an analysis of the neutron-energy spectrum it will be of interest to examine the essential features of these two scattering reactions.

Consider first the inelastic-scattering process. It was mentioned previously that when an inelastic-scattering collision occurs, a compound nucleus is formed, and this nucleus appears in an excited nuclear state. The energy of excitation is acquired from the kinetic energy of the incident neutron and the binding energy associated with the addition of this particle to the target nucleus. In the case of light nuclei, the excited energy states of the compound nucleus are far removed from the

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ground state, and these states are spaced widely apart (in the order of 1 Mev). If the struck nucleus is of large mass, however, the first excited state is closer to the ground state (10 to 100 Kev), and the adjacent states are much more closely packed[1] than in light nuclei.

These general properties of light and heavy nuclei are illustrated in Fig. 4.1. The energy E^* indicated in the figure denotes the excited state of the compound nucleus formed from the absorption of the incident neutron by the target nucleus. It is evident from the figure that the heavy nucleus, with its closely packed energy levels, has by far the greater number of accessible decay channels for returning to the ground state. One of these may involve the emission of a neutron which leaves the nucleus with some residual energy (e.g., the state E_1 in Fig. 4.1*b*). This decay channel corresponds to an inelastic collision. The light nucleus has fewer channels for decay to the ground state, for a given excitation energy E^*, and, furthermore, possesses appreciably higher threshold energies (the separation energy between the first excited state and the ground state). For example, in the case of U^{238}, the threshold energy is about 40 Kev, and the *level spacing*, in the order of several Kev. For light nuclei, such as Li and Be, however, the thresholds and level spacings are as much as 2 and 3 Mev. Because of this wider spacing, fewer energy states are available to the compound nucleus, and the opportunities for decay with neutron emission (i.e., by inelastic scattering) are greatly reduced. Consequently, the inelastic-scattering cross sections of light nuclei are relatively small, and this process is not an important mechanism for energy degradation in media which contain appreciable amounts of these materials. On the other hand, heavy nuclei have relatively large inelastic-scattering cross sections and can therefore serve as effective neutron-energy moderators so long as the neutron energy is above the threshold value. Thus inelastic collisions are an important mechanism for energy degradation only in the case of the heavy nuclei and, even then, only in the high-energy range. Although these nuclei could cause some slowing down of neutrons, it is clear that they could not be satisfactory as moderator material because of the threshold-energy limitation.

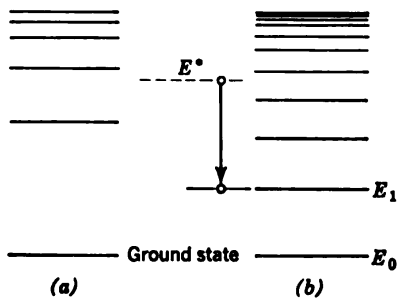

FIG. 4.1 Nuclear energy levels: (*a*) low mass number; (*b*) high mass number.

The elastic-scattering process, however, suffers no such restriction. Regardless of the energy of the incident neutron, an elastic-scattering collision can cause some reduction in the neutron energy. The extent

[1] See, for example, J. M. Blatt and V. F. Weisskopf, "Theoretical Nuclear Physics," pp. 365–374, John Wiley & Sons, Inc., New York, 1952.

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of the energy change does depend, however, on the initial energy of the neutron and on the mass of the struck nucleus. As will be shown shortly, the light nuclei are the most efficient moderators from the viewpoint of the elastic-collision process.

It is clear from the above discussion that a reactor which is designed to utilize the superior low-energy properties of its nuclear constituents must contain large amounts of moderating material. Thermal reactors are systems of this type. In accord with the remarks above, these reactors contain materials of low nuclear mass which slow down the high-energy fission neutrons by means of elastic collisions. At the higher neutron energies (above threshold) the inelastic scatterings with the heavy nuclei in the structural components of the reactor, and with the fuel itself, can assist in this slowing-down process. Although these contributions are generally small, they can be significant in some instances.[1] Inelastic scatterings are most important in systems which contain large amounts of high-nuclear-mass materials, such as iron, liquid metals, and uranium. Reactors which utilize materials of this type tend to have fast spectrums. Sometimes this property is deliberately designed into the reactor. In such systems good moderator materials are kept to a minimum in order to *reduce* the slowing down of neutrons as much as possible. In this way the average neutron energy is increased so as better to exploit the superior nuclear properties of the fuel in the higher energy range.[2] The energy spectrum of these systems is determined by their inelastic-scattering characteristics.

The greater part of the present work is directed toward the study of thermal reactors which require efficient moderator materials as principal ingredients of the core (and/or the reflector). The low-mass-number substances are best suited to this purpose. Although their inelastic-scattering cross sections are small, their elastic-scattering cross sections are relatively large, and, moreover, the energy loss per collision is substantial. Thus the elastic-scattering process is the mechanism of energy degradation of primary interest to the present study. The analysis which follows begins with an inquiry into the details of the collision mechanics.

In this treatment we examine the nuclear reactor with the aid of an *infinite-medium* model which describes the neutron-density distributions in terms of the kinetic energy of the neutrons. This model is the immediate extension of the one-velocity approximation introduced in Chap. 3. In the present analysis we discard the one-velocity restriction and attempt to describe the trajectory of the neutron in energy space as it is slowed down by elastic-scattering collisions with the nuclei of the medium. In

[1] In heterogeneous reactors and in reactors containing highly enriched fuel.

[2] The conditions for breeding and converting, for example, are improved with decreasing values of the ratio $\alpha \equiv \sigma_\gamma/\sigma_f$ in the fuel. For U^{235} this ratio is noticeably reduced in the Kev range from its value around thermal.

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order to simplify the calculation we assume that the medium is *homogeneous* and *isotropic* as well as infinite in extent. Furthermore, the neutron sources (to be consistent) are taken to be uniformly distributed throughout all space. These sources are assumed to emit neutrons isotropically in direction, with a distribution of speeds (energies) to be specified in each case. We introduce these restrictions in order to eliminate the spatial dependence of the neutron density and thereby allow us to concentrate attention on the energy distribution of the neutrons in the system.

Because of the assumption of isotropy of medium and source emission, the number of neutrons moving in a given direction per unit solid angle per unit volume will be the same for all directions. Thus for systems at steady state, the energy distribution alone will suffice to describe the neutron population. Consequently, for this calculation, we can use the functions for the neutron density and flux given by (3.49)

$$\begin{aligned}\phi(E)\,dE &= v(E)\,n(E)\,dE \\ &= \text{total track length per unit volume per unit time of neutrons with kinetic energies between } E \text{ and } E + dE\end{aligned} \tag{4.1}$$

The material of the infinite medium is characterized by neutron cross sections $\Sigma_a(E)$, $\Sigma_s(E)$, and $\Sigma_f(E)$, and ν is the average number of neutrons per fission. We define

$$\begin{aligned}\Sigma_s(E) &\equiv \text{expected number of elastic-scattering collisions between neutrons of energy } E \text{ and nuclei per unit distance of travel} \\ \Sigma_a(E) &= \Sigma_\gamma(E) + \Sigma_f(E)\end{aligned} \tag{4.2}$$

in the notation of (2.40). We shall also use the *total cross section* $\Sigma_t(E)$ according to the definition (2.50).

b. Motion in Laboratory and Center-of-mass Systems of Coordinates. If a neutron collides with a stationary nucleus (described in the same frame of motion), the neutron will emerge from the collision with a reduced kinetic energy. The nucleus in turn acquires a momentum increment which will result in its displacement (recoil) from its lattice position or an increased amplitude of vibration about its lattice site. These remarks apply, of course, to nuclear materials in solid (crystalline) form. If the medium of the struck nucleus is in the liquid or gaseous state, the nucleus will recoil from the impact, moving off with some *velocity* prescribed by the details of the collision. If the struck nucleus is a part of a molecular structure, the situation is further complicated, and the details of the event are determined principally by the relative speed of the neutron. For the present calculation, we limit the problem to the consideration of a collision between a neutron and a *free* nucleus. This situation describes in first approximation the collision process between a neutron and *any* nucleus for neutron energies in the electron-volt range or higher. The problem of predicting the energy loss of the

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neutron for these situations may be treated by the purely classical techniques of the "billiard-ball" type of collision.

Consider, then, the collision between a neutron of mass m and a nucleus of mass M. We describe the motion of these particles in two reference frames, namely, the laboratory and the center-of-mass coordinate systems. The velocities[1] of these particles in the laboratory system (L) are

$\mathbf{v}_0$ = velocity of neutron before collision
$\mathbf{V}_0$ = velocity of nucleus before collision
$\mathbf{v}_1$ = velocity of neutron after collision
$\mathbf{V}_1$ = velocity of nucleus after collision

For the center-of-mass system (C) we use the symbols $\mathbf{w}$ and $\mathbf{W}$ for the neutron and nucleus velocities, respectively. The subscripts 0 and 1 apply as before.

The immediate objective is to determine the relationship between the initial and final energies of the neutron, i.e., the energy loss suffered by the neutron of mass m in colliding with a nucleus of mass M. For this calculation we require the conservation laws for momentum and energy. The conservation of (vector) momentum in the (L) system yields

$$m\mathbf{v}_0 + M\mathbf{V}_0 = m\mathbf{v}_1 + M\mathbf{V}_1 \tag{4.3}$$

From the kinetic-energy balance we have

$$\tfrac{1}{2}mv_0^2 + \tfrac{1}{2}MV_0^2 = \tfrac{1}{2}mv_1^2 + \tfrac{1}{2}MV_1^2 \tag{4.4}$$

where $\mathbf{v}^2 \equiv \mathbf{v} \cdot \mathbf{v}$. In carrying out the proposed calculation it is convenient to transform these equations to a coordinate system in which the center of mass of the neutron-nucleus system is at rest, i.e., the (C) system. The velocity of the center of mass[2] $\mathbf{c}$ is defined by

$$(M + m)\mathbf{c} \equiv m\mathbf{v}_0 + M\mathbf{V}_0 = m\mathbf{v}_1 + M\mathbf{V}_1 \tag{4.5}$$

Since the velocities $\mathbf{v}_0$ and $\mathbf{V}_0$ are specified, $\mathbf{c}$ may be computed directly from (4.5). This expression for $\mathbf{c}$ may be used to relate the velocities in the (L) and (C) systems. It follows that the velocities in (C) before collision are

$$\begin{aligned} \mathbf{w}_0 &\equiv \mathbf{v}_0 - \mathbf{c} = \frac{M}{m + M}(\mathbf{v}_0 - \mathbf{V}_0) \\ \mathbf{W}_0 &\equiv \mathbf{V}_0 - \mathbf{c} = \frac{m}{m + M}(\mathbf{V}_0 - \mathbf{v}_0) \end{aligned} \tag{4.6}$$

and after collision

$$\begin{aligned} \mathbf{w}_1 &\equiv \mathbf{v}_1 - \mathbf{c} = \frac{M}{m + M}(\mathbf{v}_1 - \mathbf{V}_1) \\ \mathbf{W}_1 &\equiv \mathbf{V}_1 - \mathbf{c} = \frac{m}{m + M}(\mathbf{V}_1 - \mathbf{v}_1) \end{aligned} \tag{4.7}$$

[1] Boldface type denotes vector quantities.
[2] The vector $\mathbf{c}$ is defined in the (L) system.

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Thus the velocity vectors of the neutron and nucleus in system (C) are *oppositely* directed both before and after collision (cf. Fig. 4.2). If we compute the momentum in (C) before and after collision, using relations (4.6) and (4.7), it is seen that

$$m\mathbf{w}_0 + M\mathbf{W}_0 = m\mathbf{w}_1 + M\mathbf{W}_1 = 0 \tag{4.8}$$

so the total momentum of the system is zero in the center-of-mass system before and after collision. In the same way it is easily shown that the kinetic energy in (C) is also conserved. The substitution of the definitions (4.6) and (4.7) into (4.4) yields

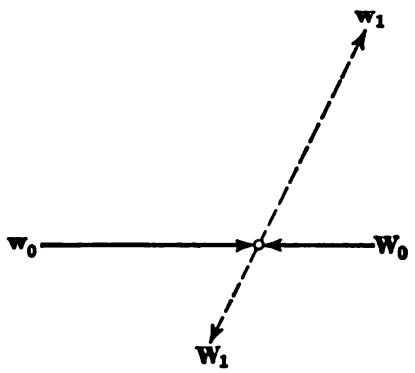

FIG. 4.2 Velocities in center-of-mass system.

$$\tfrac{1}{2}mw_0^2 + \tfrac{1}{2}MW_0^2 = \tfrac{1}{2}mw_1^2 + \tfrac{1}{2}MW_1^2 \tag{4.9}$$

But, we have from (4.8) that

$$\begin{aligned} \mathbf{w}_0 &= -\frac{M}{m}\mathbf{W}_0 \\ \mathbf{w}_1 &= -\frac{M}{m}\mathbf{W}_1 \end{aligned} \tag{4.10}$$

The substitution of these relations into (4.9) yields

$$w_0^2 = w_1^2 \qquad W_0^2 = W_1^2 \tag{4.11}\star$$

where $|\mathbf{W}| \equiv W$ and $|\mathbf{w}| \equiv w$. Thus the magnitudes of the velocities of the neutron and nucleus in the center-of-mass system do not change during collision.

c. Energy Loss from Stationary Nuclei. When the neutron kinetic energy is at least as large as 1 ev, the speed of the nucleus V_0 (due to thermal motion) is negligible in comparison to the speed of the neutron v_0. Thus a very good approximation, which may be introduced when dealing with the slowing-down process, is the assumption that the nucleus is initially at rest in the laboratory system of coordinates. This assumption simplifies the analysis somewhat; however, it may introduce appreciable error in the study of collisions between low-speed neutrons (in the order of thermal energy $\sim$0.02 ev) and nuclei. The speed of the nuclei in these situations is comparable to the speed of thermal neutrons, and the approximation is therefore poor. For the present we will not consider collisions by neutrons of electron-volt (or lower) energies and will limit our calculation to the portion of the slowing-down process above the electron-volt range. The properties of these low-energy or *thermal* neutrons will be examined later.

Consider, then, the special case in which the struck nucleus is initially at rest, that is, $\mathbf{V}_0 \equiv 0$. The velocity of the center of mass for this situation is [see Eq. (4.5)]

$$\mathbf{c} = \frac{m}{m+M}\mathbf{v}_0 \tag{4.12}$$

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and the various velocity vectors in (L) are shown in Fig. 4.3. Our purpose is to compute the neutron energy after collision, namely, $\frac{1}{2}mv_1^2$. The quantity v_1^2 is obtained from the relation [see (4.7)]

$$v_1^2 = (\mathbf{w}_1 + \mathbf{c})^2 = \mathbf{w}_1^2 + \mathbf{c}^2 + 2\mathbf{w}_1 \cdot \mathbf{c} \tag{4.13}$$

But from (4.6), (4.11), and (4.12) we have

$$\begin{aligned} \mathbf{w}_1^2 &= \mathbf{w}_0^2 = \left(\frac{M}{m+M}\right)^2 \mathbf{v}_0^2 \\ \mathbf{c}^2 &= \left(\frac{m}{m+M}\right)^2 \mathbf{v}_0^2 \end{aligned} \tag{4.14}$$

The dot product which appears on the right-hand side of (4.13) may be expressed in terms of the scattering angle in the (C) system. This quantity is defined in terms of the coordinate system of Fig. 4.4. The z axis of this system is selected in the direction of the vector $\mathbf{v}_0$. Note that $\mathbf{v}_0$, $\mathbf{v}_1$, and $\mathbf{w}_1$ are coplanar. It is evident from the figure that the dot product $\mathbf{w}_1 \cdot \mathbf{c} = w_1 c \cos\beta$. The substitution of (4.14) and (4.12) into this expression yields

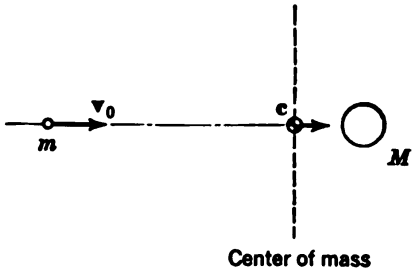

FIG. 4.3 Collision with stationary nucleus.

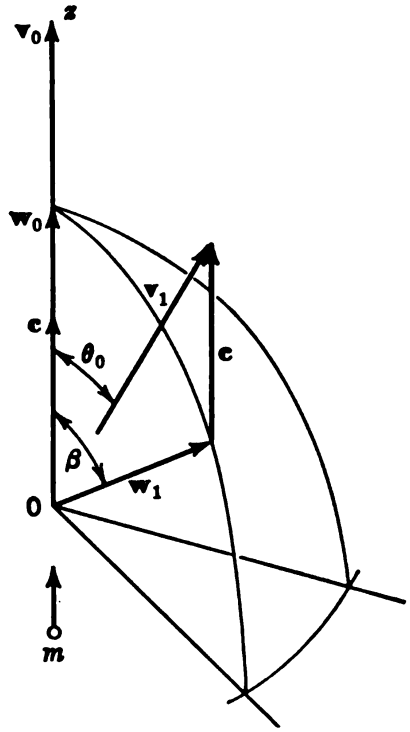

FIG. 4.4 Coordinate system for velocity vectors.

$$\mathbf{w}_1 \cdot \mathbf{c} = \left(\frac{M}{m+M}\right) v_0 \left(\frac{m}{m+M}\right) v_0 \eta \tag{4.15}$$

where we have denoted $\cos\beta \equiv \eta$. The application of (4.14) and (4.15) to (4.13) gives the relation

$$v_1^2 = w_0^2 + c^2 + 2w_0 c\eta = \left(\frac{M}{m+M}\right)^2 v_0^2 + \left(\frac{m}{m+M}\right)^2 v_0^2 + \frac{2mMv_0^2\eta}{(m+M)^2} \tag{4.16}$$

This result may be written in terms of the kinetic energies. Let

$$\begin{aligned} E_0 &= \tfrac{1}{2}mv_0^2 \equiv \text{kinetic energy of neutron in } (L) \text{ system before collision} \\ E_1 &= \tfrac{1}{2}mv_1^2 \equiv \text{kinetic energy of neutron in } (L) \text{ system after collision} \end{aligned} \tag{4.17}$$

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If (4.16) is multiplied through by $\frac{1}{2}m$ and the definitions (4.17) are applied, the result is

$$E_1 = \frac{E_0}{(A+1)^2}(A^2 + 2\eta A + 1) \tag{4.18}\star$$

where $A \equiv M/m$. Thus, in the laboratory system, the kinetic energy of the neutron after the collision is determined by the kinetic energy before collision if the scattering angle β in the center-of-mass system is known. It is clear that, as β varies from 0 to π, the final neutron energy varies from a maximum value of E_0, the initial energy, to the minimum value

$$E_{\min} = E_0\left(\frac{A-1}{A+1}\right)^2 = \alpha E_0 \tag{4.19}$$

where

$$\alpha \equiv \left(\frac{A-1}{A+1}\right)^2 \tag{4.20}\star$$

d. Frequency Function for Elastic Collision. In order to determine the distribution of final energies between the limits E_0 and $E_{\min}$ [see Eq. (4.19)], there must be some additional information as to the distribution of the scattering angles in the center-of-mass system. Or putting it in another way, we require the frequency function for the angle β. This function, call it $\mathfrak{f}$, describes the angular distribution of neutrons emerging from a collision between a neutron of given speed and a nucleus of a particular type. In general, it will depend upon the energy of the incident neutron and the nature of the struck nucleus. Its specific form is determined either from experiment or from the application of some suitable theoretical model of the nucleus. It is usually convenient to express this function analytically as an infinite series in β with the coefficients of the series to be determined from experiment or theory. Thus we define a function $\mathfrak{f}$ such that

$\mathfrak{f}(\eta)\,d\eta \equiv$ fraction of all scattering collisions which result in scattering angles in (C) whose cosines lie between η and $\eta + d\eta$

$$\equiv \sum_{n=0}^{\infty} A_n P_n(\eta)\,d\eta \tag{4.21}$$

where $P_n(\eta)$ are the Legendre polynomials[1]

$$P_0(\eta) = 1 \qquad P_1(\eta) = \eta \qquad \text{etc.}$$

The argument η has been used in the definition of $\mathfrak{f}$ since it is this quantity which enters into the energy relation (4.18) rather than β. Likewise, the expansion is carried out in terms of the Legendre polynomials in

[1] See, for example, F. B. Hildebrand, "Advanced Calculus for Engineers," pp. 171–181, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1949.

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order to accommodate the physical fact that the scattering process is uniform in azimuth about the initial direction $\mathbf{v}_0$ of the neutron.[1] Thus the argument η alone is sufficient to describe the scattering. This variable takes on all values in the interval $(-1,1)$, the very interval over which the functions P_n are defined.

If the function $\mathfrak{f}(\eta)$ is specified, the coefficients A_n are readily computed by the usual methods. Thus we determine the coefficient A_s by multiplying both sides of (4.21) with the polynomial $P_s(\eta)$ and integrating over the interval $(-1,1)$:

$$\int_{-1}^{1} P_s(\eta)\, \mathfrak{f}(\eta)\, d\eta = \sum_{n=0}^{\infty} A_n \int_{-1}^{1} P_n(\eta)\, P_s(\eta)\, d\eta = \frac{2A_s}{2s+1}$$

or

$$A_s = \frac{2s+1}{2} \int_{-1}^{1} P_s(\eta)\, \mathfrak{f}(\eta)\, d\eta \tag{4.22}$$

The simplest case of the function $\mathfrak{f}$ corresponds to isotropic scattering in the center-of-mass system. In this case, the cosine of the scattering angle η is uniformly distributed between -1 and 1. Thus we take $A_n \equiv 0$ for all $n > 0$. And from (4.22)

$$A_0 = \tfrac{1}{2} \int_{-1}^{1} P_0(\eta) C\, d\eta = C$$

where C, a constant, denotes the value of $\mathfrak{f}$ for this case. We obtain C from the requirement on the frequency function, namely,

$$\int_{-\infty}^{\infty} \mathfrak{f}(\eta)\, d\eta = C \int_{-1}^{1} d\eta \equiv 1$$

Therefore $C = A_0 = \frac{1}{2}$ and it follows that

$$\mathfrak{f}(\eta) = \tfrac{1}{2} \qquad -1 \leq \eta \leq 1 \tag{4.23}\star$$

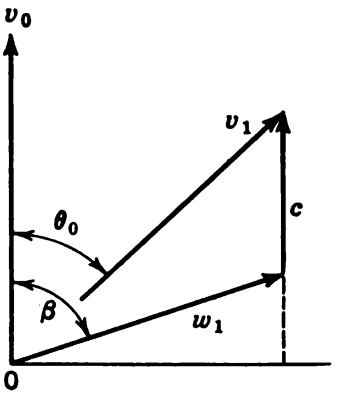

FIG. 4.5 Scattering angle θ_0 in laboratory system.

It should be noted that, even though the scattering is isotropic in the center-of-mass system, it *is not* isotropic in the laboratory system. To show this we compute the cosine of the scattering angle θ_0 in the laboratory system. The angle θ_0 is shown in Fig. 4.4. An alternate view is given in Fig. 4.5. The cosine of θ_0, call it μ_0, may be computed from the geometry of Fig. 4.5 by using the relation

$$\mu_0 = \frac{c + w_1\eta}{v_1} \tag{4.24}$$

where

$$v_1 = [w_1^2 \sin^2 \beta + (c + w_1\eta)^2]^{\frac{1}{2}} = c\left[\left(\frac{w_1}{c}\right)^2 + 2\eta\left(\frac{w_1}{c}\right) + 1\right]^{\frac{1}{2}} \tag{4.25}$$

[1] This point is discussed in Secs. 7.2b and 7.3e.

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But, from (4.14),

$$\frac{w_1}{c} = \frac{w_0}{c} = A \tag{4.26}$$

The substitution of (4.25) and (4.26) into (4.24) yields

$$\mu_0 = \frac{1 + A\eta}{(A^2 + 2\eta A + 1)^{\frac{1}{2}}} \tag{4.27}\star$$

This expression relates the cosine of the scattering angle in the laboratory system to the cosine of the scattering angle in the center-of-mass system.

In order that a scattering collision be isotropic in any reference frame, it is necessary that the average value of the cosine of the scattering angle be zero. This condition is already met for the (C) system by the functional form of $\mathfrak{f}$ selected according to (4.23). To demonstrate this fact we compute the average value of η over the interval $(-1,1)$. Now, the average value of any function of η, call it $s(\eta)$, may be defined by

$$\bar{s} = \int_{-1}^{1} s(\eta)\, \mathfrak{f}(\eta)\, d\eta \tag{4.28}$$

Thus we compute $\bar{\eta}$ and find

$$\bar{\eta} = \int_{-1}^{1} \eta\, \mathfrak{f}(\eta)\, d\eta = \tfrac{1}{2} \int_{-1}^{1} \eta\, d\eta = 0 \tag{4.29}$$

as expected. The average value of $\mu_0 \equiv \mu_0(\eta)$, on the other hand, is

$$\overline{\mu_0} = \int_{-1}^{1} \mu_0(\eta)\, \mathfrak{f}(\eta)\, d\eta = \frac{1}{2} \int_{-1}^{1} \frac{1 + A\eta}{(A^2 + 2\eta A + 1)^{\frac{1}{2}}}\, d\eta = \frac{2}{3A} \tag{4.30}\star$$

where we have used (4.23) and (4.27). Thus, the average value of μ_0 is not zero, and therefore, the scattering in the laboratory system cannot be isotropic even though it is assumed to be so in the center-of-mass system. Equation (4.30) shows that the scattering in (L) tends to be "forward," that is, the neutron tends to continue moving in its original direction. This effect is greatest for the light nuclei (A small), but as A increases $\overline{\mu_0}$ decreases. Only in the limit for infinite nuclear mass does $\overline{\mu_0} \to 0$, and the scattering becomes isotropic in (L). For the heavier nuclei, however, $\overline{\mu_0}$ is very nearly zero, and it can be assumed to good approximation that the scattering in (L) is isotropic.

The assumption of isotropic scattering in the center-of-mass system gives a reasonably accurate description of the elastic-scattering process. It serves especially well in the thermal-to-Kev range of neutron energies; however, it breaks down in the thermal region where chemical binding energies become important. This assumption is also poor in the high-energy range (>Kev). The angular distribution of highly energetic neutrons scattered from stationary nuclei is not isotropic in the center-of-mass system; the distributions show peaks in both the forward and

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backward directions. A specific nonisotropic distribution of some practical importance in reactor calculations is the so-called *p scattering*. In scattering reactions of this type, the distribution may be written in the form

$$\begin{aligned} \mathfrak{f}(\eta)\, d\eta &\equiv 0 && \eta < -1 \\ &\equiv \tfrac{1}{2}(1 + a\eta)\, d\eta && -1 \leq \eta \leq 1 \\ &\equiv 0 && \eta > 1 \end{aligned} \tag{4.31}$$

where $0 \leq |a| \leq 1$ and η is the cosine of the scattering angle in (C). As compared to the frequency function for isotropic scattering (4.23), the above function is seen to have an added term $\frac{1}{2}a\eta$. This situation is described physically by saying that there is a p component in the scattering, and this component has magnitude a. Note that if $a > 0$ the scattering is predominantly forward, and if $a < 0$, predominantly backward.

The frequency function for elastic scattering from stationary nuclei has been described in terms of the scattering angles. It is more convenient for some calculations to work with a frequency function described in terms of the neutron energy. The appropriate function is obtained directly from the energy-angle relationship (4.18). If the frequency function is specified in terms of either angle or energy, the corresponding function in the other variable is readily computed by the usual transformation of variable. The procedure to be followed is demonstrated in the following calculation. We define the frequency function in energy by

$$\mathfrak{h}(E;E_0)\, dE \equiv \text{fraction}^1 \text{ of all scattering collisions which result in final kinetic energies of neutrons in range } E \text{ to } E + dE, \text{ where } \alpha E_0 \leq E \leq E_0 \tag{4.32}$$

The energy E_0 denotes the initial energy of the scattered neutrons, and α is given by (4.20). Now, to each final energy E there is an associated scattering angle $\cos^{-1}\eta$ [cf. (4.18)]. Furthermore, to each increment change in η, call it $d\eta$, there corresponds a change dE in E. Thus if we relate E to η and dE to $d\eta$, then the probability that a neutron is scattered into a final energy range in dE about E [see (4.32)] must be exactly the probability that it be scattered into $d\eta$ about η. In other words, we require that the probability that a certain event occur be the same regardless of the independent variable (E or η) used to describe it. Thus we require that

$$|\mathfrak{h}(E;E_0)\, dE| = |\mathfrak{f}(\eta)\, d\eta| \tag{4.33}\star$$

The absolute magnitude signs are introduced so as to account for the algebraic signs of dE and $d\eta$; note that a positive increment dE corre-

[1] We use the notation $\mathfrak{h}(E;E_0)$ in order to keep in mind that the initial energy of the scattered neutron was E_0.

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sponds to a negative increment $d\eta$ in η. Now, both frequency functions $\mathfrak{f}$ and $\mathfrak{h}$ are defined as positive quantities (cf. Fig. 2.4), as are the probabilities $\mathfrak{h}(E;E_0)\,dE$ and $\mathfrak{f}(\eta)\,d\eta$; therefore,

$$\mathfrak{h}(E;E_0) = \mathfrak{f}(\eta)\left|\frac{d\eta}{dE}\right| \tag{4.34}$$

The transformation is carried out by computing the Jacobian $\left|\frac{d\eta}{dE}\right|$ from (4.18).

$$\left|\frac{d\eta}{dE}\right| = \frac{(A+1)^2}{2AE_0} \tag{4.35}$$

Note that in the present calculation E, the final neutron energy, corresponds to E_1 of Eq. (4.18). If this result is used in Eq. (4.34), the frequency function in energy for the case of *isotropic scattering in the center-of-mass system* is

$$\mathfrak{h}(E;E_0)\,dE = \frac{(A+1)^2\,dE}{4AE_0} = \frac{dE}{(1-\alpha)E_0} \qquad \alpha E_0 \le E \le E_0 \tag{4.36}$$

Thus $\mathfrak{h}(E;E_0)$ does not depend upon E, so that the kinetic energy of the scattered neutron falls with equal probability into any unit energy interval in the range $\alpha E_0 \le E \le E_0$. It is of interest to compute also the distribution function for this process. If we define

$$\mathfrak{H}(E) \equiv \text{probability that final neutron energy is less than } E \tag{4.37}$$

then by Eq. (2.18)

$$\mathfrak{H}(E) = \int_{-\infty}^{E} \mathfrak{h}(\epsilon;E_0)\,d\epsilon = \int_{\alpha E_0}^{E} \frac{d\epsilon}{(1-\alpha)E_0} = \frac{E-\alpha E_0}{(1-\alpha)E_0} \tag{4.38}$$

where the lower limit of the integral is given by $E_{\min}$ according to (4.19). Incident neutrons of energy E_0 cannot emerge from an elastic-scattering collision with kinetic energies $<\alpha E_0$; thus $\mathfrak{H}(E) \equiv 0$ for $E < \alpha E_0$. The maximum energy which a scattered neutron can possess is its initial energy E_0. From (4.38) $\mathfrak{H}(E_0) = 1$, as is to be expected from the definition of a distribution function. The distribution function and frequency function in energy for isotropic scattering in (C) are shown graphically in Fig. 4.6. The analytical expressions for these functions are summarized below.

Energy range	$\mathfrak{h}(E;E_0)$	$\mathfrak{H}(E)$
$E < \alpha E_0$	0	0
$\alpha E_0 \le E \le E_0$	$\frac{1}{(1-\alpha)E_0}$	$\frac{E-\alpha E_0}{(1-\alpha)E_0}$
$E > E_0$	0	1

(4.39)

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The energy lost by the neutron is given by the difference $\Delta E = E_0 - E$. This is the energy acquired by the nucleus and dissipated as heat in the immediate vicinity of the collision as the recoiling nucleus is brought to rest. The *average loss* in energy per collision may be computed by means of the frequency function $\mathfrak{h}(E;E_0)$ [see (4.36)].

$$\overline{\Delta E} = \int_{-\infty}^{\infty} \Delta E\, \mathfrak{h}(E;E_0)\, dE = \int_{\alpha E_0}^{E_0} \left(\frac{E_0 - E}{1 - \alpha}\right) \frac{dE}{E_0} = \left(\frac{1 - \alpha}{2}\right) E_0 \quad (4.40)$$

The *average energy* of the scattered neutron after collision is

$$\bar{E} = \int_{\alpha E_0}^{E_0} E\, \mathfrak{h}(E;E_0)\, dE = \left(\frac{1 + \alpha}{2}\right) E_0 \quad (4.41)$$

Some idea of the magnitude of these changes in energy can be obtained from the following sample computation: Consider a 1-Mev neutron which

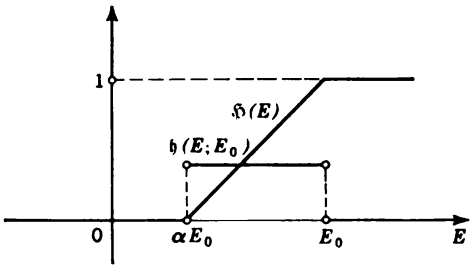

FIG. 4.6 Frequency and distribution functions for isotropic scattering in center-of-mass system.

is released in a medium of beryllium ($A = 9$). For this nucleus, $\alpha = 0.64$ and $\frac{1}{2}(1 + \alpha) = 0.82$. After the first collision the neutron energy will lie somewhere between 0.64 and 1 Mev. On the average it will be 0.82 Mev. If we assume that the neutron energy after each collision is the average value given by (4.40), then after the second collision it will be

$$\overline{E_2} = \left(\frac{1 + \alpha}{2}\right) \overline{E_1} = \left(\frac{1 + \alpha}{2}\right)^2 E_0 = 0.67 \text{ Mev}$$

and $\overline{E_3} = 0.55$, $\overline{E_4} = 0.449$, and so on. On the average, each elastic collision with Be reduces the neutron energy by 18 per cent.

e. Lethargy Variable. A quantity of greater interest in reactor analysis than the average loss in energy per collision is the average loss in the logarithm of the energy, that is, the average value of $\ln E_0 - \ln E (\equiv \xi.)$ We compute the number ξ, using the frequency function (4.39).

$$\xi \equiv \overline{\ln \frac{E_0}{E}} = \int_{\alpha E_0}^{E_0} \ln\left(\frac{E_0}{E}\right) \frac{dE}{(1 - \alpha)E_0} = 1 + \frac{\alpha \ln \alpha}{1 - \alpha} \quad (4.42)\star$$

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where α is given by (4.20). The utility of the quantity ξ rests with the fact that it is independent of the initial energy of the neutron. A graph of this function in terms of the mass number A is given in Fig. 4.7.

The average logarithmic energy loss may also be obtained by another approach. In this connection we introduce the *lethargy*, which we define in the following way:

$$u \equiv \text{lethargy of a neutron of kinetic energy } E$$
$$\equiv \ln \frac{E_\star}{E} \tag{4.43}$$

The number $E_\star$ is a reference energy (selected for the convenience of any

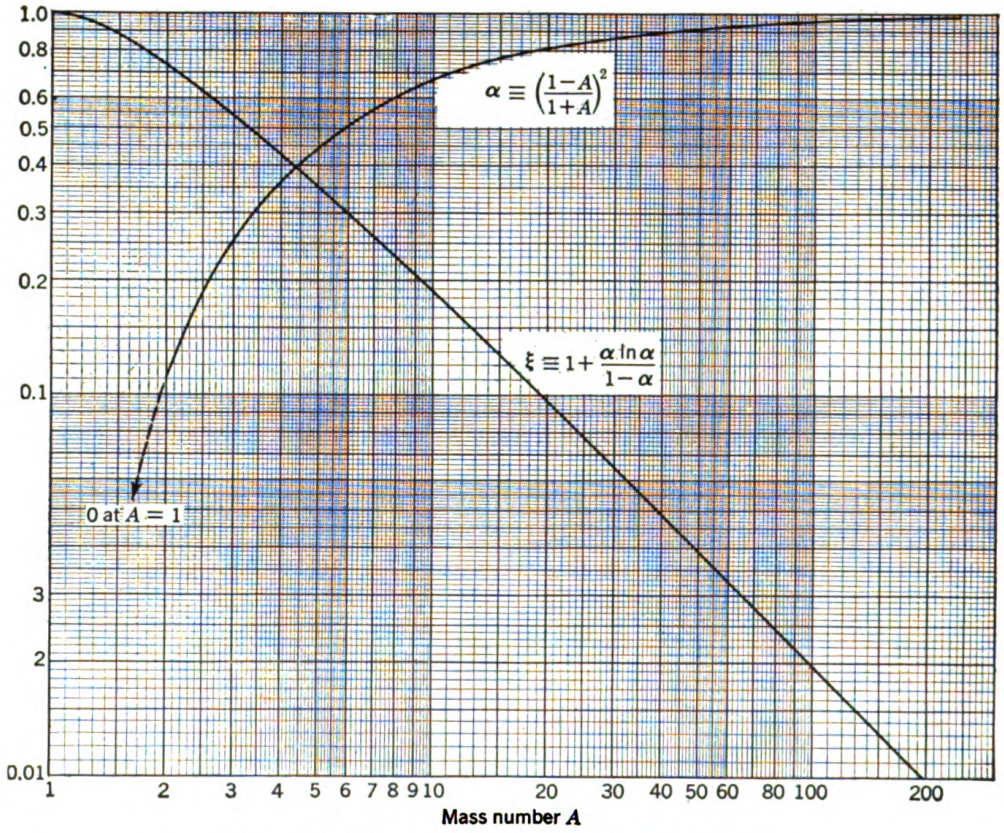

FIG. 4.7 The average logarithmic energy loss ξ.

given problem) which corresponds to the zero of the lethargy. Usually, $E_\star$ is taken to be some very high energy (say 10 Mev) such that practically all neutrons of interest have energies below $E_\star$; thus, it is not necessary to work with negative values of the lethargy. It should be noted from (4.43) that

$$du = -\frac{dE}{E} \tag{4.44}\star$$

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From the definitions of ξ and u, it is clear that ξ is the average *gain* in the lethargy of a neutron per collision. From (4.42) we have

$$\xi = \overline{\ln \frac{E_0}{E}} = \overline{\ln \frac{E_\star}{E} - \ln \frac{E_\star}{E_0}} = \overline{u - u_0} \tag{4.45}$$

where u corresponds to energy E, and u_0 to E_0.

In a typical reactor situation, neutrons appear from fission at energies which are very high compared to the thermal energy of the medium. Consequently, we can think of all fission neutrons as appearing at some energy $E_0 \gg E_{th}$ (thermal energy). The history of these neutrons may be described in the following way: (1) Neutrons appear at high energy E_0. (2) Each neutron makes a first collision with a nucleus of the medium, and this collision may be either an elastic scattering or an absorption (in this discussion, only scattering collisions are considered, since we are interested in only the energy changes of the neutron). (3) After the first collision, the neutron will have an energy E_1 somewhere between αE_0 and E_0. (4) The neutron makes a second collision, after which it will have energy E_2 between αE_1 and E_1. (5) The collisions continue, and at each collision the neutron loses some kinetic energy. (6) The neutron finally attains some low energy at which its speed is comparable to the speeds of the nuclei in thermal motion in the medium.[1]

Neglecting absorption losses and thermal motion, each neutron travels from energy E_0 down to energy zero, or from zero lethargy up to infinite lethargy. While each neutron traverses this lethargy range by a different path (i.e., with collisions at different points on the lethargy scale), it is possible, nevertheless, to compute some average quantities during the slowing-down process. For example, one quantity which is of frequent interest in reactor analysis is the number of collisions made by a neutron in traversing a lethargy interval u_1 to u_2, where $0 \leq u_1 \leq u_2 < \infty$. S. M. Dancoff[2] has shown that the average gain in lethargy $\overline{\Delta u}$ by a neutron after n collisions with nuclei of mass A is given by

$$\overline{\Delta u} = n\xi \tag{4.46}$$

with ξ defined by (4.42). This is not properly the average number of collisions to traverse an interval Δu; however, we will use the form of this result and assume that

$$\frac{u_2 - u_1}{\xi} \equiv \begin{array}{l}\text{average number of collisions made by a neutron in} \\ \text{passing from } u_1 \text{ to } u_2\end{array} \tag{4.47}$$

[1] An analysis of the behavior of the neutrons in a medium wherein the nuclei are in thermal motion is given in Sec. 4.7.

[2] One form of the distribution function for collision from which this result was obtained is given by E. U. Condon and G. Breit, *Phys. Rev.*, **49**, 229 (1936).

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We justify this seemingly crude assumption by arguing that the expression (4.47) is identical in form with the exact solution which has been obtained in the case of hydrogen ($\xi = 1$), namely,

$$u_2 - u_1 = \text{average number of collisions made by a neutron in passing from } u_1 \text{ to } u_2 \text{ in a medium of hydrogen}$$

Needless to say, similar calculations for the general case ($A > 1$) are difficult to carry out and have led to complicated expressions for which simple approximating formulas have not yet been found. However, it is reasonable to assume that, if $u_2 - u_1$ is small compared to $\ln (1/\alpha)$ (the maximum jump in lethargy per collision), Eq. (4.47) should hold, at least approximately. (Here we are considering single-element media only.) Note that this condition corresponds to the situation in hydrogen wherein $\ln (1/\alpha)$ is infinite and *any* $(u_2 - u_1)$ is small in comparison. For media in which $A > 1$, and $u_2 - u_1 \sim \ln (1/\alpha)$, Eq. (4.47) is commonly used even though it is at best only an approximation for which the limits of validity are not known. Since the results of using (4.47) over the extended lethargy range have generally been good, we shall accept the approximation in this work; thus, for all A and for any lethargy interval $u_1 \to u_2$, however large, we will take $(u_2 - u_1)/\xi$ to be the average number of collisions in traversing the interval $(u_2 - u_1)$ and recognize that the approximation is particularly good for

$$\frac{u_2 - u_1}{\ln \dfrac{1}{\alpha}} \ll 1 \tag{4.48}$$

We conclude this section with a calculation of the frequency function in terms of the lethargy variable for the case of isotropic scattering in the center-of-mass system. The computation is carried out by transforming the energy function $\mathfrak{h}(E;E_0)$ given in (4.39) to the lethargy space by means of a relation of the type (4.33). We define

$$\mathfrak{g}(u;u_0)\,du \equiv \text{probability that a neutron of lethargy } u_0 \text{ when scattered emerges with a lethargy in } du \text{ about } u \tag{4.49}$$

where $u_0 \equiv \ln \dfrac{E_\star}{E_0}$. Then

$$|\mathfrak{g}(u;u_0)\,du| = |\mathfrak{h}(E;E_0)\,dE|$$

and it is easily shown that

$$\mathfrak{g}(u;u_0) = \frac{e^{-(u-u_0)}}{1-\alpha} \tag{4.50}\star$$

The corresponding distribution function $\mathfrak{G}(u;u_0)$ is

$$\mathfrak{G}(u;u_0) = \int_{u_0}^{u} \mathfrak{g}(u';u_0)\,du' = \frac{1 - e^{-(u-u_0)}}{1-\alpha} \tag{4.51}$$

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It is important to note that this result gives the probability that a neutron scattered at u_0 emerges with lethargy $\leq u$, that is [cf. Eq. (2.8)],

$$\mathfrak{G}(u;u_0) = P(w \leq u) \quad (4.52)$$

where w is a random variable. Furthermore [see Eq. (4.39)]

$$1 - \mathfrak{G}(u;u_0) \equiv \frac{e^{-(u-u_0)} - \alpha}{1 - \alpha} \leftrightarrow \mathfrak{H}(E) \quad (4.53)\star$$

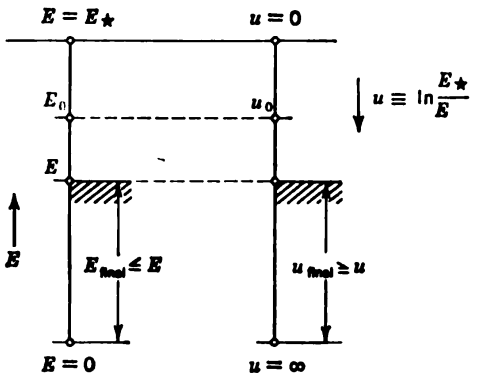

FIG. 4.8 Final lethargy and energy of scattered neutron.

This relation simply states that the probability that a neutron of lethargy u_0 (or energy E_0) emerge with lethargy $> u$ is identically the probability that the neutron emerge with energy $\leq E$. This relationship between the distribution functions in lethargy and energy is evident in Fig. 4.8.

4.2 Slowing-down Density in Pure Scatterer

a. Neutron Flux. In this section we consider the problem of finding the energy distribution of neutrons slowing down in an infinite, homogeneous, isotropic medium. We have already noted that, if the neutron sources in this medium are distributed uniformly in space and time, the neutron population may be described by means of the neutron flux or density functions given in terms of the neutron energy alone. These functions may also be described in terms of the neutron speed, or as we will shortly demonstrate, in terms of the lethargy. Which one of these independent variables, lethargy, speed, or energy, is selected for a specific calculation is entirely a matter of convenience. In any case, if the neutron population is described in terms of one of these three variables, the appropriate function in terms of the other two is easily obtained by a transformation of variable. For example, in the case of the energy and speed variables, the fluxes (or densities) satisfy a relation of the type (3.47). Thus we write

$$|\phi(E)\, dE| = |\phi(v)\, dv| \quad (4.54)$$

where $\phi(E)\, dE \equiv$ total track length of neutrons per unit volume per unit time with energies in dE about E
$\phi(v)\, dv \equiv$ total track length of neutrons per unit volume per unit time which have speeds in dv about v (4.55)

As previously mentioned, the symbol ϕ denotes neutron flux and the argument following indicates the variable in terms of which it is defined.

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Properly, the functions $\phi(E)$ and $\phi(v)$ should be given distinct symbols, but this practice has been discarded here in the interests of simplicity in notation. Equation (4.54) states that, if the intervals dE and dv correspond, then the neutron flux described by either coordinate system (E or v) must be identical.

A similar relationship may be written in terms of the lethargy. If we define

$$\phi(u)\,du = \begin{matrix}\text{total track length of neutrons per unit volume per}\\ \text{unit time which have lethargies in } du \text{ about } u\end{matrix} \tag{4.56}$$

then,

$$|\phi(u)\,du| = |\phi(E)\,dE| \tag{4.57}$$

The neutron densities may also be related by equations of this type [see Eq. (3.42)]. Throughout this work we will use all three independent variables E, u, and v freely and transfer from one to the other as the need arises. For completeness we list below the various collision densities in terms of the new variable u:

$$\left.\begin{matrix}\Sigma_s(u)\ \phi(u)\,du\\ \Sigma_a(u)\ \phi(u)\,du\\ \Sigma_f(u)\ \phi(u)\,du\end{matrix}\right\} = \text{number of neutron-nucleus} \begin{Bmatrix}\text{elastic}\\ \text{scattering,}\\ \text{absorption,}\\ \text{fission}\end{Bmatrix} \begin{matrix}\text{reactions per unit}\\ \text{volume per unit time}\\ \text{due to neutrons with}\\ \text{lethargies in } du\\ \text{about } u\end{matrix} \tag{4.58}$$

It is important to recognize that, whereas the neutron fluxes (or densities) are related by expressions of the type (4.54) and (4.57) which require a specification of the interval, the cross sections are identified directly. That is,

$$\Sigma(E) = \Sigma(v) = \Sigma(u) \tag{4.59}\star$$

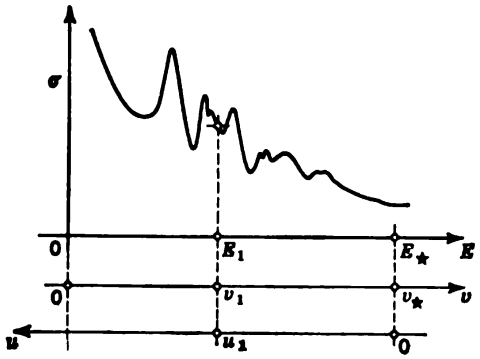

FIG. 4.9 Typical neutron cross-section curve.

The distinction arises from the definition of these two types of functions. The fluxes and densities are defined *per unit energy* (or lethargy, or speed) and therefore require that the width of the interval in question be specified. The cross sections, on the other hand, specify the interaction characteristics between nuclei and neutrons of a particular energy. Figure 4.9 shows a typical cross-section curve given as a function of neutron energy. Note that all three cross sections $\sigma(E_1)$, $\sigma(v_1)$, and $\sigma(u_1)$ refer to the same point on the curve which may be specified by either E_1, v_1, or u_1.

b. Continuous Slowing-down Model. It was remarked in Sec. 2.1a that the neutron trajectory in energy space is a discontinuous function.

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If the neutron energy were to be given as a function of its position along its path in actual space, the resulting function would be of the form shown in Fig. 4.10. Thus, the "slowing-down track" is a step function which has a unique form for a given neutron history. Three such histories are sketched in the figure. Although no two can be alike, most of the neutron trajectories for a given medium follow closely a general form. In fact, it is entirely meaningful to talk about an "average" trajectory or an average neutron slowing-down track. Whatever the form of this function, it will describe the average behavior of a neutron in the slowing-down process. As in the case of the neutron density, the specific slowing-down function for every neutron is not required (or desired). For the purposes of reactor analysis, a function which describes the average behavior is sufficient.

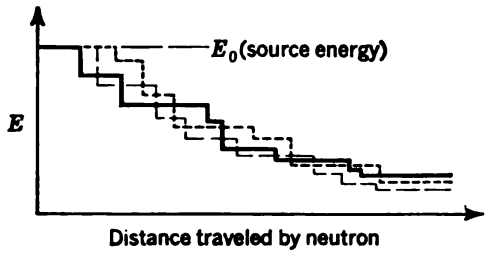

FIG. 4.10 Slowing-down track.

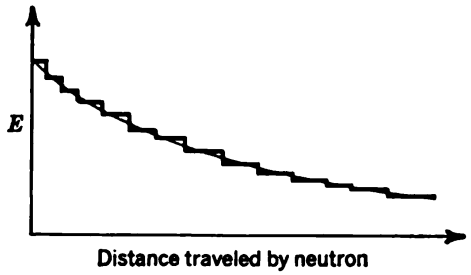

FIG. 4.11 Slowing-down track in a medium of heavy nuclear mass.

The actual discontinuous function of the neutron-energy track illustrated in Fig. 4.10 is very difficult to handle in the study of the slowing-down process. Consequently, much attention has been devoted to finding some analytically continuous function which will serve as a good average over all the possible step functions of Fig. 4.10 and thereby represent the average behavior of the neutrons. From a physical viewpoint such a continuous function is easily imagined in the case of heavy nuclei. For these nuclei the maximum jump in energy that the neutron can experience is small relative to the energy of the colliding neutron. For example, in the case of deuterium ($A = 2$), the maximum fractional jump in energy is $\frac{4}{9}$, for helium ($A = 4$), $\frac{8}{25}$, and for lithium ($A = 7$), $\frac{7}{32}$ [cf. Eq. (4.40)]. Thus the *maximum allowable* jump decreases rapidly with increasing nuclear mass. For the heavier nuclei, then, a continuous function could describe to good accuracy the slowing-down track of the neutron (cf. Fig. 4.11). In the case of the light nuclei, the concept of a continuous function is not so useful, and a more accurate description which accounts for the discrete jumps in neutron energy must be made. For the present we will confine our attention to a study of the slowing-down process which utilizes the concept of a *continuous slowing-down model.* In this model, the energy-loss process is a continuous one, with changes in neutron energy due to collisions occurring in arbitrarily small

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(differential) decrements. To describe this process we introduce the *slowing-down density* $q(E)$ and define

$$q(E) \equiv \text{slowing-down density at } E \equiv \text{number of neutrons per unit volume per unit time whose energies are changed from some value above } E \text{ to some value below } E \tag{4.60}$$

Thus this function gives the *rate* at which neutrons are slowing down *past* some given energy E. It should be observed that this concept is similar to that of a flow (current) in a fluid. One may think of the function q as giving the flow rate of neutrons down the energy scale. In the fluid analogy, q could be used to specify a continuity condition on the neutron population in terms of the factors which affect the slowing-down process. This we propose to do shortly.

The slowing-down density may also be defined in terms of the neutron speed or lethargy. As in the case of the neutron cross sections [cf. Eq. (4.59)],

$$q(E) = q(v) = q(u) \tag{4.61}$$

That is, the slowing-down rate is the same on the energy, speed, or lethargy scales if the measurement is made at corresponding values of E, v, or u.

It is important to note that the concept of a slowing-down density is *not* limited to the continuous slowing-down model. This function has a much wider application, and it will be used later in connection with the analysis of the slowing-down process which allows for the step-function behavior illustrated in Fig. 4.10 (cf. Sec. 4.2c).

We discuss now a simple model which will adequately display several basic features of the slowing-down process. Consider an infinite medium into which neutrons of energy E_0 are introduced continuously in time and uniformly in space at a rate q_0 neutrons per unit volume per unit time. Eventually a steady state is achieved in this medium so that the number of neutrons in any given energy interval ΔE in a given volume remains constant in time. Let us further idealize the medium by assuming that the nuclei are pure scatterers, i.e., $\Sigma_a(E) = 0$. Then each neutron of energy E_0 which appears in the medium moves off from its point of origin until it makes a first scattering collision. Since the distribution of sources is uniform, the distribution of first scatterings will also be uniform.

In this model the slowing-down density is different from zero for all energies $E \leq E_0$, but is identically zero for all $E > E_0$ [since by (4.39) no neutrons are speeded up by scattering collisions, and no neutrons are introduced at $E > E_0$]. All neutrons introduced into the medium at E_0 slow down to some energy below E_0; thus, in steady state

$$q(E_0) = q_0 \tag{4.62}$$

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Also, in the steady-state condition, since there are no losses by absorption or leakage in this model, the number of neutrons slowing down into a given energy interval must be equal to the number slowing down out of that interval; i.e., neutrons do not accumulate at any one energy over a period of time. Thus, for $E < E_0$,

$$q(E + dE) = q(E)$$

or

$$\frac{d}{dE} q(E) = 0 \tag{4.63}$$

and

$$q(E) = \text{constant}$$

If we apply the initial condition (4.62),

$$q(E) = q_0 \tag{4.64}$$

The slowing-down density in this model, then, is constant in energy.

One of the important problems to be considered in this chapter is the determination of the relationship between the flux and the slowing-down density for various physical situations. As an introductory treatment of this problem we carry out the following calculation, namely, compute the number of scattering collisions which occur per unit volume per unit time, using first the concept of the neutron flux and second the concept of the slowing-down density. We write the scattering-collision density in terms of the flux by means of Eq. (4.58).

$$\Sigma_s(u)\, \phi(u)\, du = \begin{matrix}\text{total number of scatterings per unit volume per unit} \\ \text{time due to neutrons whose lethargies lie in } du \text{ about } u\end{matrix}$$

This quantity may also be expressed in terms of the slowing-down density $q(u)$. According to (4.47), du/ξ is the average number of scatterings made by a neutron in traversing du. But, $q(u)$ neutrons per unit volume per unit time slow past lethargy u; therefore,

$$q(u) \frac{du}{\xi} = \begin{matrix}\text{total number of scatterings per unit volume per unit} \\ \text{time by neutrons with lethargies in } du \text{ about } u\end{matrix} \tag{4.65}$$

If these two expressions are equated, it follows that

$$q(u) = \xi \Sigma_s(u)\, \phi(u) \tag{4.66}$$

This relationship between $q(u)$ and $\phi(u)$ applies in an infinite medium of pure scatterer. It will be our purpose later to develop suitable expressions of this type which can be used in finite systems containing neutron-absorbing materials.

It is of interest to compare the expression (4.65) with the corresponding relation given in terms of the neutron energy. The symbol ξ denotes the average gain in lethargy per collision; its counterpart in energy space is $(1 - \alpha)E/2$ [see Eq. (4.40)]. Thus if dE corresponds to du, then

$$\left| q(u) \frac{du}{\xi} \right| = \left| \frac{2q(E)\, dE}{(1 - \alpha)E} \right| \tag{4.67}$$

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This statement follows directly from the physical fact that the number of collisions which occur in traversing dE must be identically the number which occur in traversing du if dE implies du. But, by (4.61), $q(u) = q(E)$, and by (4.44), $du = -dE/E$; therefore,

$$\left|\frac{q(u)\,du}{\xi}\right| = \left|\frac{-2q(u)\,du}{1-\alpha}\right|$$

and it follows that

$$\xi = \frac{1-\alpha}{2} \tag{4.68}$$

which appears to be in contradiction to the definition of ξ given by (4.42); however, this is not the case. In fact, the above result is the limiting form of ξ for values of α approaching unity, i.e., $A \to \infty$ (see Fig. 4.12). Thus the scattering-collision density may be computed by either of the two forms given in (4.67) provided that α is near unity. Note that if this condition is to be satisfied, and condition (4.48) as well, then clearly $du \to 0$; i.e., (4.67) applies if Δu is small and the nuclear mass large.

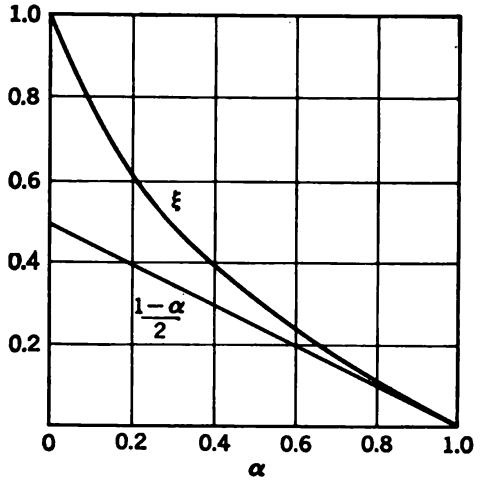

FIG. 4.12 Approximation for ξ.

c. Scattering Density by Collision Intervals. We turn our attention now to a more realistic treatment of the slowing-down process which takes into account the discontinuous form of the neutron trajectory in energy space. We carry out the analysis by considering the collisions at each stage of the slowing-down process and determine the distribution in energy of the neutrons for the physical situation in question.[1] For the present calculation we assume that the scattering collisions are isotropic in the center-of-mass system of coordinates and recognize that our results will be of questionable value, therefore, when applied to neutrons in the thermal range, or to very fast neutrons (see Sec. 4.1d).

A situation of fairly general interest in reactor analysis is one in which neutrons of some relatively high energy E_0 (which may be in the order of 1 Mev) are introduced into a uniform infinite medium and subsequently slow down by elastic collisions. Suppose, as a particular case, that the medium consists of a single nuclear species and that the following set of assumptions apply: (1) q_0 neutrons of energy E_0 appear per unit volume per unit time from uniformly distributed sources throughout the medium.

[1] From our knowledge of the scattering phenomenon as defined by the frequency function.

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(2) All nuclei in the medium are stationary. (3) The system is at steady state.

It must be observed that in this system a finite number of neutrons at any given instant have exactly energy E_0; thus the neutron density $n(E)$ has a singularity at $E = E_0$. The mathematical difficulties inherent in this single-source energy model may be overcome by regarding the function $n(E)$ as defined up to, but not including, E_0; then the condition at $E = E_0$ may be used as an initial condition (in energy) on the slowing-down density $q(E)$.

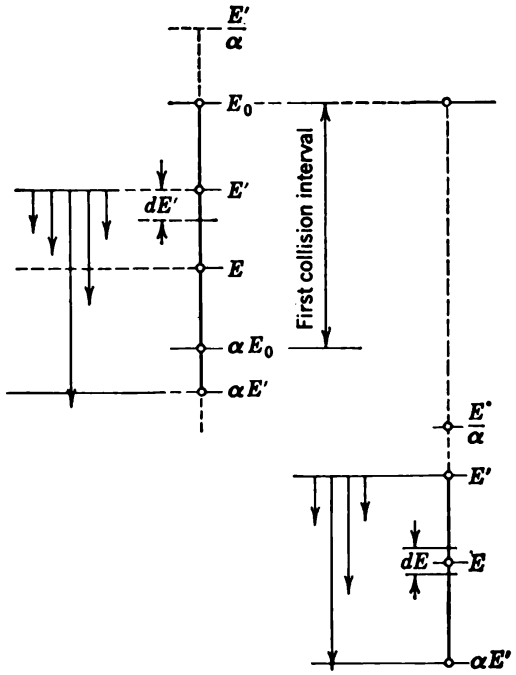

FIG. 4.13 First collision interval.

We begin the calculation by considering the distribution of neutrons whose energies fall within the first collision interval, namely, in the interval αE_0 to E_0. Our purpose is to determine the neutron balance and the slowing-down density at any energy E for $\alpha E_0 \leq E \leq E_0$. Consider first the slowing-down density $q(E)$. The rate at which neutrons are slowing down past energy E is determined by the collision density at all energies $E' \geq E$ (see Fig. 4.13). Thus we compute the number of collisions in dE' about E' which yield neutrons of energy $\leq E$. Only these neutrons can contribute directly to $q(E)$. The reasoning here is clear if it is recalled that $q(E)$ represents a "flow rate" in energy space. Thus, if a collision around E' produces a neutron of energy greater than E, this neutron will not be observed passing energy E and therefore cannot be counted in the density $q(E)$.

The number of scattering collisions per unit volume per unit time in dE' about E' is $\Sigma_s(E')\,\phi(E')\,dE'$. Of this total number a certain fraction

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will produce neutrons with energies $\leq E$. But this number is precisely the probability that a neutron which suffers a collision at energy E' will emerge with some energy $\leq E$. We obtain this from the distribution function $\mathfrak{P}(E)$ [see Eq. (4.39)]; thus the fraction of neutrons scattered to energies $\leq E$ is

$$P(\epsilon \leq E) = \mathfrak{P}(E) = \frac{E - \alpha E'}{(1 - \alpha)E'} \tag{4.69}$$

The total number of neutrons scattered from dE' which emerge with energy less than, or equal to, E is

$$\Sigma_s(E')\,\phi(E')\left[\frac{E - \alpha E'}{(1 - \alpha)E'}\right]dE' \tag{4.70}$$

In order to obtain the total contribution to $q(E)$ from all such intervals dE' we integrate the above result from E to E_0. To be correct, the integration should extend to E/α because this is the highest energy which a neutron can possess and still "reach" E by a single scattering. However, the integration need extend only to E_0, for by definition there are no neutrons in this system with energies greater than E_0.

The contributions to $q(E)$ so far considered are due to scattering of neutrons of energies $E' \geq E$ but less than E_0. As previously mentioned, the flux is not defined at $E = E_0$, and the contributions to $q(E)$ from the first collisions must be taken into account by a separate computation. For the pure scattering medium considered here, there are exactly q_0 first scatterings per unit volume per unit time. Of this total number, a fraction $(E - \alpha E_0)/(1 - \alpha)E_0$ yield neutrons of energy $\leq E$. The total number of neutrons from first collisions which can contribute to $q(E)$ is therefore

$$\frac{q_0(E - \alpha E_0)}{(1 - \alpha)E_0} \tag{4.71}$$

We obtain the total contribution to $q(E)$ by performing the integration described above and adding the contribution from the first scatterings; thus, for the first collision interval

$$q(E) = \int_E^{E_0} \Sigma_s(E')\,\phi(E')\left[\frac{E - \alpha E'}{(1 - \alpha)E'}\right]dE' + \frac{q_0(E - \alpha E_0)}{E_0(1 - \alpha)} \tag{4.72}$$

This expression gives the slowing-down density $q(E)$ for all energies in the interval $\alpha E_0 \leq E \leq E_0$, that is, (4.72) applies for all $E > \alpha E_0$. When $E \leq \alpha E_0$, the source cannot contribute directly to $q(E)$ and the q_0 term is omitted; thus,

$$q(E) = \int_E^{E/\alpha} \Sigma_s(E')\,\phi(E')\left[\frac{E - \alpha E'}{(1 - \alpha)E'}\right]dE' \qquad \text{for } E \leq \alpha E_0 \tag{4.73}$$

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Note that the integration now extends to E/α due to the fact that $E/\alpha < E_0$, and the flux is therefore defined over the entire interval.

The above results involve two unknowns $\phi(E)$ and $q(E)$. In order to obtain a complete definition of the neutron distribution in energy, we require a second relation. This can be obtained from the neutron-balance condition. Consider, therefore, an energy interval dE about E (see Fig. 4.13). The number of neutrons disappearing from this interval (due to scatterings which change the neutron energy to a value which falls outside the interval) must be balanced by the number scattered into dE from energies above E, in the steady-state condition. The rate at which neutrons are removed from dE is $\Sigma_s(E)\ \phi(E)\ dE$. The rate at which neutrons enter dE is given by the number of collisions which occur at energies above E times the probability that these scatterings "throw" the neutron into the interval dE. We compute this number by considering some interval dE' about E', where $E' \geq E$. As before, the scattering rate in dE' is $\Sigma_s(E')\ \phi(E')\ dE'$. The fraction of these collisions which yield neutrons that have energy in dE about E is just the frequency function $\mathfrak{h}(E;E')$ [see Eq. (4.39)]; this follows from the definition (4.32). If $E \leq \alpha E_0$, the neutron source cannot contribute directly to the neutron population in dE; thus, the neutron balance in dE about E is given by

$$\Sigma_s(E)\ \phi(E)\ dE = \int_E^{E/\alpha} \Sigma_s(E')\ \phi(E') \left[\frac{dE}{(1-\alpha)E'}\right] dE' \qquad \text{for } E \leq \alpha E_0 \tag{4.74}$$

On the other hand, if $E > \alpha E_0$, then the contributions from the source at E_0 must be included, and the neutron balance is

$$\Sigma_s(E)\ \phi(E)\ dE = \int_E^{E_0} \Sigma_s(E')\ \phi(E') \left[\frac{dE}{(1-\alpha)E'}\right] dE' + \frac{q_0\, dE}{(1-\alpha)E_0} \tag{4.75}$$

The results (4.74) and (4.75) define the neutron flux $\phi(E)$ for all values of $E < E_0$. The solution of these integral equations when substituted into the expressions (4.72) and (4.73) yields the appropriate forms for $q(E)$ over the entire range $0 \leq E \leq E_0$.

We demonstrate the procedure for solving these equations by carrying out the calculation for the first two collision intervals, namely, $\alpha E_0 \to E_0$ and $\alpha^2 E_0 \to \alpha E_0$. For convenience call the scattering density $\Sigma_s\phi$ in the first interval $F_1(E) \equiv \Sigma_s(E)\ \phi_1(E)$. Then (4.75) may be written

$$F_1(E) = \int_E^{E_0} \frac{F_1(E')\ dE'}{(1-\alpha)E'} + \frac{q_0}{(1-\alpha)E_0} \tag{4.76}$$

and

$$\frac{d}{dE} F_1(E) = -\frac{F_1(E)}{(1-\alpha)E}$$

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This is a first-order differential equation in $F_1(E)$; its solution is

$$F_1(E) = F_1(E_0)\left(\frac{E_0}{E}\right)^{1/(1-\alpha)} \tag{4.77}$$

We obtain $F_1(E_0)$ by taking the limit of (4.76) as $E \to E_0$

$$\lim_{E \to E_0} F_1(E) = \frac{q_0}{(1-\alpha)E_0} \equiv F_1(E_0)$$

Substitution into (4.77) yields the scattering density in the first collision interval ($\alpha E_0 \leq E \leq E_0$)

$$F_1(E) \equiv \Sigma_s(E)\,\phi_1(E) = \frac{q_0}{(1-\alpha)E_0}\left(\frac{E_0}{E}\right)^{1/(1-\alpha)} \tag{4.78}\star$$

For the second collision interval we use (4.74) and write

$$F_2(E) \equiv \Sigma_s(E)\,\phi_2(E) = \int_E^{\alpha E_0} \frac{F_2(E')\,dE'}{(1-\alpha)E'} + \int_{\alpha E_0}^{E/\alpha} \frac{F_1(E')\,dE'}{(1-\alpha)E'} \tag{4.79}$$

Note that the integration over the interval E to E/α has been separated into two parts so as to accommodate the change in the functional form of $\phi(E)$ as the integration proceeds from the second collision interval to the first (see Fig. 4.14). As in the case of the first interval above, we transform this integral equation into a differential equation; the result is

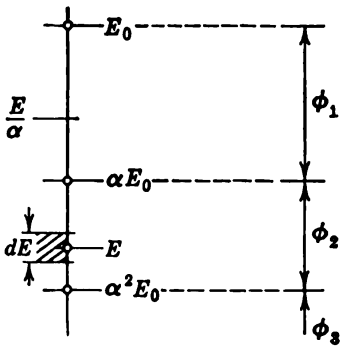

FIG. 4.14 Second collision interval.

$$\frac{d}{dE}F_2(E) = -\frac{F_2(E)}{(1-\alpha)E} + \frac{q_0 E_0^{\alpha/(1-\alpha)}\alpha^{1/(1-\alpha)}}{(1-\alpha)^2(E)^{1/(1-\alpha)+1}} \tag{4.80}$$

where we have used the solution (4.78). The solution to (4.80) is

$$F_2(E) = \Sigma_s(E)\,\phi_2(E) = \frac{q_0 E_0^{\alpha/(1-\alpha)}}{(1-\alpha)E^{1/(1-\alpha)}}\left\{\frac{\alpha^{1/(1-\alpha)}}{1-\alpha}\ln\left(\frac{E}{\alpha E_0}\right) + [1-\alpha^{1/(1-\alpha)}]\right\} \tag{4.81}\star$$

We have used here the limiting value

$$\lim_{E \to \alpha E_0} F_2(E) = \frac{q_0[1-\alpha^{1/(1-\alpha)}]}{(1-\alpha)E_0\alpha^{1/(1-\alpha)}}$$

This computational procedure may be continued indefinitely to obtain the collision densities $F_3(E)$, $F_4(E)$, . . . and the corresponding fluxes $\phi_3(E)$, $\phi_4(E)$, ... Although the calculation is somewhat laborious, the method is straightforward.

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d. Asymptotic Solution. It is of interest to compare the collision densities F_1 and F_2 at $E = \alpha E_0$. The appropriate expressions are

$$F_1(\alpha E_0) = \frac{q_0}{(1-\alpha)E_0\alpha^{1/(1-\alpha)}} \qquad F_2(\alpha E_0) = \frac{q_0[1-\alpha^{1/(1-\alpha)}]}{(1-\alpha)E_0\alpha^{1/(1-\alpha)}}$$

and a comparison of these values shows that

$$F_2(\alpha E_0) = F_1(\alpha E_0)[1-\alpha^{1/(1-\alpha)}]$$

Evidently the collision density $F(E)$ is discontinuous at $E = \alpha E_0$; therefore, the flux $\phi(E)$ must also be discontinuous at αE_0. If the procedure mentioned above is extended to the third collision interval and beyond, one finds that $F(E)$ is continuous at $E = \alpha^2 E_0$, but $F'(\alpha^2 E_0)$ is discontinuous. At $E = \alpha^3 E_0$, both $F(\alpha^3 E_0)$ and $F'(\alpha^3 E_0)$ are continuous, but $F''(\alpha^3 E_0)$ is discontinuous, and so on; at energy $\alpha^n E_0$, the discontinuity first appears in the $(n-1)$st derivative of $F(E)$. Thus as the neutrons progress down the energy scale, the collision densities of any two adjoining collision intervals, say $F_n(E)$ and $F_{n-1}(E)$, which describe the energy distribution of the neutrons, approach the same analytical form, and dissimilarities between the two functions recede to higher and higher derivatives. As the ratio $E/E_0 \to 0$, $F(E)$ approaches a limiting form. This limit can be obtained from the solution of the nth interval. The collision density for this interval is

$$F_n(E) = \int_E^{\alpha^{n-1}E_0} \frac{F_n(E')\,dE'}{(1-\alpha)E'} + \int_{\alpha^{n-1}E_0}^{E/\alpha} \frac{F_{n-1}(E')\,dE'}{(1-\alpha)E'} \tag{4.82}$$

But, as we have already observed, for large values of n, $F_n(E) \simeq F_{n-1}(E)$; thus approximately

$$F_n(E) \simeq \int_E^{E/\alpha} \frac{F_n(E')\,dE'}{(1-\alpha)E'} \tag{4.83}$$

A solution to this equation is

$$F_n(E) = \frac{C}{E} \equiv F(E) \tag{4.84}$$

where C is some constant. The symbol $F(E) \equiv \Sigma_s(E)\,\phi(E)$ denotes the limiting form of the collision density. The constant C may be obtained for the prescribed condition on the slowing-down density $q(E)$. The appropriate equation for $q(E)$ for neutron energies far removed from the source energy E_0 is given in (4.73). If we use (4.84) for the collision density in this equation, then

$$q(E) = \int_E^{E/\alpha} \frac{F(E')(E-\alpha E')\,dE'}{(1-\alpha)E'} = C\int_E^{E/\alpha} \frac{(E-\alpha E')\,dE'}{(1-\alpha)(E')^2} = q_0 \tag{4.85}$$

The last equality follows from the fact that no neutrons can be lost in a medium of pure scattering material, and the number of neutrons slowing

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down past energy E must be the same for all E, namely, q_0, the source value. The solution of (4.85) for the constant yields $C = q_0/\xi$; thus,

$$F(E) = \Sigma_s(E)\,\phi(E) = \frac{q_0}{\xi E} \tag{4.86}$$

or, if we use $q(E) = q_0$,

$$q(E) = \xi\Sigma_s(E)\,\phi(E)E \tag{4.87}\star$$

The asymptotic solution (4.87) is a very good approximation for all energies below $E = \alpha^2 E_0$, and for many purposes it will be acceptable

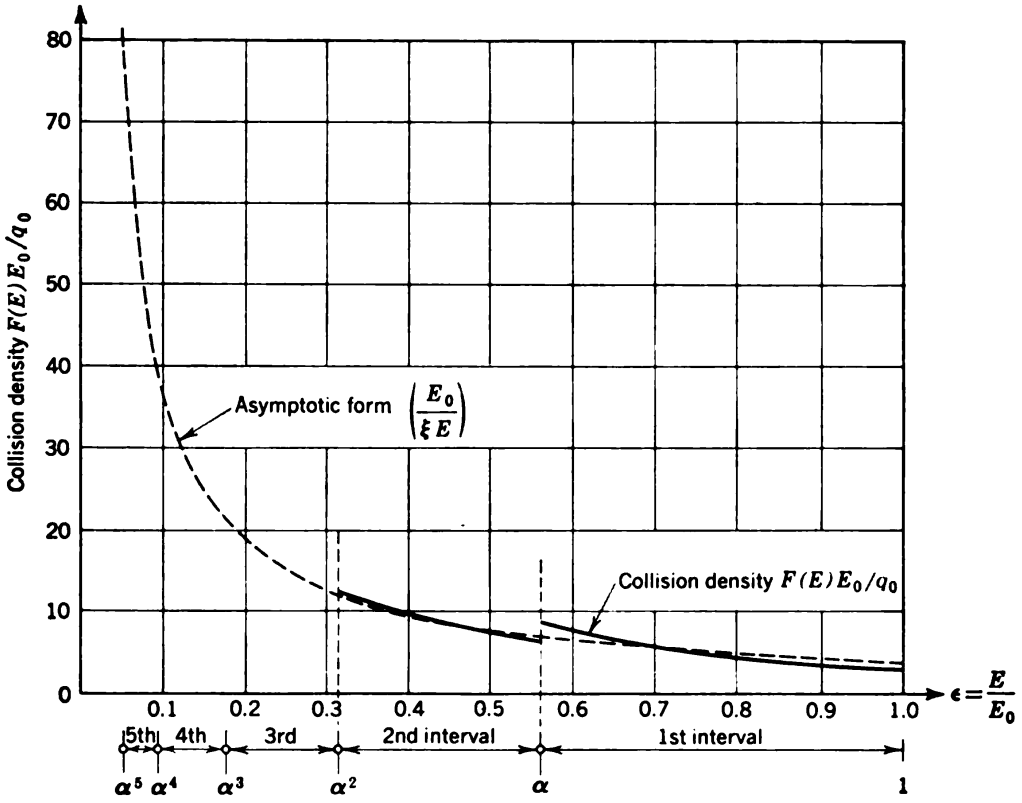

FIG. 4.15 Collision density in pure scattering medium of mass $A = 7$. (E_0 = energy of source neutrons.)

up to E_0. Hydrogen is a special case; since $\alpha = 0$ for hydrogen, the entire energy range lies between E_0 and αE_0. However, for deuterium $\alpha = \frac{1}{9}$, and if E_0 is chosen as 2 Mev (a representative energy for neutrons from fission), $\alpha^2 E_0 \simeq 25{,}000$ ev. Thus the most important part of the slowing-down range even in thermal reactors with deuterium lies in the region in which (4.87) is a good approximation. For other substances, the situation is even more favorable.

Figure 4.15 shows the collision density $F(E)\,E_0/q_0$ in a medium of pure scattering material of mass $A = 7$. This function is plotted in terms

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of the energy ratio $\epsilon \equiv E/E_0$, where E_0 is the energy of the source neutrons. The discontinuous form of the collision density is clearly displayed in the first two collision intervals. It is of interest to note that, in the third interval and beyond, $F(E)$ "smooths out" and rapidly approaches the asymptotic solution $E_0/E\xi$ (given by the broken-line curve). In the preceding discussion of the asymptotic solution, it was remarked that this form could be used as an approximate description of $F(E)$ even in the first and second intervals, especially for the heavier nuclei. The relatively good match of the asymptotic form to the actual function $F(E)$ in this energy range is well demonstrated in Fig. 4.15.

We close this section with a brief discussion of the solutions for the slowing-down density and the collision density in terms of the lethargy variable u. The set of equations which correspond to the results (4.72) through (4.75) are readily obtained by introducing the distribution function $\mathfrak{G}(u;u')$ and the frequency function $\mathfrak{g}(u;u')$ from Eqs. (4.50) and (4.51) and making a suitable change of variable from E to u. It can be shown that the slowing-down density $q(u)$ is given by [see also (4.53)]

$$q(u) = \frac{1}{1-\alpha}\int_{u_0}^{u} \Sigma_s(u')\,\phi(u')(e^{u'-u} - \alpha)\,du' + \frac{q_0(e^{u_0-u} - \alpha)}{1-\alpha} \qquad u \leq u_0 + \ln\frac{1}{\alpha}$$

$$= \frac{1}{1-\alpha}\int_{u-\ln\frac{1}{\alpha}}^{u} \Sigma_s(u')\,\phi(u')(e^{u'-u} - \alpha)\,du' \qquad u > u_0 + \ln\frac{1}{\alpha} \tag{4.88}$$

where u_0 corresponds to the energy E_0. As a general rule when dealing with a source of monoenergetic neutrons, it is convenient to define u in terms of the source energy. Thus, if we take $u \equiv \ln E_0/E$, then $u_0 \equiv 0$. The collision density $F(u)$ when written in this notation takes the form

$$F(u) = \frac{1}{1-\alpha}\int_0^u F(u')\,e^{u'-u}\,du' + \frac{q_0 e^{-u}}{1-\alpha} \qquad u \leq \ln\frac{1}{\alpha}$$

$$= \frac{1}{1-\alpha}\int_{u-\ln\frac{1}{\alpha}}^{u} F(u')\,e^{u'-u}\,du' \qquad u > \ln\frac{1}{\alpha} \tag{4.89}$$

These integral equations (4.88) and (4.89) for $q(u)$ and $F(u)$ may be solved by the same procedure as used in the case of the energy variable. It is easy to show, for example, that the collision density in the first interval $F_1(u)$ is given by

$$F_1(u) = \frac{q_0 e^{\alpha u/(1-\alpha)}}{1-\alpha} \tag{4.90}$$

and so on for the other intervals. It is of interest to determine the

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asymptotic solution for $F(u)$ in the lethargy range far removed from the source at $u = 0$. In this range we use the form

$$F_n(u) = \frac{1}{1-\alpha}\int_{u-\ln\frac{1}{\alpha}}^{(n-1)\ln\frac{1}{\alpha}} F_{n-1}(u')\, e^{u'-u}\, du' + \frac{1}{1-\alpha}\int_{(n-1)\ln\frac{1}{\alpha}}^{u} F_n(u')\, e^{u'-u}\, du' \simeq \frac{1}{1-\alpha}\int_{u-\ln\frac{1}{\alpha}}^{u} F_n(u')\, e^{u'-u}\, du' \tag{4.91}$$

in the nth collision interval. The approximate form used at the right follows from the same arguments as used in the calculation of $F_n(E)$. The solution to Eq. (4.91) is

$$F_n(u) = \frac{q_0}{\xi} \tag{4.92}$$

as can be verified by direct substitution. Thus in the *asymptotic range*, the slowing-down density $q(u)$ and the flux $\phi(u)$ are related by

$$q(u) = \xi\Sigma_s(u)\,\phi(u) \tag{4.93}$$

This result states that, if the collision density $F(E)$ in a medium varies like $1/E$, then $F(u)$ is a constant [cf. Eqs. (4.86) and (4.92)].

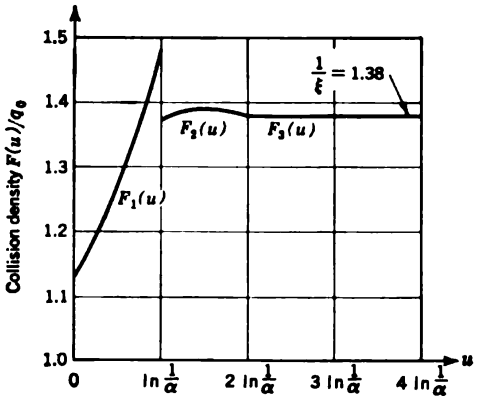

FIG. 4.16 Collision density $F(u)$ in pure scattering medium of mass $A = 2$.

Figure 4.16 shows $F(u)/q_0$ in a medium of pure scattering material of mass $A = 2$. It should be noted that, although there are relatively wide variations in the function $F(u)$ through the first two collision intervals, the function rapidly "smooths out" and approaches the asymptotic solution $1/\xi$.

4.3 Slowing Down in Hydrogen

a. Collision Density. In the preceding section we examined the slowing-down process in an infinite medium of pure scattering material. In all problems of practical interest to reactor physics, however, the neutrons slow down in media which contain some neutron-absorbing materials. The amount of absorber present depends, of course, on the reactor region in question. The reflector, for example, usually consists of materials of relatively small absorption cross section. In such regions, the results based on the model of a pure scattering medium would give a first approximation to the actual conditions. Even here, however, the pure scattering model is of questionable value. In the reactor core, the situation is still more unfavorable for the use of this model because of the presence of large amounts of fuel materials which have large absorption cross sections.

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A typical history of a neutron slowing down in a medium which contains both absorbing and scattering materials was described in Sec. 2.1a. An important feature of this process was that at every collision between a neutron and a nucleus there was a finite probability that the neutron could be absorbed. The probability that a neutron is absorbed upon making a collision is $\Sigma_a(E)/\Sigma_t(E)$, and that it be scattered, $\Sigma_s(E)/\Sigma_t(E)$. Thus at each stage in the slowing-down process, the neutron can be removed from the slowing-down "stream." The physical consequence of this fact is that the slowing-down density $q(E)$ will *not* be constant in energy. The process is further complicated if the medium contains fissionable material. In this case when a neutron is removed from the slowing-down stream because of an absorption collision, other neutrons may appear from fission sources. Thus in developing the neutron-balance relations for the slowing-down process in an absorbing medium, additional terms must be introduced which take into account the absorption losses and fission sources [see Eqs. (4.74) and (4.75)].

As a first step in deriving the expressions for the neutron density in an absorbing medium we consider the relatively simple case of an infinite medium of pure hydrogen, for which a rigorous solution is possible. The model for this calculation is defined by the following set of assumptions: (1) infinite medium of pure hydrogen ($A = 1$); (2) $\Sigma_a(E) \neq 0$; (3) q_0 neutrons of energy E_0 appear per unit volume per unit time throughout all points of the medium; and (4) the hydrogen nuclei are stationary. We begin the analysis with the calculation of the total collision density $F(E)$ defined by

$$F(E) \equiv \Sigma_t(E)\ \phi(E) \equiv [\Sigma_a(E) + \Sigma_s(E)]\ \phi(E)$$

The appropriate neutron-balance condition for a medium of hydrogen is seen to be [cf. Eq. (4.75)]

$$\Sigma_t(E)\ \phi(E) = \int_E^{E_0} \Sigma_s(E')\ \phi(E')\ \frac{dE'}{E'} + \frac{q_0}{E_0} \qquad E < E_0 \qquad (4.94)$$

The term at the left gives the rate at which neutrons are removed, by both absorption and scattering collisions, from the unit energy interval around E. The integral, as in Eq. (4.75), gives the rate at which neutrons enter unit energy interval about E by being scattered down from energies above E. Note that the frequency function for scattering in hydrogen takes the simple form $1/E'$ if E' is the neutron energy before collision. Because $\alpha = 0$ for hydrogen, a neutron can be scattered from E_0 to any energy $0 \leq E \leq E_0$. Consequently, the neutron source can contribute directly to the neutron population at *every* energy from E_0 down to zero. Thus the term q_0/E_0 appears in the balance relation for all $E \leq E_0$. This form for the source term is only an approximation, however. Actually, all neutrons that appear from the source do not

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enter into the slowing-down process; some source neutrons are absorbed at the first collision, and therefore the q_0 factor should be corrected to account for these losses. It would be better, then, to use the form

$$\begin{pmatrix}\text{Number of neutrons}\\ \text{slowing down from}\\ E_0 \text{ per unit volume}\\ \text{per unit time}\end{pmatrix} = \lim_{E\to E_0} q(E) = q(E_0) = \left[\frac{\Sigma_s(E_0)}{\Sigma_a(E_0) + \Sigma_s(E_0)}\right] q_0 \tag{4.95}$$

But in many cases, $\Sigma_a(E_0)/\Sigma_s(E_0) \ll 1$, and

$$q(E_0) \simeq q_0$$

will often be taken as an approximation.

Equation (4.94) may be solved by reducing it to a differential equation. Thus, if we differentiate (4.94), we obtain

$$\frac{d}{dE} F(E) = -\frac{F(E)\,\Sigma_s(E)}{\Sigma_t(E)E}$$

The solution to this differential equation is

$$F(E) = F(E_0) \exp\left[\int_E^{E_0} \frac{\Sigma_s(E')\,dE'}{\Sigma_t(E')E'}\right] = F(E_0)\,\frac{E_0}{E} \exp\left[-\int_E^{E_0} \frac{\Sigma_a(E')\,dE'}{\Sigma_t(E')E'}\right]$$

But from (4.94), $F(E_0) = q_0/E_0$; therefore,

$$F(E) = \frac{q_0}{E} \exp\left[-\int_E^{E_0} \frac{\Sigma_a(E')\,dE'}{\Sigma_t(E')E'}\right] \tag{4.96}$$

b. Nonabsorption Probability for Hydrogen. The slowing-down density $q(E)$ may be computed using the result (4.96). The appropriate integral equation is [see Eq. (4.72)]

$$q(E) = E\int_E^{E_0} \Sigma_s(E')\,\phi(E')\,\frac{dE'}{E'} + \frac{q_0 E}{E_0} \equiv EF(E) \tag{4.97}$$

The identity follows from (4.94). This result may be written in terms of $F(E)$ given in (4.96); thus

$$q_H(E) = q_0 \exp\left[-\int_E^{E_0} \frac{\Sigma_a(E')\,dE'}{\Sigma_t(E')E'}\right] \equiv q_0 p_H(E;E_0) \tag{4.98}\star$$

where the subscript H has been added to denote hydrogen. A verbal statement of this result might be expressed as follows: The number of neutrons which succeed in slowing down from E_0 as far as E is given by the number of neutrons starting at E_0 (namely, q_0) multiplied by the probability that they are *not* absorbed between E and E_0. Thus the exponential in (4.98), which we denote by $p_H(E;E_0)$, is identified as the *nonabsorption probability:*

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$$p_H(E;E_0) \equiv \exp\left[-\int_E^{E_0} \frac{\Sigma_a(E')\,dE'}{\Sigma_t(E')E'}\right] \qquad (4.99)\star$$

$\equiv$ probability that a neutron will slow down from E_0 to E without being absorbed (in a medium of pure hydrogen)

This function may also be written in terms of the lethargy. If we define $u \equiv \ln E_0/E$, then it follows from (4.99) that

$$p_H(u;0) \equiv \exp\left[-\int_0^u \frac{\Sigma_a(u')\,du'}{\Sigma_t(u')}\right] \qquad (4.100)$$

which gives the probability that a neutron of lethargy zero released in a medium of pure hydrogen slows down to lethargy u without being absorbed.

It can be demonstrated by means of a crude probability argument that these results (4.99) and (4.100) are indeed the nonabsorption probabilities defined above. For convenience, we will carry out the calculation in terms of the lethargy variable. Suppose that the interval 0 to u (which corresponds to the interval E_0 to E) is divided into N subintervals Δu_i which are sufficiently small so that the cross sections may be given one particular value $[\Sigma_a(u_i),\Sigma_s(u_i), \ldots]$ over each subinterval (see Fig. 4.17). Then let p_i denote the probability that a neutron will pass through Δu_i without being absorbed. But, each p_i must be an independent probability;[1] therefore,

$$p_H(u;0) = \prod_{i=1}^{N} p_i \qquad (4.101)$$

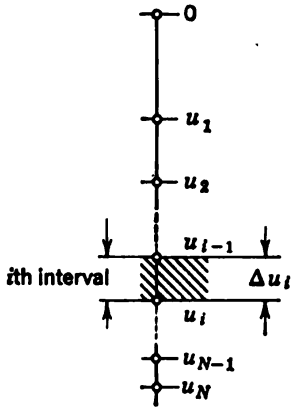

FIG. 4.17 Lethargy subintervals.

If we call p_i^a the probability that a neutron *is* absorbed in traversing Δu_i, then by definition

$$p_i = 1 - p_i^a \qquad (4.102)$$

The absorption probability p_i^a may be given as the product of the probability that the neutron will make a collision in Δu_i and the probability that this one collision will be an absorption. Now it was shown in Sec. 4.1e that Δu is the average number of collisions made by a neutron with hydrogen in traversing the interval Δu. Thus if the subintervals Δu_i are very small, then the probability of more than one collision in each interval becomes negligibly small. In the limit as the $\Delta u_i \to 0$, Δu_i gives the probability that the neutron will make a collision in traversing this subinterval. The probability that a particular collision at u_i is an absorption is simply $\Sigma_a(u_i)/\Sigma_t(u_i)$; therefore, $p_i^a = \Delta u_i \Sigma_a(u_i)/\Sigma_t(u_i)$, and

[1] The probability of collision in any subinterval Δu_i cannot depend upon the past history of the neutron.

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$$p_H(u;0) = \prod_{i=1}^{N} \left[1 - \frac{\Delta u_i \Sigma_a(u_i)}{\Sigma_t(u_i)} \right] \tag{4.103}$$

We compute the logarithm of $p_H(u,0)$ and note that for small Δu_i

$$\ln p_H(u;0) = \ln \prod_{i=1}^{N} \left[1 - \frac{\Delta u_i \Sigma_a(u_i)}{\Sigma_t(u_i)} \right] \equiv \sum_{i=1}^{N} \ln \left[1 - \frac{\Delta u_i \Sigma_a(u_i)}{\Sigma_t(u_i)} \right]$$

$$\simeq - \sum_{i=1}^{N} \frac{\Delta u_i \Sigma_a(u_i)}{\Sigma_t(u_i)}$$

If we pass to the limit $\Delta u_i \to 0$, then the summation may be replaced by an integral over the variable u, and we obtain

$$\ln p_H(u;0) = - \int_0^u \frac{\Sigma_a(u')\, du'}{\Sigma_t(u')}$$

which is in agreement with Eq. (4.100).

The functions $p_H(E;E_0)$ and $p_H(u;0)$ are defined for pure hydrogen, and these are special cases of a more general pair of functions $p(E;E_0)$ and $p(u;0)$ called the *resonance escape probabilities*. These functions may be applied to more general media; an approximate treatment is discussed in the next section.

4.4 Slowing-down Density with Absorption

a. Integral Equations. We turn now to a treatment of the more general problem of slowing down in a medium which contains absorbing nuclei. For the present we consider a medium of a single nuclear species and leave for a later analysis the problem of slowing down in a mixture of nuclear materials. In this more general treatment both α and $\Sigma_a(E)$ are different from zero. As before, we assume that the nuclei are stationary and that the neutron source (q_0 neutrons per unit volume per unit time) is monoenergetic (source neutrons have lethargy u_0).

The integral equation for the slowing-down density is identical in form to the relation (4.88). If we take for convenience, $u_0 = 0$ then

$$q(u) = \frac{1}{1-\alpha} \int_0^u \Sigma_s(u')\, \phi(u')(e^{u'-u} - \alpha)\, du' + \frac{q_0(e^{-u} - \alpha)}{1-\alpha} \qquad u \le \ln \frac{1}{\alpha} \quad (a)$$

$$= \frac{1}{1-\alpha} \int_{u - \ln \frac{1}{\alpha}}^{u} \Sigma_s(u')\, \phi(u')(e^{u'-u} - \alpha)\, du' \qquad u > \ln \frac{1}{\alpha} \quad (b) \tag{4.104}$$

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The collision density in an absorbing medium $F(u)$ differs from the expression (4.89) by the addition of a term $\Sigma_a(u)\ \phi(u)$ to the left-hand side. Thus,

$$F(u) \equiv \Sigma_t(u)\ \phi(u) = \frac{1}{1-\alpha}\int_0^u \Sigma_s(u')\ \phi(u')\ e^{u'-u}\,du' + \frac{q_0 e^{-u}}{1-\alpha} \qquad u \leq \ln\frac{1}{\alpha} \quad (a)$$

$$= \frac{1}{1-\alpha}\int_{u-\ln\frac{1}{\alpha}}^u \Sigma_s(u')\ \phi(u')\ e^{u'-u}\,du' \qquad u > \ln\frac{1}{\alpha} \quad (b) \tag{4.105}$$

The term at the left gives the rate at which neutrons are removed from the unit lethargy interval about u by both scatterings and absorptions. The terms at the right give the rate at which neutrons are added to this interval by scattering in from above.

b. Solution in Asymptotic Range. A general solution of Eqs. (4.104) and (4.105) has not been found; however, an approximate solution may be obtained which is valid for *small absorption*. This approximation is based on the assumption that the asymptotic solution applies for all values of $u > 0$. We select then Eq. (4.104*b*) for the slowing-down density. This is one equation in the two unknowns $q(u)$ and $\phi(u)$. As a second relation we introduce

$$\frac{d}{du}\,q(u) = -\Sigma_a(u)\ \phi(u) \tag{4.106}$$

with the initial condition $q(0) = q_0$. This relation is entirely satisfactory from physical requirements inasmuch as the only mechanism for the loss of neutrons from the slowing-down stream in an infinite medium is the absorption collision. However, even with this simplification the set of equations (4.104*b*) and (4.106) cannot be solved rigorously. Therefore we introduce the next approximation, which is to regard the absorption as small so that we have nearly the solution for the pure scattering case. Under these restrictions there can be no rapid changes in $\phi(u)$; such changes could arise only from the high rate of neutron removals due to the presence of strong absorbers. Thus we assume that the scattering density $\Sigma_s(u)\ \phi(u)$ changes only slightly over each collision interval $u - \ln\frac{1}{\alpha}$ to u. By making this assumption it becomes possible to evaluate the integral of Eq. (4.104*b*). Accordingly we expand $\Sigma_s(u')\ \phi(u')$ in a Taylor series about the point u and use the first two terms; then

$$\Sigma_s(u')\ \phi(u') \simeq \Sigma_s(u)\ \phi(u) + (u'-u)\frac{d}{du}\left[\Sigma_s(u)\ \phi(u)\right] \tag{4.107}$$

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The substitution of this expression into the integral of (4.104*b*) yields, upon integration,

$$q(u) \simeq \xi\Sigma_s(u)\,\phi(u) - \gamma\xi\frac{d}{du}[\Sigma_s(u)\,\phi(u)] \tag{4.108}$$

where

$$\gamma \equiv 1 - \frac{\alpha \ln^2 \alpha}{2\xi(1-\alpha)} \tag{4.109}\star$$

As a first approximation to (4.108)

$$q(u) \simeq \xi\Sigma_s(u)\,\phi(u) \qquad \text{and} \qquad \frac{d}{du}q(u) \simeq \xi\frac{d}{du}[\Sigma_s(u)\,\phi(u)]$$

We use this estimate of the derivative $[\Sigma_s(u)\,\phi(u)]'$ in (4.108); thus,

$$q(u) \simeq \xi\Sigma_s(u)\,\phi(u) - \gamma\frac{d}{du}q(u)$$

This equation may be solved by using the relation (4.106) for $q'(u)$. The result is

$$q(u) = [\xi\Sigma_s(u) + \gamma\Sigma_a(u)]\,\phi(u) \tag{4.110}$$

which relates the slowing-down density $q(u)$ to the flux $\phi(u)$ in a medium of light absorption. One of these functions can be eliminated by using this relation in (4.106). If we solve for $q(u)$, we obtain the expression

$$q(u) = q(0)\exp\left[-\int_0^u \frac{\Sigma_a(u')\,du'}{\xi\Sigma_s(u') + \gamma\Sigma_a(u')}\right] \tag{4.111}$$

It should be noted that as $\alpha \to 0$ both ξ and γ approach unity, and this approximate expression for the slowing-down density approaches the result for pure hydrogen [see Eq. (4.98)].

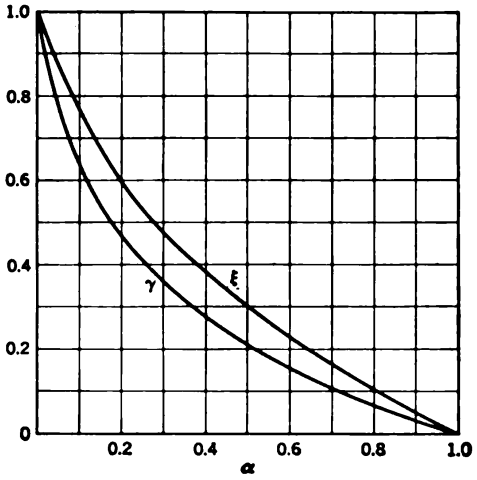

FIG. 4.18 Comparison of ξ and γ.

Equations (4.110) and (4.111) are applicable only when the term $\gamma\Sigma_a(u)$ is small compared to $\xi\Sigma_s(u)$. Thus, although ξ and γ differ by quantities of the order of 25 to 50 per cent of their values (see Fig. 4.18), it is often acceptable to assume that $\xi \simeq \gamma$, so that

$$q(u) \simeq \xi[\Sigma_a(u) + \Sigma_s(u)]\,\phi(u) = \xi\Sigma_t(u)\,\phi(u) \tag{4.112}\star$$

since this approximation, even though poor, is being made in a correction term. For much of the material in this work, it will be assumed that the flux and slowing-down density are related by Eq. (4.112); however,

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it should be borne in mind that (4.112) is valid only if the quantity $\Sigma_s(u)\ \phi(u)$ is slowly varying over any range of length ln $1/\alpha$ in lethargy and that the use of (4.110) in place of (4.112) may give better results.

c. Resonance-escape Probability. The nonabsorption probability for a system containing some absorbing material is given by the exponential factor in Eq. (4.111). In general for any interval u_0 to u_1, we define the nonabsorption probability, or as it is commonly called the "resonance-escape probability," $p(u_1;u_0)$ by

$$p(u_1;u_0) \equiv \exp\left[\int_{u_0}^{u_1} \frac{\Sigma_a(u)\ du}{\xi\Sigma_t(u)}\right] \qquad (4.113)\star$$

This nomenclature has been adopted because the principal contribution to the integral in (4.113) would [if (4.113) were applicable in those regions] come from the energy range of the absorption cross-section curve

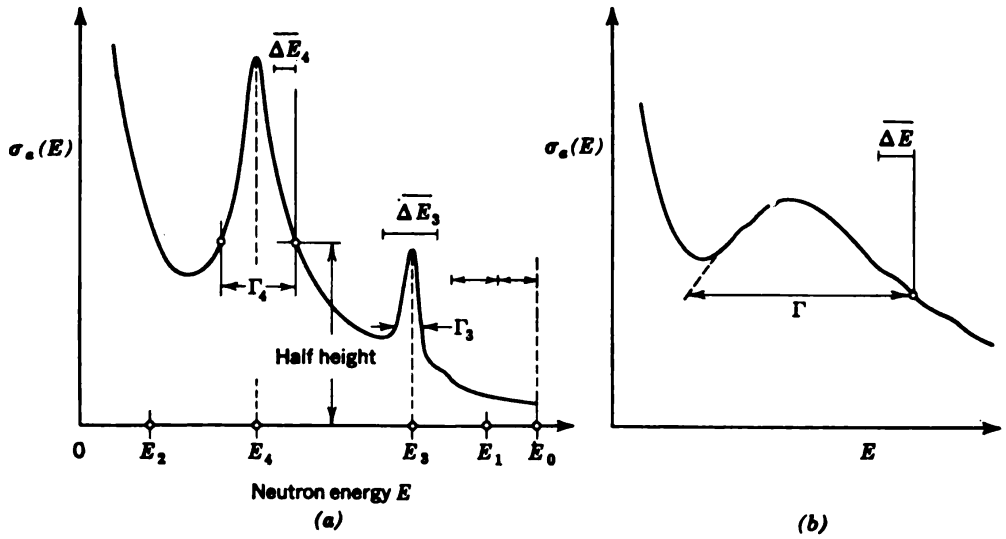

FIG. 4.19 Resonances in absorption cross section. (*a*) Resonance structure; (*b*) broad resonance.

which included the resonance peaks (see Fig. 4.19). Since the absorption cross section is very large in the region of a resonance peak, the requirement that the scattering collision density be a slowly varying function over a collision interval (ln $1/\alpha$) will not be satisfied, and separate consideration must be given to the problem of evaluating the nonabsorption probability in regions which include an absorption resonance. It would appear, then, that the term "resonance-escape probability" is misapplied to $p(u_1;u_0)$; however, in many practical cases (4.113) may be used to obtain a good approximation even in the presence of absorption resonances, since the macroscopic absorption cross section of a medium consisting of several nuclear species may not exhibit a particularly large value at a resonance of any one of the nuclear types present. More-

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over, the mere existence of a resonance in the absorption cross-section curve does not invalidate, a priori, the applicability of Eq. (4.113).

The usefulness of this relation for estimating the resonance-escape probability in an absorbing medium depends upon the "size" of the resonances. The size of a resonance is determined by the maximum value of $\sigma_a(E)$ at the resonance peak and by the *width* of the resonance. The width of a resonance Γ is customarily defined as the distance across (in energy units) the resonance at a height of one-half the maximum value of the $\sigma_a(E)$ curve (see Fig. 4.19*a*). On the basis of a physical argument, it is clear that, if the resonance is high and broad,[1] many neutrons will be captured in this energy range (see Fig. 4.19*b*). Consequently, the neutron density, and therefore the flux, will be highly depleted in this vicinity, and the assumption on which the result (4.111) is based would be violated. Thus (4.113) could not be expected to give a good estimate of the resonance-escape probability in such a system. On the other hand, if the absorption resonances are "sharp," (4.113) may still be applicable. The sharpness of a resonance is determined by the ratio $\overline{\Delta E_i}/\Gamma_i$, where $\overline{\Delta E_i}$ denotes the average loss in the energy of a neutron per collision around the resonance energy E_i. The resonance is sharp if this ratio is much greater than unity. By this definition the resonance at energy E_3 in Fig. 4.19*a* is sharp inasmuch as $\overline{\Delta E_3}/\Gamma_3 \gg 1$. The resonance at E_4, however, is broad because in this instance $\overline{\Delta E_4}/\Gamma_4 \simeq 1$.

Equation (4.113) is applicable to systems which exhibit sharp resonances due to the fact that in these situations a large fraction of the neutron population will never "see" some of the resonances. So long as the average jump in energy per collision at the resonance energy is much larger than the resonance width, there is a strong likelihood that the neutron will be scattered from some energy above the resonance to some value below without ever occupying the energy band in which the large resonance cross section appears. Thus many of the neutrons in the system will avoid capture by never having been exposed to the resonances and thereby enjoy a greater opportunity to reach any specified energy level below the source energy. However, if the resonances are broad in comparison to the average energy jump in the vicinity of the resonance energy, no neutrons will succeed in jumping clear across the resonance in one collision, and all neutrons will spend some time in the resonance energy band. As a result, the probability for capture increases, and the resonance-escape probability for a system which possesses broad resonances will be significantly less than one for a system which has sharp resonances.

These arguments are not meant to imply, however, that the result (4.113) for the resonance-escape probability is rigorously correct for systems which have strong but sharp resonances. The intention here is to

[1] And the concentration of absorber large, i.e., when Σ_a is of the order of Σ_s.

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offer some justification for the use of this formula in systems which contain appreciable amounts of absorber. In this connection it is appropriate to note that (4.113) takes the rigorously correct form in the limit as $\xi \to 1$, i.e., in a system of pure hydrogen [cf. Eq. (4.99)]. Furthermore, we know that the presence of a high sharp resonance in the cross-section curve does not act as an impenetrable barrier to the slowing-down process. Many neutrons will get through such a barrier in a single collision. Equation (4.113) shows that, although the appearance of such a resonance will decrease the resonance-escape probability (the integrand increases

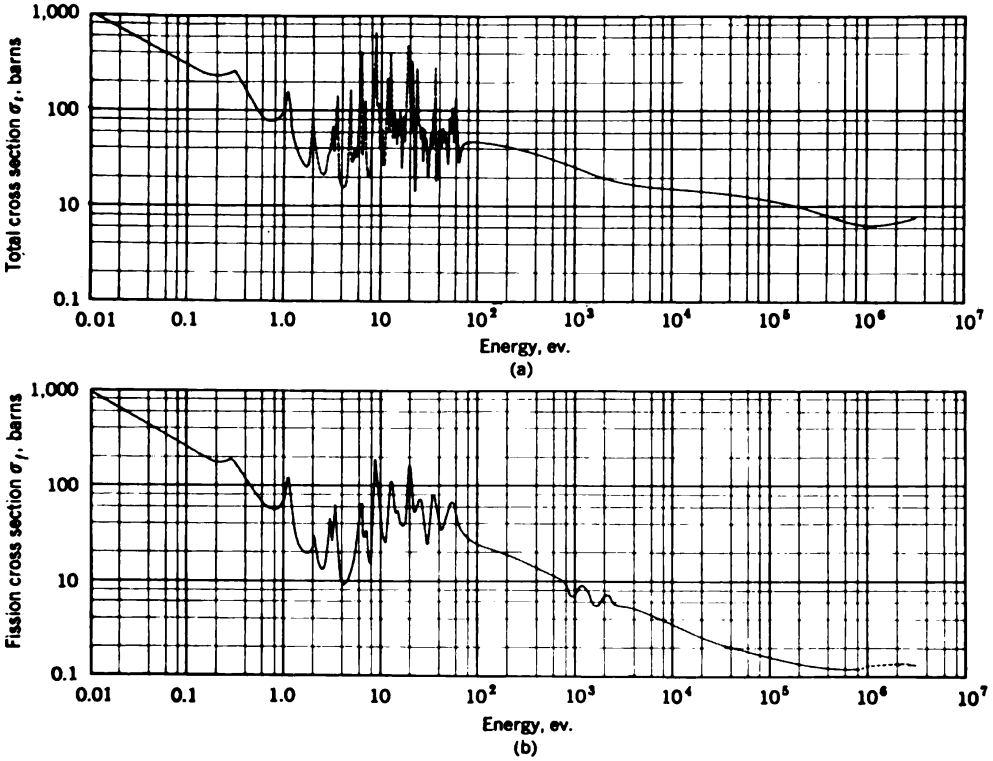

FIG. 4.20 Cross-section curve for a nuclear fuel: (*a*) total cross section for U^{235}; (*b*) fission cross section for U^{235}.

as Σ_a increases), the rate of decrease is limited by the presence of the Σ_t term in the denominator of the integrand. This result exhibits, then, some of the features that would be required of such a function on simple physical grounds. Equation (4.113) should be most useful in analyzing media with light absorption and may be extended to systems which have sharp resonances.

It was pointed out previously that the preceding analysis was inapplicable to media which contained heavy absorbers ($\Sigma_a \sim \Sigma_s$), especially if large, broad resonances appeared in the absorption cross-section curves. Unfortunately these are the materials of greatest interest to reactor technology. Reactor cores invariably possess relatively high concentrations

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of fissionable material; these materials are characterized by absorption cross sections which exhibit many large resonances (see, for example, Fig. 4.20). Thus, although some preliminary estimates can be made for such systems with the aid of Eq. (4.113), other methods must be utilized for accurate determinations of the resonance-escape probability in these cases. The principal difficulty in analyzing media with large absorption is that the assumption of a slowly varying collision density cannot be applied. We found that in pure scattering media the collision density varied as $1/E$. This is not the case, however, in absorbing media. As the macroscopic absorption cross section increases, the collision density departs further and further from the $1/E$ behavior, even in the absence of resonances. If resonances appear, the collision density is further distorted and significant dips appear in the vicinity of the resonances (see Fig. 4.21).

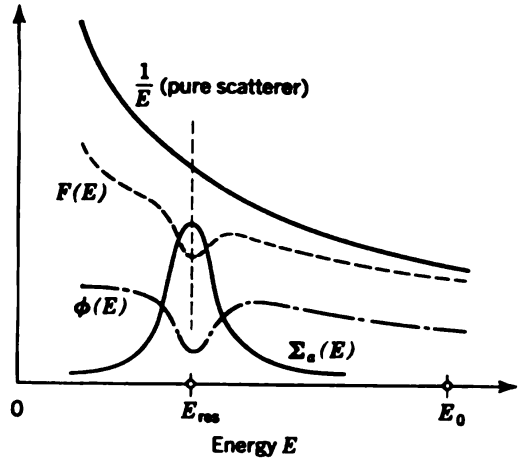

FIG. 4.21 Distortions in collision density due to resonance absorptions.

These features of the problem complicate the calculation of the neutron flux, and the integral equations for highly absorbing media cannot be treated by the elementary methods used above. In these instances it is more practical to resort to experimental methods. Thus much of the information on the resonance-escape probability of the various mixtures of nuclear fuels and moderators is presently available in experimental form.[1] The resonance-escape probabilities in these media are usually obtained by measuring the slowing-down density as a function of energy for a given neutron-source distribution. The ratios of these data to the known source strength yield the resonance-escape probability [see, for example, Eq. (4.111)].

An advantage which this method offers over a purely analytical approach is that the details of the resonance structure need not be known in order to determine the resonance-escape probability. In view of the

[1] A detailed discussion of this subject is given in Secs. 10.1c and 10.3.

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considerable difficulties involved in making cross-section measurements in the resonance range, we have here another argument for abandoning the analytical approach in favor of the experimental.

d. Collision Density by Monte Carlo Method. The analyses of the preceding sections showed the inherent difficulties associated with the computation of the neutron distributions in a slowing-down medium containing absorbing materials. Analytical solutions were obtained only for the special case of a medium of small absorption cross section. Exceptions to this restriction were allowed for media which possessed large absorption cross sections over short energy intervals, i.e., sharp resonances. Even in these somewhat limited cases, it was necessary to introduce several assumptions and approximations into the analysis in order to obtain useful results. The solutions obtained were limited, moreover, to the asymptotic range of neutron energies [see Eqs. (4.110) and (4.111)].

Two problems of importance to reactor physics which were not treated in these analyses are the slowing-down distribution of neutrons in the energy range close to the energy of the source neutrons and the general slowing-down problem in strongly absorbing media. These calculations cannot be carried out by elementary analytical methods. The experimental approach mentioned in connection with the calculation of the resonance-escape probability (see Sec. 4.4c) is an alternate method for treating these problems. Unfortunately, in many instances it is as difficult to perform good experiments as it is to carry out more exact analytical studies.

A computational method which is eminently suited to the analysis of the slowing-down distribution in strong absorbers and in regions close to the source energy is the *Monte Carlo* procedure.[1] This method is essentially a statistical compilation. The average behavior of a system of many particles (or events) is predicted from a knowledge of a limited number of *individual histories* obtained by a Monte Carlo calculation. In some applications (e.g., the resonance-escape probability) the Monte Carlo calculations can be performed "by hand." For many reactor-analysis problems, however, extremely large numbers of histories are required (greater than 10^4), and these can be handled only by means of high-speed computing machines. It is not within the scope of this work to examine the details of the method; however, we will have frequent occasion to refer to results which have been obtained from machine calculations using this method.

The Monte Carlo method has been successfully applied to a variety of reactor-physics problems. An application of interest to the present topic is the analysis of the slowing-down problem in an absorbing medium

[1] See, for example, Herman Kahn, "Applications of Monte Carlo," Rand Corporation, Santa Monica, Calif., AECU-3259, Apr. 19, 1954.

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carried out by Coveyou, Bate, and Osborn.[1] A part of this treatment was devoted to the calculation of the neutron-scattering collision densities within the first two collision intervals below the energy of the source neutrons. The physical model on which these computations were based was defined by the following set of assumptions: (1) neutrons slow down in an infinite, homogeneous medium from a monoenergetic source of strength q_0 neutrons per unit volume per unit time, (2) the nuclei of the medium are unbound and (3) are distributed in energy according to a Maxwell-Boltzmann distribution, (4) the scattering cross section is independent of energy, and (5) the absorption cross section is inversely proportional to the relative speed between the neutron and the nucleus. These assumptions are similar to the set used in Sec. 4.2c. The principal difference appears in 3 and 5. Assumption 5 is a convenient specification of the absorption cross section; however, it does have some physical significance since in general $\sigma_a(v) \sim 1/v$ (if one omits the contributions from the resonances). Assumption 3 is an attempt to describe the thermal motion of the nuclei in the medium; further remarks on this phenomenon will be deferred until Sec. 4.7.

The application of the Monte Carlo technique to the present problem may be briefly outlined in terms of the following steps: (1) a neutron is introduced into the system with a preassigned source speed; (2) a decision is then made as to the nature of the first event to be experienced by the neutron, i.e., is it to be absorbed or scattered, and by what species;[2] (3) if the event is an absorption, a new neutron is introduced at the source speed and the procedure repeated; (4) if the event is a scattering, a decision must be made regarding the speed of the target particle and the relative angle of approach of the neutron to the target. Having made this decision it is then a simple matter, on the assumption of isotropic scattering in the center-of-mass system, to select the new speed of the neutron; (5) these steps are repeated until an absorption collision occurs, thus terminating a history. By tracing many such histories it is possible to compute the average number of the various events (such as scatterings and absorptions) per energy interval for all energies per neutron introduced into the system. Thus the collision densities, neutron flux, and the slowing-down density can be obtained. Note that the resonance-escape probability is an especially easy quantity to compute by this method; clearly, $p(E)$ is the number of neutrons which survive to E divided by the total number of histories examined.

A specific calculation for the scattering collision density was carried out for the first two collision intervals using a slowing-down medium of

[1] R. R. Coveyou, R. R. Bate, and R. K. Osborn, *J. Nuclear Energy*, **2**, 153–167, (1956).

[2] These decisions are made by a suitable procedure which involves the selection of random numbers.

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mass number $A = 9$. The results of this calculation are presented in Fig. 4.22. The ordinate is the function $\Sigma_s(x)\ \phi(x)/q_0$ where

$$x \equiv \frac{\delta v}{v_T} \qquad v_T \equiv \left(\frac{3kT_N}{m}\right)^{\frac{1}{2}} \tag{4.114}$$

and v is the neutron speed, m the neutron mass, T_N the moderator temperature, k the Boltzmann constant,[1] and $\delta = 16$ (a convenient scale factor). Thus the collision density is given as a function of a speed ratio. The source condition for this computation was taken as $x_0 = 512$. The open circles in the figure represent the points computed by the Monte Carlo method. The discontinuity in the collision density at the lower

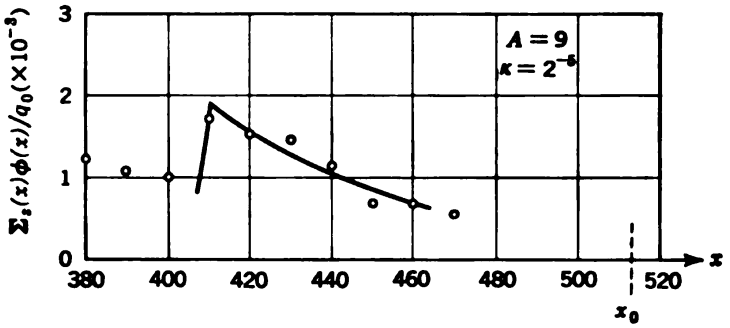

FIG. 4.22 Scattering collision density in $1/v$ absorber.

end of the first collision interval[2] $x = x_1$ is clearly discernible (cf. Fig. 4.15). The solid-line curve is the solution for the collision density in the first interval for a $1/v$ absorber. This function is

$$\frac{\Sigma_s(x)\ \phi(x)}{q_0} \equiv \frac{2}{(1-\alpha)x_0}\left(\frac{x}{x_0}\right)^2\left(\frac{x_0+\kappa\delta}{x+\kappa\delta}\right)^{(3-\alpha)/(1-\alpha)} \tag{4.115}$$

where

$$\kappa \equiv \left(\frac{2T_*}{3T_N}\right)^{\frac{1}{2}}\left(\frac{N_a}{N_s}\right)\frac{\sigma_a(T_*)}{\sigma_s} \tag{4.116}$$

and $T_* \equiv 293°\text{K}$, N_a is the nuclear density of absorbers, and N_s the nuclear density of scatterers. It should be noted that the parameter κ is a measure of the absorption strength, and $\kappa = 0$ corresponds to the case of a pure scatterer; whence,

$$\frac{\Sigma_s(x)\ \phi(x)}{q_0} = \frac{2}{(1-\alpha)x_0}\left(\frac{x_0}{x}\right)^{(1+\alpha)/(1-\alpha)} \tag{4.117}$$

It is easily shown that this function is the direct transformation[3] of expression (4.78) when the variable E is transformed to the variable x. The results for the Monte Carlo calculation were carried out for $\kappa = 2^{-5}$, which corresponds to a medium with relatively low absorption. The

[1] $k = 1.3804 \times 10^{-16}$ erg/°K $= 8.6275 \times 10^{-5}$ ev/°K.

[2] $x_1 = \alpha^{\frac{1}{2}}x_0$, where α is given by Eq. (4.20).

[3] By using a relation of the type (3.43).

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method is not limited, however, by the value of κ, and the flux distribution in more highly absorbing media can be obtained as readily as in the present case. It is seen from the figure that the Monte Carlo data show some scatter; this is due to the poor statistics in the high-energy range. The essential features of the collision density in this range are well exhibited, nevertheless.

4.5 Mixtures of Nuclear Species

a. Asymptotic Equations. In the present section we consider the slowing down of neutrons from a monoenergetic source in an infinite medium which is a uniform mixture of several nuclear species. The calculations which follow apply to the asymptotic range (well below the energies of the source neutrons). We noticed in our previous calculations for mononuclear media that this energy range was the most important for the analysis of thermal reactors. Therefore, even though we ignore the details of the neutron distributions in the vicinity of the source energy, the results we obtain will be of some immediate value for reactor calculations.

The appropriate equations for describing the neutron distributions in a medium of several nuclear species are obtained from a generalization of Eqs. (4.104) and (4.105). The only change which is required in these results involves the integral term which gives the rate at which neutrons are slowed down into the lethargy range of interest from collisions above. For the mixture, this term must include the contributions from collisions with all the various nuclear types present; it is necessary, then, to add a scattering integral for each nuclear species i. If we define ($i = 1, 2, \ldots,$ N species)

$\Sigma_s^{(i)} \equiv$ macroscopic scattering cross section of ith nuclear species

$\alpha_i \equiv \left(\frac{A_i - 1}{A_i + 1}\right)^2$ where $A_i \equiv$ nuclear mass of ith nuclear species

then it follows that for the asymptotic range $\left(u > \ln \frac{1}{\alpha_i}\right)$

$$q(u) = \sum_{i=1}^{N} \frac{1}{1 - \alpha_i} \int_{u - \ln \frac{1}{\alpha_i}}^{u} \Sigma_s^{(i)}(u')\, \phi(u')(e^{u'-u} - \alpha_i)\, du' \quad (4.118)$$

$$\Sigma_t(u)\, \phi(u) = \sum_{i=1}^{N} \frac{1}{1 - \alpha_i} \int_{u - \ln \frac{1}{\alpha_i}}^{u} \Sigma_s^{(i)}(u')\, \phi(u')\, e^{u'-u}\, du' \quad (4.119)$$

Equation (4.118) is a statement of the physical fact that each neutron which slows down past lethargy u has made its last scattering collision with some particular nuclear type at some lethargy $u' < u$. The prob-

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ability that these collisions increase the lethargy of the neutron to a value above u must be calculated separately for each nuclear type; finally, the combined contribution from all scattering collisions must be summed up with the length of the scattering interval ($\ln 1/\alpha_i$) varying from one nucleus to another. Similarly, Eq. (4.119) is a balance condition for the unit interval around lethargy u.

b. Mixture of Pure Scatterers. In the special case wherein all nuclei in the mixture are pure scatterers, Eqs. (4.118) and (4.119) have a simple solution. The integral equation (4.119), with

$$\Sigma_a^{(i)} \equiv 0 \left[\text{so that } \Sigma_t = \Sigma_s = \sum_{i=1}^{N} \Sigma_s^{(i)} \right]$$

has the solution

$$\Sigma_s^{(i)}(u)\, \phi(u) = C_i \qquad i = 1, 2, \ldots, N$$

where the C_i are a set of constants. The use of this result in the integrand of (4.118) gives

$$q(u) = \sum_{i=1}^{N} \frac{C_i}{1-\alpha_i} \int_{u-\ln \frac{1}{\alpha_i}}^{u} (e^{u'-u} - \alpha_i)\, du' = \sum_{i=1}^{N} C_i \xi_i$$

$$= \sum_{i=1}^{N} \xi_i \Sigma_s^{(i)}(u)\, \phi(u) \quad (4.120)$$

If we define the average ξ for a mixture by

$$\bar{\xi}(u) \equiv \frac{1}{\Sigma_s(u)} \sum_{i=1}^{N} \xi_i \Sigma_s^{(i)}(u) \quad (4.121)$$

Eq. (4.120) may be written

$$q(u) = \bar{\xi}(u) \Sigma_s(u)\, \phi(u) \equiv q(0) \quad (4.122)$$

The last equality follows from the physical requirement that, in the absence of absorbers, $q(u) \equiv q(0)$. A mixture of pure scatterers is treated, then, as a single scatterer, with the quantity ξ replaced by the average defined in (4.121). This definition is plausible from a physical standpoint since a given nuclear species is effective in slowing down neutrons only in proportion to the fraction of the scattering collisions which are due to that species. It should be noted that, since $\Sigma_s^{(i)}(u)$ is lethargy dependent, $\bar{\xi}(u)$ must be also; but the dependence of $\bar{\xi}$ tends to be less sensitive because of its functional form. Thus a particular ξ is usually chosen to represent the entire lethargy range of interest.

The neutron flux for a mixture of pure scatterers is obtained directly from (4.122).

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$$\phi(u) = \frac{q(0)}{\bar{\xi}(u)\,\Sigma_s(u)} \tag{4.123}$$

Note that if one assumes that all the scattering cross sections have the same functional dependence, then

$$\bar{\xi}(u) \equiv \bar{\xi} = \text{constant} \tag{4.124}$$

and we obtain for the flux

$$\phi(u) = \frac{q(0)}{\bar{\xi}\Sigma_s(u)} \tag{4.125}$$

The simplest case of this is, of course, the assumption that all $\Sigma_s^{(i)}$ are independent of lethargy; then

$$\bar{\xi} = \frac{1}{\Sigma_s} \sum_{i=1}^{N} \xi_i N_i \sigma_s^{(i)} \tag{4.126}$$

The flux is also a constant $\phi(u) = \phi_0 = q(0)/\bar{\xi}\Sigma_s$.

c. Mixture of Absorbers. In the presence of absorption, Eqs. (4.118) and (4.119) do not have simple solutions, and even the limitation that $\Sigma_a \ll \Sigma_s$ for all the nuclear species present does not lead easily to a solution. If we introduce this assumption, then the variation in the scattering-collision density over a collision interval should be small, and we can use the approximation

$$\Sigma_s^{(i)}(u')\,\phi(u') \simeq \Sigma_s^{(i)}(u)\,\phi(u) + (u' - u)\,\frac{d}{du}\,[\Sigma_s^{(i)}(u)\,\phi(u)] \tag{4.127}$$

The substitution of this expression into the integrand of Eq. (4.118) yields

$$q(u) \simeq \sum_{i=1}^{N} \left\{ \xi_i \Sigma_s^{(i)}(u)\,\phi(u) - \xi_i \gamma_i \frac{d}{du}\,[\Sigma_s^{(i)}(u)\,\phi(u)] \right\} \tag{4.128}$$

where

$$\gamma_i \equiv 1 - \frac{\alpha_i \ln^2 \alpha_i}{2\xi_i(1 - \alpha_i)} \tag{4.129}$$

If the derivative term in (4.128) is neglected, we obtain as a first approximation

$$\frac{d}{du}\,q(u) \simeq \sum_{i=1}^{N} \xi_i \frac{d}{du}\,[\Sigma_s^{(i)}(u)\,\phi(u)]$$

We use this result in (4.128) to obtain

$$q(u) = \sum_{i=1}^{N} \xi_i \Sigma_s^{(i)}(u)\,\phi(u) - \bar{\gamma}(u)\,\frac{d}{du}\,q(u) \tag{4.130}$$

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where
$$\bar{\gamma}(u) \equiv \frac{\sum_{i=1}^{N} [\Sigma_s^{(i)}(u)\ \phi(u)]'\ \xi_i\gamma_i}{\sum_{i=1}^{N} [\Sigma_s^{(i)}(u)\ \phi(u)]'\ \xi_i} \tag{4.131}$$

As in the calculation for the mononuclear case, we assume that $q(u)$ and $\phi(u)$ are related by

$$\frac{d}{du}\, q(u) = -\Sigma_a(u)\ \phi(u) \tag{4.106}$$

and substitute this relation into (4.130). Thus

$$q(u) = \sum_{i=1}^{N} \xi_i\Sigma_s^{(i)}(u)\ \phi(u) + \bar{\gamma}(u)\ \Sigma_a(u)\ \phi(u)$$
$$= [\bar{\xi}(u)\ \Sigma_s(u) + \bar{\gamma}(u)\ \Sigma_a(u)]\ \phi(u) \tag{4.132}$$

where we have used (4.121). Again the approximation is often used that

$$\bar{\xi}(u) \simeq \bar{\gamma}(u) \tag{4.133}$$

so that
$$q(u) = \bar{\xi}(u)\ \Sigma_t(u)\ \phi(u) \tag{4.134}\star$$

This relationship between $q(u)$ and $\phi(u)$, which seems to be related to the accurate equations only through several very limiting assumptions, has been used in many calculations on actual reactors with very good results. It will be generally assumed throughout this work.

An explicit expression for the slowing-down density is obtained by substituting (4.134) into (4.106). The solution is

$$q(u) = q(0) \exp\left[- \int_0^u \frac{\Sigma_a(u')\ du'}{\bar{\xi}(u')\ \Sigma_t(u')} \right] \tag{4.135}\star$$

wherein the exponential gives the resonance-escape probability in an absorbing medium.

It should be recognized that if it is desired to use the relation (4.132) in an actual computation, instead of the simpler form (4.134), it is necessary to carry out an iterative procedure. This complication arises from the definition of $\bar{\gamma}(u)$ which involves the neutron flux [see Eq. (4.131)]. This difficulty may be overcome by following the procedure outlined below: As a first approximation use Eq. (4.134), and call the corresponding expression for the flux $\phi_1(u)$; thus,

$$\phi_1(u) = \frac{q_1(u)}{\bar{\xi}(u)\ \Sigma_t(u)} = \frac{q(0)}{\bar{\xi}(u)\ \Sigma_t(u)} \exp\left[- \int_0^u \frac{\Sigma_a(u')\ du'}{\bar{\xi}(u')\ \Sigma_t(u')} \right] \tag{4.136}$$

where $\bar{\xi}(u)$ is computed from (4.121). This expression for the flux may now be used to compute the first estimate $\bar{\gamma}_1(u)$. We obtain from (4.131)

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$$\bar{\gamma}_{\text{I}}(u) = \frac{\sum_{i=1}^{N} [\Sigma_s^{(i)}(u)\ \phi_{\text{I}}(u)]'\ \xi_i \gamma_i}{\sum_{i=1}^{N} [\Sigma_s^{(i)}(u)\ \phi_{\text{I}}(u)]'\ \xi_i} \tag{4.137}$$

This relation may be used for a second estimate of ϕ. From (4.132),

$$\begin{aligned}\phi_{\text{II}}(u) &= \frac{q_{\text{II}}(u)}{\bar{\xi}(u)\ \Sigma_s(u) + \bar{\gamma}_{\text{I}}(u)\ \Sigma_a(u)} \\ &= \frac{q(0)}{\bar{\xi}(u)\ \Sigma_s(u) + \bar{\gamma}_{\text{I}}(u)\ \Sigma_a(u)} \\ &\quad \times \exp\left[- \int_0^u \frac{\Sigma_a(u')\ du'}{\bar{\xi}(u')\ \Sigma_s(u') + \bar{\gamma}_{\text{I}}(u')\ \Sigma_a(u')} \right]\end{aligned} \tag{4.138}$$

The substitution of this result into (4.131) yields the second estimate $\bar{\gamma}_{\text{II}}(u)$, which in turn gives $\phi_{\text{III}}(u)$, and so on. This procedure should converge quite rapidly. As pointed out previously, however, for most practical systems of interest, Eq. (4.134) should be entirely adequate. In view of the numerous assumptions imposed on the preceding analysis, the refinements suggested above are somewhat questionable and the considerable labor required in performing the iteration seems hardly justified.

d. Moderating Ratio. It was shown in Sec. 4.1 that the effectiveness of a material as a slowing-down medium was determined in part by its value of ξ, the average gain in lethargy per collision. Another factor which can affect this characteristic of a substance is, of course, its macroscopic scattering cross section Σ_s. The product of these two parameters gives an indication of the moderating properties of a material. This quantity is sometimes called the moderating power:

$$\xi\Sigma_s \equiv \text{moderating power} \tag{4.139}$$

Thus the larger the moderating power of a material, the greater its effectiveness as a slowing-down medium. This number, however, is not a complete measure of the slowing-down effectiveness. A better quantity is the *moderating ratio*, which takes into account the adverse effect of a high-absorption cross section. The definition of this quantity is

$$\frac{\xi\Sigma_s}{\Sigma_a} \equiv \text{moderating ratio} \tag{4.140}$$

The influence of this parameter on the slowing-down density may be demonstrated by means of a simple example.

Consider two uniform infinite media of dissimilar composition. It is immaterial whether these media consist of mononuclear materials or are uniform mixtures; the following arguments apply in either case. Let q_0 monoenergetic neutrons per unit volume per unit time be introduced into

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each of these media, and let $u = 0$ correspond to the energy of these source neutrons. From our previous analyses [see Eqs. (4.111), (4.113), and (4.135)] we know that

$$\begin{aligned} q_1(u) &= q_0 p_1(u;0) \\ q_2(u) &= q_0 p_2(u;0) \end{aligned} \tag{4.141}$$

where the resonance-escape probabilities $p_i(u;0)$ are defined

$$p_i(u;0) \equiv \exp\left[- \int_0^u \frac{\Sigma_a^{(i)}(u')\, du'}{\xi_i \Sigma_t^{(i)}(u')} \right] \tag{4.142}$$

For convenience we define an average value of the integrand such that

$$p_i(u;0) \equiv \exp\left[- \left(\overline{\frac{\Sigma_a}{\xi \Sigma_t}} \right)_i u \right] \tag{4.143}$$

If we call $m_i \equiv (\overline{\xi \Sigma_t / \Sigma_a})_i$, then from (4.141)

$$\frac{q_1(u)}{q_2(u)} = e^{-u(1/m_1 - 1/m_2)} \tag{4.144}$$

This ratio gives a useful criterion for comparing the moderating effectiveness of two materials. Clearly if $m_1 > m_2$, then $q_1(u) > q_2(u)$, that is, more neutrons succeed in slowing down to lethargy u in medium 1 than in 2. Note that the parameters m_i are essentially the moderating ratios defined by (4.140), since for slight absorption $\Sigma_a \ll \Sigma_s$, and we have

$$m \equiv \left(\overline{\frac{\xi \Sigma_t}{\Sigma_a}} \right) \simeq \left(\overline{\frac{\xi \Sigma_s}{\Sigma_a}} \right)$$

Thus (4.144) states that $q_1(u) > q_2(u)$ if the moderating ratio of medium 1 is greater than that of medium 2. Good moderating materials, then, possess large ξ (small A), large Σ_s, and small Σ_a. It should be recognized that, in using the moderating ratio as a criterion for moderating effectiveness, it is necessary to establish unambiguously the cross sections which enter into the calculation. It is evident from (4.143) that the proper cross sections to use are a suitably averaged set over the lethargy range of interest. In many instances, however, a rough estimate is sufficient, and it has become customary to compare moderating media in terms of these properties at thermal energy (0.025 ev). The moderating ratios at thermal energy of some efficient moderators are listed in Table 4.1.

TABLE 4.1 MODERATING RATIOS

Material	*Moderating ratio*
H_2O	164
D_2O	13,200
Be	144
C	236

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4.6 Infinite Homogeneous Multiplying Media

a. Arbitrary Source Distribution. Our considerations of the slowing-down problem so far have been limited to infinite media with monoenergetic sources. In an actual reactor the neutron sources are in main part the fission reactions which take place; these source neutrons appear in the system according to a particular distribution in energy, the *fission spectrum*. A somewhat more realistic analysis of the slowing-down problem can be obtained by taking this source distribution into account. We begin the calculation by setting up the following general problem: (1) neutrons appear in the medium isotropically at the rate $S(u)$, where $S(u)\,du$ is the number of neutrons which are born per unit volume per unit time with lethargies in du about u; (2) the nuclei of the medium are stationary; and (3) the medium is isotropic and homogeneous. The flux and the slowing-down density are assumed to be related by the equations

$$\frac{d}{du}\,q(u) = -\Sigma_a(u)\,\phi(u) + S(u) \tag{4.145}$$

$$q(u) = \xi\Sigma_t(u)\,\phi(u) \tag{4.112}$$

Equation (4.145) is an immediate extension of (4.106) and states that any change in $q(u)$ must be due to the combined effect of the removals

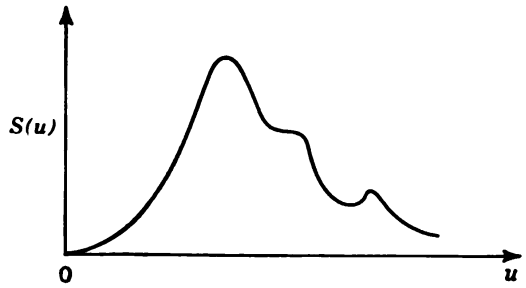

FIG. 4.23 Source function $S(u)$.

$\Sigma_a(u)\,\phi(u)$ and of the source $S(u)$. Equation (4.112) is taken from the analysis of Sec. 4.4b and therefore applies to the asymptotic range and to media with slight absorption. These relations replace the more general form (4.104). Equations (4.112) and (4.145) are to be used with the initial conditions

$$q(0) = 0 \qquad S(0) = 0 \tag{4.146}$$

which imply that the lethargy scale is so selected that the origin is well above all neutron sources (see Fig. 4.23); thus no neutrons appear in the system at energies above the value which corresponds to $u = 0$. The solution to (4.112) and (4.145) is readily shown to be

$$q(u) = \int_0^u S(u')p(u;u')\,du' \tag{4.147}\star$$

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where $$p(u;u') \equiv \exp\left[- \int_{u'}^{u} \frac{\Sigma_a(w)\, dw}{\xi \Sigma_t(w)} \right] \tag{4.148}$$

The physical statement of (4.147) is that the number of neutrons slowing past lethargy u (in an infinite medium, per unit volume per unit time) is given by the number of neutrons produced at u', $S(u')\, du'$, multiplied by the probability $p(u;u')$ that a neutron at u' will reach u without being absorbed and the product integrated over all $u' < u$ [for which $S(u')$ has a nonzero value].

b. Monoenergetic Source. A special case of (4.147) is the medium with a monoenergetic source; thus, suppose that all neutrons enter the medium with lethargy $u = 0$ (corresponding to some fixed energy E_0). Then, we take

$$S(u) \equiv q_0\, \delta(u) \tag{4.149}$$

where $\delta(u)$ is the Dirac delta function.[1] Note that the total number of neutrons which appear at *all* lethargies per unit volume per unit time is given by the integral

$$\int_{-\infty}^{\infty} S(u)\, du = q_0 \int_0^{\infty} \delta(u)\, du = q_0 \tag{4.150}$$

The slowing-down density in this system is obtained by the direct substitution of (4.149) into the general result (4.147). The solution is

$$q(u) = q_0 \int_0^u \delta(u')\, p(u;u')\, du' = q_0 \exp\left[- \int_0^u \frac{\Sigma_a(w)\, dw}{\xi \Sigma_t(w)} \right] \tag{4.151}$$

as previously obtained [cf. Eq. (4.111)].

c. Fission Source. As a second special case of Eq. (4.147), suppose that the infinite medium in question contains a uniform distribution of fissionable material, so that the nuclear characteristics of the composition are now ξ, $\Sigma_s(u)$, $\Sigma_f(u)$, $\Sigma_a(u)$, and $\Sigma_t(u)$. Suppose further that, as a result of each fission, ν neutrons appear at the point of the fission reaction and that these neutrons are distributed in lethargy according to the function $\chi(u)$, where

$$\int_0^{\infty} \chi(u)\, du = 1 \tag{4.152}$$

Thus $\chi(u)$ is the normalized *fission spectrum* in lethargy. Note that $\chi(u)\, du$ gives the fraction of neutrons which appear from fission with lethargies in du about u. It is sometimes more convenient to work with the corresponding function in energy space $z(E)$, which may also be

[1] H. Margenau and G. M. Murphy, "The Mathematics of Physics and Chemistry," p. 325, D. Van Nostrand Company, Inc., Princeton, N.J., 1943.

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normalized; thus,

$$\int_0^\infty z(E)\,dE \equiv \int_0^\infty \chi(u)\,du = 1 \tag{4.153}$$

The fission spectrum is obtained from experiment. Figure 4.24 shows some experimental data along with an analytical approximation.

One of the initial assumptions (2) was that the nuclei were stationary; thus the neutrons may slow down all the way to $u = \infty$. The source

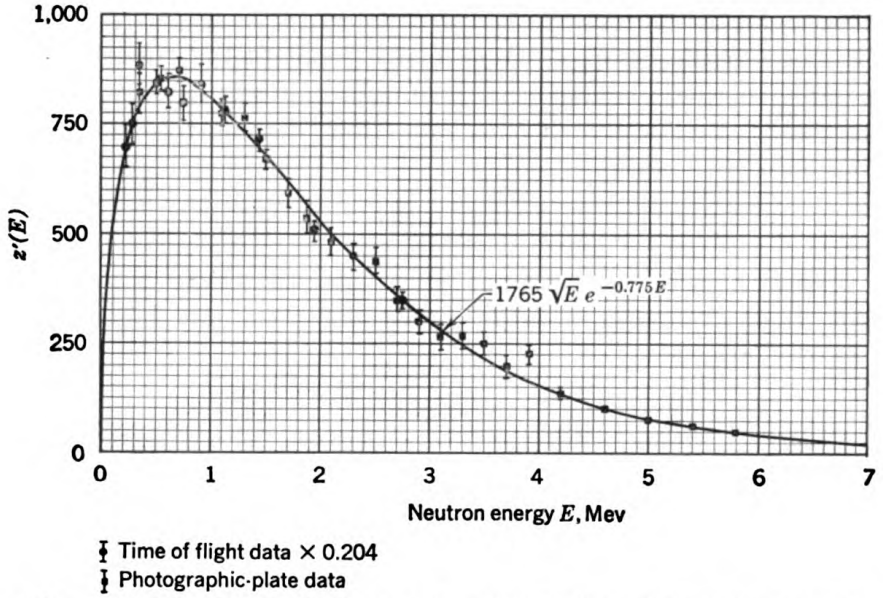

FIG. 4.24 Fission spectrum in energy for U^{235}; indicated formula is good up to $E = 9$ Mev. [*From L. Cranberg, G. Frye, N. Nereson, and L. Rosen, Phys. Rev.*, **103**(3): 662–670 (1956).]

function for this system is therefore

$$S(u) = \nu\,\chi(u) \int_0^\infty \Sigma_f(u')\,\phi(u')\,du' \tag{4.154}$$

The slowing-down density may be obtained by using this expression in (4.147).

$$q(u) = \nu \int_0^\infty \Sigma_f(u'')\,\phi(u'')\,du'' \int_0^u \chi(u')\,p(u;u')\,du' \tag{4.155}$$

If the fissionable material is to be the only source of neutrons, and if the neutron flux in the infinite medium is to be in the steady state, we require that

$$\begin{pmatrix}\text{Number of neutrons}\\ \text{introduced per unit}\\ \text{volume per unit time}\end{pmatrix} = \begin{pmatrix}\text{number of neutrons}\\ \text{produced from fissions per}\\ \text{unit volume per unit time}\end{pmatrix}$$

The equivalent mathematical statement is

$$\int_0^\infty S(u)\,du = \nu \int_0^\infty \chi(u)\,du \int_0^\infty \Sigma_f(u')\,\phi(u')\,du' \tag{4.156}$$

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If we introduce one neutron per unit volume per unit time, then $\int_0^\infty S(u)\,du = 1$, and

$$1 = \nu \int_0^\infty \chi(u)\,du \int_0^\infty \Sigma_f(u')\,\phi(u')\,du' = \nu \int_0^\infty \Sigma_f(u')\,\phi(u')\,du' \quad (4.157)$$

by virtue of (4.152). The equality at the right states that if the system is to be at steady state then exactly one neutron per unit volume per unit time must be produced as a result of all the fissions at all lethargies in that volume. The application of this result to the relation for $q(u)$ given in (4.155) yields

$$q(u) = \int_0^u \chi(u')\,p(u;u')\,du' \quad (4.158)$$

and for the flux we have [from (4.112)]

$$\phi(u) = \frac{1}{\xi\Sigma_t(u)} \int_0^u \chi(u')\,p(u;u')\,du' \quad (4.159)$$

d. Criticality Equation. The expression for the steady-state condition (4.157) may be written in terms of the result (4.159); thus

$$1 = \nu \int_0^\infty \frac{\Sigma_f(u)\,du}{\xi\Sigma_t(u)} \int_0^u \chi(u')\,p(u;u')\,du' \quad (4.160)$$

For convenience we define[1]

$$\gamma(u) \equiv \frac{\Sigma_f(u)}{\xi\Sigma_t(u)} \qquad \beta(u) \equiv \frac{\Sigma_a(u)}{\xi\Sigma_t(u)} \quad (4.161)$$

Then (4.160) gives

$$1 = \nu \int_0^\infty \gamma(u)\,du \int_0^u \chi(u') \exp\left[-\int_{u'}^u \beta(w)\,dw\right] du' \quad (4.162)\star$$

This is the condition for the steady-state operation of a multiplying medium with only fission sources of neutrons and slowing down allowed over the infinite lethargy range; i.e., in the terminology of the next section, there is no *thermal group*. Condition (4.162) is called the *criticality condition*, and it is a requirement on the nuclear characteristics of the multiplying medium. We will have more to say of this later.

The fission spectrum $z(E)$, for the particular fuel U^{235}, has a maximum value at about 1 Mev. For many purposes it is adequate to group all neutrons from fission at some one energy in the neighborhood of the position of the maximum point. The same can be done with $\chi(u)$. If for convenience we choose $u = 0$ in the vicinity of the maximum point (see Fig. 4.25), then the fission spectrum is approximated by a monoenergetic source at $u = 0$.

$$\chi(u) = \delta(u) \quad (4.163)$$

[1] Not to be confused with γ from (4.109) or $\gamma(u)$ from (4.131).

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The substitution of this relation into (4.160) yields

$$1 = \nu \int_0^\infty \gamma(u)\, p(u;0)\, du = \nu \int_0^\infty \gamma(u) \exp\left[- \int_0^u \beta(u')\, du' \right] du \quad (4.164)\star$$

Since the cross sections which appear in (4.162) and (4.164) are generally obtained from experiment, simple analytical representations usually are not available for these quantities, and the evaluation of the integrals must be done numerically.

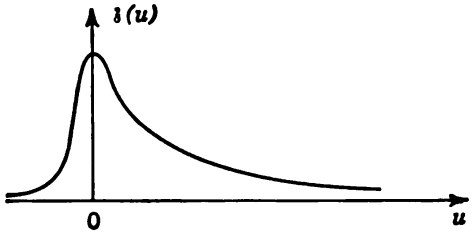

FIG. 4.25 Fission spectrum in lethargy.

A physical interpretation of the integrals in (4.164) may be demonstrated by writing the integrands in the form

$$\int_0^\infty (1) p(u;0) \frac{\Sigma_f(u)}{\Sigma_t(u)} \left(\frac{du}{\xi}\right) \nu = 1$$

That is, the number of neutrons produced per unit volume per unit time (which must be unity) is the sum over all lethargy intervals du of the product

$$\begin{bmatrix}\text{Number of neutrons}\\ \text{produced per unit}\\ \text{volume per unit}\\ \text{time at } u = 0\end{bmatrix}\begin{bmatrix}\text{probability that a}\\ \text{neutron at } u = 0\\ \text{reaches lethargy}\\ u \text{ successfully}\end{bmatrix}\begin{bmatrix}\text{fraction of}\\ \text{collisions}\\ \text{at } u \text{ which}\\ \text{are fissions}\end{bmatrix}$$

$$\times \begin{bmatrix}\text{average}\\ \text{number}\\ \text{collisions}\\ \text{in } du \text{ at } u\end{bmatrix}\begin{bmatrix}\text{average}\\ \text{number}\\ \text{neutrons}\\ \text{per fission}\end{bmatrix}$$

There is an alternative way of regarding (4.160) which may be more instructive. The ratio of the number of neutrons in a given generation to the number in the preceding generation is the multiplication constant [see Eq. (3.16)]; then

$$k_\infty = \nu \int_0^\infty \gamma(u)\, du \int_0^u ȝ(u')\, p(u;u')\, du' \quad (4.165)$$

where k_∞ is defined as the multiplication constant of the infinite medium. If the composition of the medium is specified, then all the cross sections are known [and therefore $\gamma(u)$ and $\beta(u)$] and the integrals may be evaluated. If the right-hand side of (4.165) is unity, then the system is critical; if its value is less than unity, the system is subcritical; and if its value is greater than unity, the reactor is supercritical.

Alternatively, whatever the value of the right-hand side, some number ν_c may be found such that

$$\nu_c \int_0^\infty \gamma(u)\, du \int_0^u ȝ(u')\, p(u;u')\, du' = 1 \quad (4.166)$$

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i.e., for *any* composition of the infinite medium, there exists some (fictitious) average number of neutrons per fission which would make the system critical. This value is given the symbol ν_c and is to be compared with ν, the experimentally determined number of neutrons from fission. It follows from (4.165) and (4.166) that

$$k_\infty = \frac{\nu}{\nu_c} \tag{4.167}\star$$

e. Slowing Down by Average Jumps. The actual slowing-down process experienced by a neutron consists of a series of discrete jumps in its kinetic energy; thus the neutron trajectory in energy space is a discontinuous function with a random distribution of "steps" (see Fig. 4.10). The magnitude of these steps (or jumps) is given by the frequency function for the elastic-scattering collision [see Eqs. (4.39) and (4.50)]. In each of the preceding analyses an attempt was made to take this behavior into account. Only in the case of a hydrogenous medium, however, was it possible to obtain exact solutions.

An alternative analysis of the slowing-down process, which may be more attractive in an intuitive sense, treats this process as a sequence of "average jumps." In this treatment the actual frequency function is replaced by a function which allows the scattered neutron to acquire exactly the *average gain* in lethargy per collision ξ [or lose the average change in energy $\overline{\Delta E}$ per collision; see Eq. (4.40)]. The neutron trajectory in lethargy space, then, is a step function with a series of jumps, all of magnitude ξ. This approach to the slowing-down problem is at best a first approximation; however, it does have some merit in that it retains a number of the essential features of the physical phenomenon, although in crude form. Moreover, it allows a solution of the exact equations [see Eqs. (4.105)].

The average-jump collision process described above requires that the initial and final lethargies of a scattered neutron (u' and u, respectively) be related by the equation

$$u = u' + \xi \tag{4.168}$$

The frequency function for a slowing-down process consisting of these average jumps is defined

$$\begin{aligned} l(u;u')\,du &\equiv \text{probability that a neutron of lethargy } u' \text{ when scattered emerges from the collision with lethargy in } du \text{ about } u \\ &\equiv \delta[u' - (u - \xi)]\,du \end{aligned} \tag{4.169}$$

where δ is again the Dirac delta function. The physical statement of this function is that the scattered neutron will emerge from the collision with lethargy in du about u only if u differs from u' by exactly the amount ξ.

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As an example of the application of these concepts, consider the analysis of the slowing-down problem in an infinite, homogeneous multiplying medium which is at steady state, i.e., critical. Let the distribution of fission neutrons be given by the normalized spectrum $\chi(u)$ defined in Eq. (4.152), and let $\Sigma_a(u)$, $\Sigma_f(u)$, $\Sigma_t(u)$, ξ, and $\Sigma_s(u)$ denote the nuclear constants of the medium. The total collision density for this system is given by

$$\Sigma_t(u)\,\phi(u)\,du = \int_0^u \Sigma_s(u')\,\phi(u')\,l(u;u')\,du'\,du + \nu\chi(u)\int_0^\infty \Sigma_f(u')\,\phi(u')\,du'\,du \quad (4.170)$$

The first integral gives the rate at which neutrons are scattered into the interval du about u, and the second, the rate at which they appear from fissions. The system described by (4.170) is at steady state; therefore, the criticality condition must be satisfied:

$$1 = \nu\int_0^\infty \Sigma_f(u')\,\phi(u')\,du' \quad (4.157)$$

If this relation is used in (4.170), along with the definition (4.169), then

$$\begin{aligned}\Sigma_t(u)\,\phi(u) &= \chi(u) + \int_0^u \Sigma_s(u')\,\phi(u')\,\delta[u' - (u - \xi)]\,du' \\ &= \chi(u) + \Sigma_s(u - \xi)\phi(u - \xi)\end{aligned} \quad (4.171)$$

This relation is a difference equation[1] in the unknown function ϕ. If we define the origin of the lethargy scale such that $\chi(u) = 0$ for all $u \leq 0$, then the appropriate initial condition to be used with (4.171) is

$$\phi(0) = 0 \quad (4.172)$$

The solution to the difference equation may be carried out in steps by starting with the known value of the flux at the origin (4.172). In general, if ϕ_n denotes the flux at the lower limit of the nth collision interval, then

$$\phi_n = \frac{1}{\Sigma_t(n\xi)}\{\chi(n\xi) + \Sigma_s[(n-1)\xi]\phi_{n-1}\} \quad (4.173)$$

Thus, for example,

$$\begin{aligned}\phi_1 \equiv \phi(\xi) &= \frac{\chi(\xi)}{\Sigma_t(\xi)} \\ \phi_2 = \phi(2\xi) &= \frac{1}{\Sigma_t(2\xi)}\left[\chi(2\xi) + \frac{\Sigma_s(\xi)\,\chi(\xi)}{\Sigma_t(\xi)}\right]\end{aligned} \quad (4.174)$$

and so on. An approximate solution to the difference equation can be obtained in the case of a slowly varying flux. Thus we expand

[1] See, for example, Theodore v. Kármán and Maurice A. Biot, "Mathematical Methods in Engineering," chap. XI, McGraw-Hill Book Company, Inc., New York, 1940.

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$$\phi(u - \xi) = \phi(u) - \xi\phi'(u) + \cdots \tag{4.175}$$

and use the first two terms on the condition that $\xi\phi'(u) \ll \phi(u)$. The substitution of this expression into (4.171) reduces the difference equation to a first-order differential equation. The result is (Σ_s is taken to be independent of lethargy)

$$\Sigma_t(u)\ \phi(u) \simeq \zeta(u) + \Sigma_s\phi(u) - \xi\Sigma_s\ \phi'(u) \tag{4.176}$$

which has the solution

$$\phi(u) = \frac{1}{\xi\Sigma_s}\int_0^u \zeta(u') \exp\left[-\int_{u'}^u \frac{\Sigma_a(w)\ dw}{\xi\Sigma_s}\right] du' \tag{4.177}$$

It is seen that this result is in good agreement with the relation (4.159). Note that the resonance-escape probability, given by the exponential of (4.177), is defined in terms of the scattering cross section, whereas the form in (4.159) involves the total cross section. The discrepancy here is not significant however, inasmuch as (4.159) applies to media with slight absorption. This is also the case in the present calculation; otherwise we could not have assumed that the flux was a slowly varying function over a single collision interval.

4.7 Thermal-group Cross Sections

a. Thermal Motion of Neutrons and Nuclei. In all the preceding analyses dealing with the interaction between neutrons and nuclei it was assumed that the nuclei were stationary. Actually, the nuclei in any material at a temperature above absolute zero will be in thermal motion. It is known from the kinetic theory[1] that the average energy associated with the translational motion of such particles is given by $\frac{3}{2}kT$ (where k is Boltzmann's constant and T is the absolute temperature of the medium). So long as the neutron speeds are much greater than those of the nuclei (i.e., the neutron energy is much greater than $\frac{3}{2}kT$), there is no important effect of the nuclear motion, and this is the case for materials at ordinary temperatures (20 to 1000°C) and neutrons with energies of a few electron volts or greater. However, when neutron speeds are comparable to nuclear speeds, the nuclear motion cannot be ignored.

Nuclear motion becomes important, then, when neutron energies are in the order of kT ($kT = 0.025$ ev for $T = 20$°C). The effects of the nuclear motion are significant in connection with two different aspects of the interaction phenomenon, namely, the frequency function for scattering collision and the specification of neutron cross sections for neutron energies around kT. It will be recalled that the frequency functions for

[1] See, for example, J. E. Mayer and M. G. Mayer, "Statistical Mechanics," pp. 1–17, John Wiley & Sons, Inc., New York, 1948.

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collision used in the previous calculations [see Eqs. (4.39) and (4.50)] were defined for stationary nuclei; these functions do not apply, therefore, to scattering collisions by *thermal neutrons* (neutrons of energy around kT). In fact, a scattering collision experienced by a thermal neutron will result in an *increase* in its kinetic energy in a large fraction of the collisions. Because of the thermal motion of the nuclei, some energy can be transferred from the nucleus to the neutron, thereby increasing the neutron speed. This is in contrast to collisions between neutrons and stationary nuclei which always result in a decrease in the energy of the neutron (and an increase in the energy of the nucleus).

Elastic-scattering collisions between thermal neutrons and nuclei do not, on the average, raise or lower the kinetic energy of the neutron. Once a neutron has been slowed to the thermal-energy range, it wanders through the medium from one scattering collision to another, sometimes gaining energy, sometimes losing energy, until it is finally absorbed (or escapes from the medium if the system is finite). The effect of the nuclear motion on the slowing-down process is to decrease the average logarithmic energy loss (decrease ξ) as the kinetic energy of the neutron decreases toward kT; thus, the neutron actually makes more collisions in a small energy interval (a few kT) above $E = kT$ than would be predicted by a theory which treated ξ as a constant in energy. The use of a constant ξ, then, underestimates the number of absorptions in a small energy interval at low energies, just where the absorption cross sections are usually high; this may, in part, be compensated for by the fact that the number of fissions in the same interval is also underestimated.

The second effect of the nuclear motion appears in the definition of the neutron cross sections for thermal neutrons and the specification of the neutron and nuclear densities as a function of energy. In the case of media with stationary nuclei the cross sections could be readily defined inasmuch as the *relative speed* between neutron and nucleus was the *absolute* speed of the neutron. When the nuclei are in motion, however, the neutron cross sections must be expressed in terms of the relative speed between the two moving particles. Thus in computing collision densities for thermal neutrons and nuclei it is necessary to use the relative speed, and this requires a complete description of the motion of both neutrons *and* nuclei. In general, these motions are extremely complicated, and detailed descriptions are difficult to specify and even more difficult to apply.

The behavior of neutrons at the lower end of the energy scale is much different, then, from the behavior in the high-energy range. Because of the basic difference in the nature of the two phenomena, slowing-down and thermal motion, it is convenient to separate the neutron trajectory in energy space into two parts. The first part deals with the slowing-down process from the source energy down to the thermal range, and the

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second, with the behavior of the neutrons around $E = kT$. The slowing-down process has been studied in detail in the preceding sections, and various models for treating this portion of the neutron trajectory have been developed. The problem which remains, and to which we now address ourselves, is the selection of suitable functions for describing the energy distribution of neutrons and nuclei in the thermal-energy range. Given these distributions, expressions for the nuclear-reaction rates can be readily obtained.

To a fair approximation the physical situation in the thermal-energy range may be regarded as that of two gases (neutrons and nuclei) in thermal equilibrium. On the basis of this picture it is possible to derive by the methods of the kinetic theory the functional form of the distribution functions for both the neutrons and nuclei. The present derivation of these functions follows the treatment of Chapman and Cowling.[1]

Consider, then, an infinite medium consisting of neutrons and nuclei in thermal equilibrium, and for convenience let us assume that only one nuclear species is present. Further, let us assume that only scattering reactions occur between the neutrons and nuclei so that no neutrons are lost from the system by absorption. Let the distribution of neutrons and the nuclei be given by the functions

$$\begin{aligned} \mathfrak{m}(\mathbf{v})\, d\mathbf{v} &\equiv \text{fraction of neutrons in unit volume whose velocities lie between } \mathbf{v} \text{ and } \mathbf{v} + d\mathbf{v} \\ \mathfrak{M}(\mathbf{V})\, d\mathbf{V} &\equiv \text{fraction of nuclei in unit volume whose velocities lie between } \mathbf{V} \text{ and } \mathbf{V} + d\mathbf{V} \end{aligned} \tag{4.178}$$

The derivation of the functions $\mathfrak{m}$ and $\mathfrak{M}$ is developed from the neutron-balance relation for all reactions which can affect the density of neutrons in a differential element $d\mathbf{v}$. For this purpose we require two additional functions, namely,

$$\begin{aligned} \boldsymbol{\varrho} &\equiv \mathbf{v} - \mathbf{V} \equiv \text{relative velocity between neutron and nucleus} \\ \sigma(|\mathbf{v} - \mathbf{V}|,\omega)\, d\omega &\equiv \text{microscopic scattering cross section for a neutron of velocity } \mathbf{v} \text{ and a nucleus of velocity } \mathbf{V} \text{ which will rotate relative velocity vector } \boldsymbol{\varrho} \text{ through angle } \omega \\ &\equiv \sigma(\rho,\omega) \qquad \text{with } \rho \equiv |\boldsymbol{\varrho}| \end{aligned} \tag{4.179}$$

The vector diagram for a typical scattering collision in the laboratory system of coordinates is shown in Fig. 4.26. The unprimed quantities refer to the velocities before collision, the primed, after collision. As indicated in the definition above for σ, ω is the angle between the relative velocity vector before collision $\boldsymbol{\varrho}$ and the relative velocity vector after collision $\boldsymbol{\varrho}'$.

[1] Sydney Chapman and T. G. Cowling, "The Mathematical Theory of Non-uniform Gases," 2d ed., pp. 374–377, Cambridge University Press, New York, 1952.

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We begin the neutron-balance calculation by computing first the total number of neutrons scattered out of the differential element $d\mathbf{v}$. Since the speeds of the nuclei are comparable to those of the neutrons, it is necessary to compute the scattering densities in terms of the relative speed between each pair of colliding particles. Thus for each nuclear velocity $\mathbf{V}$ there is a corresponding relative speed ρ and a cross section $\sigma(\rho,\omega)$. In order to compute the total scatterings out of $d\mathbf{v}$ we integrate the product of $\mathfrak{m}(\mathbf{v})\,d\mathbf{v}$ and $\rho\sigma\mathfrak{M}$ over all $\mathbf{V}$ and ω; thus,

$$\text{Number of neutrons scattered out of element } d\mathbf{v} \text{ per unit time} = \int_{\mathbf{V},\omega} \mathfrak{m}(\mathbf{v})\,d\mathbf{v}\,\rho\,\sigma(\rho,\omega)\,\mathfrak{M}(\mathbf{V})\,d\mathbf{V}\,d\omega \quad (4.180)$$

In order to complete the neutron-balance relation for steady state, we require also the rate at which neutrons are scattered into the element $d\mathbf{v}$. For this purpose we introduce the density of nuclei of any velocity $\mathbf{V}'$ before collision $\mathfrak{M}(\mathbf{V}')$ and the density of neutrons of some velocity $\mathbf{v}'$ before collision $\mathfrak{m}(\mathbf{v}')$. It is important to recognize that the velocity $\mathbf{v}'$ is not entirely arbitrary since given $\mathbf{V}'$, a scattering angle ω', and a final velocity $\mathbf{v}$ there exists only one velocity $\mathbf{v}'$ which the neutron may possess if the neutron is to emerge from the collision with velocity in $d\mathbf{v}$ about $\mathbf{v}$. Or saying it in another way, given $\mathbf{V}'$, ω', and $\mathbf{v}$, $\mathbf{v}'$ is completely defined. This may be seen from the figure. Thus to compute the total number of scatterings into $d\mathbf{v}$ we sum over all possible $\mathbf{V}'$ and ω' and select each $\mathbf{v}'$ such that

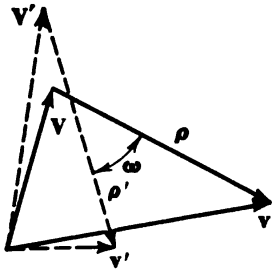

FIG. 4.26 Velocity vectors before and after collision in a laboratory system of coordinates.

$$\mathbf{v}' \equiv \mathbf{v}'(\mathbf{v},\mathbf{V}',\omega') \quad (4.181)$$

causes the emergent neutron to appear with velocity in $d\mathbf{v}$ about $\mathbf{v}$. This sum is given by the integral

$$\int_{\mathbf{v}',\omega'} \mathfrak{m}[\mathbf{v}'(\mathbf{v},\mathbf{V}',\omega')]\,d\mathbf{v}'\,\rho'\sigma(\rho',\omega')\,\mathfrak{M}(\mathbf{V}')\,d\mathbf{V}'\,d\omega' \quad (4.182)$$

Note that

$$\sigma(\rho',\omega')\,d\omega' \equiv \text{microscopic scattering cross section for a neutron of velocity } \mathbf{v}' \text{ and a nucleus of velocity } \mathbf{V}' \text{ which will rotate relative velocity vector } \boldsymbol{\rho}' \text{ through angle } \omega' \quad (4.183)$$

Also

$$\sigma(\rho,\omega) \equiv \sigma(\rho',\omega') \quad (4.184)$$

which states the physical fact that the scattering cross section to rotate the vector $\boldsymbol{\rho}$ (corresponding to $\mathbf{v}$ and $\mathbf{V}$) through angle ω into $\boldsymbol{\rho}'$ must be the same as that which rotates $\boldsymbol{\rho}'$ (due to $\mathbf{v}'$ and $\mathbf{V}'$) through ω' into $\boldsymbol{\rho}$. In the present generalized notation, $\omega \equiv -\omega'$.

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The next step is to transform the integral (4.182) by a change of integration variables. Thus we write (4.182) in the form

$$\int_{\mathbf{V},\boldsymbol{\omega}} \mathfrak{m}(\mathbf{v}')\, d\mathbf{v}\, \rho\, \sigma(\rho,\omega)\, \mathfrak{M}(\mathbf{V}')\, d\mathbf{V}\, d\boldsymbol{\omega} \equiv \begin{matrix}\text{number of neutrons scat-}\\ \text{tered into } d\mathbf{v} \text{ per unit time}\end{matrix} \tag{4.185}$$

Note that we still denote velocity vectors before collision by the symbols $\mathbf{v}'$ and $\mathbf{V}'$; however, the integration variables which correspond to the coordinates of $\mathbf{v}'$ and $\mathbf{V}'$ have been changed from the primed symbols to the unprimed; thus, $d\mathbf{v}' \to d\mathbf{v}$, $d\mathbf{V}' \to d\mathbf{V}$, and $d\boldsymbol{\omega}' \to d\boldsymbol{\omega}$. Also, observe that

$$d\mathbf{v}'\, d\mathbf{V}'\, d\boldsymbol{\omega}' \equiv d\mathbf{v}\, d\mathbf{V}\, d\boldsymbol{\omega} \tag{4.186}$$

In transforming from (4.182) to (4.185) we have used the fact that $\rho \equiv \rho'$; i.e., the relative speed between neutron and nucleus remains unchanged through an elastic-scattering collision. The proof of this statement follows directly from the results obtained in Sec. 4.1b. We note first that in terms of the velocities in the center-of-mass system (C), the relative velocities may be written [using (4.6) and (4.7)]

$$\boldsymbol{\rho} \equiv \mathbf{w} - \mathbf{W} \qquad \boldsymbol{\rho}' \equiv \mathbf{w}' - \mathbf{W}' \tag{4.187}$$

If we square each of these expressions, subtract ρ^2 from ρ'^2, and apply the results (4.10) and (4.11), it follows immediately that

$$\rho = \rho' \tag{4.188}$$

At steady state the expressions (4.180) and (4.185) must be equal. Since the integration variables are the same we may write the balance condition in the form

$$\int_{\mathbf{V},\boldsymbol{\omega}} [\mathfrak{m}(\mathbf{v}')\, \mathfrak{M}(\mathbf{V}') - \mathfrak{m}(\mathbf{v})\, \mathfrak{M}(\mathbf{V})]\rho\, \sigma(\rho,\omega)\, d\mathbf{v}\, d\mathbf{V}\, d\boldsymbol{\omega} = 0 \tag{4.189}$$

A physically acceptable solution to this equation is

$$\mathfrak{m}(\mathbf{v}')\, \mathfrak{M}(\mathbf{V}') \equiv \mathfrak{m}(\mathbf{v})\, \mathfrak{M}(\mathbf{V}) \tag{4.190}$$

The next step in the analysis is to define two new functions f and F such that

$$f(\mathbf{v}) \equiv \ln \mathfrak{m}(\mathbf{v}) \qquad F(\mathbf{V}) \equiv \ln \mathfrak{M}(\mathbf{V}) \tag{4.191}$$

If we use these in (4.190), it follows that

$$f(\mathbf{v}') + F(\mathbf{V}') \equiv f(\mathbf{v}) + F(\mathbf{V}) \tag{4.192}$$

which states that there exists some *scalar* function of the velocities before and after collision which is unchanged by the collision. The only scalar invariant in the present system is the total kinetic energy of the colliding particles. Thus the functions f and F may be immediately identified with the kinetic energies of the neutron and nucleus, respectively.

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Therefore, we set

$$f(\mathbf{v}) \equiv a(\tfrac{1}{2}mv^2) \qquad F(\mathbf{V}) \equiv a(\tfrac{1}{2}MV^2) \tag{4.193}$$

where a is some constant, m the neutron mass, and M the nuclear mass.

Finally, we substitute these expressions into (4.191) and obtain

$$\mathfrak{m}(\mathbf{v}) \equiv be^{a\frac{1}{2}mv^2} \qquad \mathfrak{M}(\mathbf{V}) \equiv Be^{a\frac{1}{2}MV^2} \tag{4.194}$$

The coefficients b and B are simply normalization factors and are easily computed by setting the integrals of $\mathfrak{m}$ and $\mathfrak{M}$ over $\mathbf{v}$ and $\mathbf{V}$, respectively, equal to unity. The determination of the quantity a which appears in the above equations is accomplished with the aid of certain thermodynamic relations. The computation is straightforward but of no particular interest to the present study. Without further comment, then, we note that it can be shown that[1]

$$a \equiv \frac{1}{kT} \tag{4.195}$$

where k is the Boltzmann constant and T is the temperature of the medium. If we use this result in (4.194) and carry out the normalization of the functions $\mathfrak{m}$ and $\mathfrak{M}$, we obtain

$$\begin{aligned} \mathfrak{m}(\mathbf{v}) &= \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-mv^2/2kT} \\ \mathfrak{M}(\mathbf{V}) &= \left(\frac{M}{2\pi kT}\right)^{\frac{3}{2}} e^{-MV^2/2kT} \end{aligned} \tag{4.196}$$

which are the well-known *Maxwell-Boltzmann* distributions. There are two essential properties of these functions which are of particular interest to the present development. (1) The temperature parameter which appears in each of the exponentials is the same quantity for both the distribution of the neutrons and of the nuclei. (2) These functions are of Gaussian shape. When the functions (4.196) are plotted against speed, they will exhibit peaks around the value of the speed corresponding to the energy kT, and with increasing values of temperature these peaks will shift to higher values of the speed. Thus higher temperatures correspond to higher average speeds for both neutron and nuclei.

The fact that the same parameter T appears in both distributions given in (4.196) is a direct consequence of the assumption that the medium in question is a *purely scattering* material. We know, however, that many media of practical importance to reactor technology possess appreciable absorption cross sections. This is especially true, of course, for media comprising the core of thermal reactors. In these situations the results (4.196) are not generally valid, and some modification must be made in

[1] Mayer and Mayer, *op. cit.*, pp. 8–14.

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the definitions of the distribution functions which accounts for the effects of the absorption losses on the neutron spectrum. The principal effect of the absorption collisions is to depress the neutron population over the energy interval wherein the absorption cross section is relatively large. Since for most materials the microscopic absorption cross section varies more or less as $1/v$, this means that the neutron densities are increasingly reduced, from the pure scattering media values, as one progresses down the energy (speed) scale. As a result, the neutron distribution $\mathfrak{m}$ is distorted and tends to peak at higher values of the neutron speed corresponding to the range wherein the absorption cross section is relatively small. This shift in the peak, and distortion in the shape, of the neutron-distribution function is called *hardening* of the spectrum, and it is the more pronounced, the greater the value of Σ_a. These trends are illustrated in Fig. 4.27.

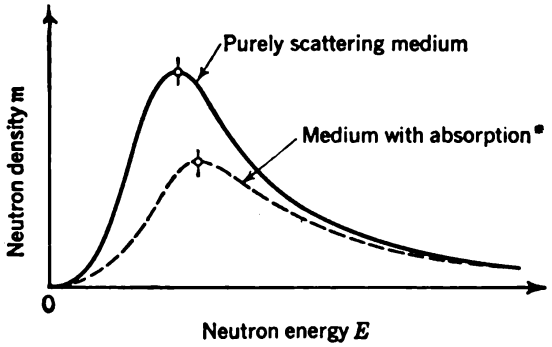

FIG. 4.27 Distortion in thermal-neutron spectrum due to the presence of absorbing materials (*normalized so that each system contains the same total number of neutrons).

One way in which this shift in the neutron density may be taken into account is by introducing an *effective neutron temperature* T_n. Thus we define some fictitious temperature T_n which, when used in the function $\mathfrak{m}$ of (4.196), gives a new Gaussian that yields a better fit to the distorted density distribution arising from the presence of the absorber. The determination of this parameter is deferred until a later section. For the present, all we require is that such a number can be specified for a given system.

The results (4.196) may now be written in terms of this parameter. It will be convenient for subsequent treatments to introduce also a temperature T_N which describes the thermal state of the nuclei in the system. Thus a suitable set of functions for the neutron and nuclear densities in an absorbing medium may be given by

$$\mathfrak{m}(\mathbf{v})\, d\mathbf{v} \equiv \text{fraction of neutrons in thermal range whose } \textit{velocities} \text{ lie between } \mathbf{v} \text{ and } \mathbf{v} + d\mathbf{v}$$
$$\equiv \left(\frac{m}{2\pi k T_n}\right)^{\frac{3}{2}} e^{-m\mathbf{v}^2/2kT_n}\, d\mathbf{v} \qquad (4.197)\star$$

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$$\mathfrak{M}(\mathbf{V})\,d\mathbf{V} \equiv \frac{\text{fraction of nuclei whose } \textit{velocities} \text{ lie between } \mathbf{V} \text{ and } \mathbf{V} + d\mathbf{V}}{} \equiv \left(\frac{M}{2\pi kT_N}\right)^{\frac{3}{2}} e^{-MV^2/2kT_N}\,d\mathbf{V} \tag{4.198}\star$$

It should be noted that these distributions are defined in terms of *velocity space*, which is not to be confused with *speed space*.[1] The distributions in velocity space are more detailed than those in speed space inasmuch as they specify not only the neutron speed but also its direction of motion. The transformation from the velocity function to the corresponding speed function is readily obtained by integration. Thus in the case of the neutron distribution (4.197) we compute the corresponding function in speed space by integrating (4.197) over all directions of motion[2] Ω. If we define

$$\mathfrak{m}(v)\,dv \equiv \frac{\text{fraction of neutrons whose } \textit{speeds} \text{ lie between } v \text{ and } v + dv}{} \tag{4.199}$$

then

$$\mathfrak{m}(v)\,dv \equiv \int_{\Omega} \mathfrak{m}(\mathbf{v})\,d\mathbf{v} = v^2\,dv \int_{\Omega} \mathfrak{m}(\mathbf{v})\,d\Omega = 4\pi v^2 \left(\frac{m}{2\pi kT_n}\right)^{\frac{3}{2}} e^{-mv^2/2kT_n}\,dv \tag{4.200}$$

Also, if n_{th} is the total number of thermal neutrons per unit volume, then

$$n(v)\,dv = n_{th}\,\mathfrak{m}(v)\,dv = \frac{\text{number of neutrons per unit volume whose speeds lie between } v \text{ and } v + dv}{} \tag{4.201}$$

b. Definition of Thermal-group Cross Section. The Maxwell-Boltzmann distributions (4.197) and (4.198) may be used to construct suitable average cross sections for the neutrons in the thermal-energy range. These average cross sections will be useful in subsequent analyses wherein the entire thermal neutron population is represented by the one-velocity model. If Σ^{th} denotes some average cross section and ϕ_{th} the total track length per unit time (in some specified volume) of the thermal group, then a convenient definition of the group cross section is given by

$$\Sigma^{th}\phi_{th} \equiv \int_{\mathbf{v}} n_{th}\,\mathfrak{m}(\mathbf{v})\,d\mathbf{v} \int_{\mathbf{V}} \sigma^{\infty}(|\mathbf{v} - \mathbf{V}|)|\mathbf{v} - \mathbf{V}|N\,\mathfrak{M}(\mathbf{V})\,d\mathbf{V} \tag{4.202}$$

where N is the number of nuclei per unit volume, n_{th} the number of thermal neutrons per unit volume, and $\mathfrak{M}$ and $\mathfrak{m}$ are given by (4.197) and (4.198). The symbol $\sigma^{\infty}(|\mathbf{v} - \mathbf{V}|)$, which represents the neutron-nucleus

[1] *Ibid.*, p. 14.

[2] The symbol Ω denotes a unit vector in actual space (i.e., a direction) and can be specified by giving two angles, a co-latitude ϑ and an azimuth ψ; therefore, $\Omega \equiv \Omega(\vartheta,\psi)$, and the differential solid angle (i.e., area on surface of unit sphere) is $d\Omega \equiv \sin\vartheta\,d\vartheta\,d\psi$.

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interaction phenomenon of interest, is a function only of the relative speed between the colliding particles. It is *not*, in general, the cross section measured by the experimenter in the laboratory. The relationship between the measured σ and the quantity which appears in the above equation is easily exhibited upon introducing two additional definitions and rearranging the form (4.202). Let

$$\Sigma^{\text{th}} \equiv \sigma^{\text{th}} N \tag{4.203}$$

$$\phi_{\text{th}} \equiv n_{\text{th}} \int_{\mathbf{v}} v\, \mathfrak{m}(\mathbf{v})\, d\mathbf{v} \tag{4.204}$$

then, (4.202) reduces to

$$\sigma^{\text{th}} = \frac{\int_{\mathbf{v}} \mathfrak{m}(\mathbf{v})\, d\mathbf{v} \int_{\mathbf{V}} \sigma^{\infty}(|\mathbf{v} - \mathbf{V}|)|\mathbf{v} - \mathbf{V}|\, \mathfrak{M}(\mathbf{V})\, d\mathbf{V}}{\int_{\mathbf{v}} v\, \mathfrak{m}(\mathbf{v})\, d\mathbf{v}} \tag{4.205}$$

The $\mathbf{V}$ integral in this expression is in fact the quantity obtained by the experimenter. Thus we identify the reaction rate $v\sigma(v)$ as the integral

$$v\sigma(v) \equiv \int_{\mathbf{V}} \sigma^{\infty}(|\mathbf{v} - \mathbf{V}|)|\mathbf{v} - \mathbf{V}|\, \mathfrak{M}(\mathbf{V})\, d\mathbf{V} \tag{4.206}\star$$

The physical interpretation of this relation is discussed in the section which follows. If we use this definition in (4.205), we obtain

$$\sigma^{\text{th}} = \frac{\int_{\mathbf{v}} v\sigma(v)\, \mathfrak{m}(\mathbf{v})\, d\mathbf{v}}{\int_{\mathbf{v}} v\, \mathfrak{m}(\mathbf{v})\, d\mathbf{v}} \tag{4.207}\star$$

This result may be written in an alternative form by separating the integration over velocity space into an integration over speed space and an integration over all directions of neutron motion $\boldsymbol{\Omega}$. Then, for example,

$$\int_{\mathbf{v}} v\sigma(v)\, \mathfrak{m}(\mathbf{v})\, d\mathbf{v} = \int_0^{\infty} v^2 \sigma(v)\, dv \int_{\boldsymbol{\Omega}} \mathfrak{m}(\mathbf{v})\, d\boldsymbol{\Omega} \equiv \int_0^{\infty} v\sigma(v)\, \mathfrak{m}(v)\, dv \tag{4.208}$$

where we have used the definition (4.200) for the neutron distribution in speed space $\mathfrak{m}(v)$. If a similar operation is performed on the denominator, then (4.207) reduces to

$$\sigma^{\text{th}} = \frac{\int_0^{\infty} v\sigma(v)\, \mathfrak{m}(v)\, dv}{\int_0^{\infty} v\, \mathfrak{m}(v)\, dv} \tag{4.209}\star$$

We find then that the computation of the thermal-neutron group cross sections may be carried out by integrating the *measured* cross-section data weighted by the neutron distribution in speed space (4.200) over the range $0 \leq v \leq \infty$. It would appear that the details of the computation introduce some ambiguity from a physical viewpoint since the integration

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extends over the entire speed range. Conceivably, a thermal neutron could acquire extremely large speeds by suffering successive scattering collisions each of which caused an increase in its kinetic energy. This situation might raise some question as to how this neutron is to be classified, i.e., is it a thermal neutron or is it a "fast" neutron. From a practical standpoint, however, this causes no complication inasmuch as the number of such high-speed neutrons is negligibly small due to the Gaussian form of the function $\mathfrak{m}(v)$.

c. Reaction Rates for Neutrons and Nonstationary Nuclei. We have already noted that the cross sections measured in the laboratory are the quantities $\sigma(v)$, and these are related to the interaction functions $\sigma^{\infty}(|\mathbf{v} - \mathbf{V}|)$ through the integral equation (4.206). Actually, the precise forms of the functions σ^{∞} are not generally known. The only information about a specific reaction rate which is obtained from a cross-section experiment is the integral quantity $\sigma(v)$. In fact, the experiment itself consists in performing the integration indicated in (4.206). This may be more easily recognized if one recalls the experimental procedure involved in making a cross-section measurement. A brief description of this procedure was given in Sec. 2.4c. Such measurements are performed for neutrons of various speeds (energies) by irradiating a sample of the material in question with a monoenergetic beam. For each neutron speed, the experimenter obtains a measurement of the beam intensity, with and without the specimen. The difference between these intensities, which is usually given in terms of the *reaction rate* $R(v)$, where

$$R(v) \equiv v\sigma(v) = \int_{\mathbf{V}} \sigma^{\infty}(|\mathbf{v} - \mathbf{V}|)|\mathbf{v} - \mathbf{V}|\, \mathfrak{M}(\mathbf{V})\, d\mathbf{V} \tag{4.210}$$

is attributed to the reactions in the specimen. Thus neutrons of a particular speed (established by the beam) are allowed to interact with the nuclei of the sample. These nuclei, however, are in a random motion determined by the temperature of the material. Consequently the experiment consists, ultimately, of measuring the interaction between neutrons of a given speed and nuclei of all speeds. The integrated effect of all these interactions is the quantity $\sigma(v)$, and the graphic representation of these functions is the cross-section curves to which we have made frequent reference. Finally, it should be pointed out that Eq. (4.206) is a mathematical statement (based on the model of a two-body collision) of the physical process involved in performing the experiment.

We have noted that the reaction rate R given by (4.210) is dependent upon the nuclear composition of the specimen and upon the neutron speed. There is a third factor, however, which can affect this function significantly, namely, the temperature of the material. In the present model, the influence of the temperature upon the reaction rate is seen through the distribution function $\mathfrak{M}(\mathbf{V})$, which appears in (4.210). Thus,

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properly, R should be written as $R(v;T_N)$; we will make this distinction later when this effect is examined in more detail. For the present it will suffice to point out the physical significance of this dependence.

It is customary when making cross-section measurements to perform the experiment at standard temperature (20°C). However, thermal-group cross-section data will be required, in general, at temperatures different from this value, depending upon the operating conditions of the reactor in question. Thus, if an accurate calculation of the σ^{th} is required, this difference must be taken into account. This means, then, that a procedure must be developed for determining the reaction rate R at the temperature of interest from the available data which has been obtained at some known reference temperature. The temperature-corrected rate R may then be used in (4.209) to compute the corresponding σ^{th}. A suitable procedure for generating these data from the laboratory measurements is discussed in Sec. 4.7e.

d. Effective Neutron Temperature. The neutron distributions around thermal energy in both velocity and speed space involved an effective neutron temperature T_n. This number appears as a parameter in the functions (4.197) and (4.200) and establishes the position and shape of the Maxwell-Boltzmann distribution which has been assumed to describe the thermal-neutron population. The calculation of this quantity has received considerable attention from various investigators. For the present discussion we draw from the work of Coveyou, Bate, and Osborn previously cited in connection with the calculation of collision densities near source energy (see Sec. 4.4d).

The work of Coveyou, Bate, and Osborn was aimed at determining the effect of moderator temperature upon the neutron flux in an infinite capturing medium. The calculations were carried out for a uniform medium of unbound nuclei of constant scattering cross section and $1/v$ absorption cross section. It was assumed that the nuclei were distributed according to the Maxwell-Boltzmann relation (4.198) and that neutrons were introduced into the system from a monoenergetic source. The details of the slowing-down process and the scattering around thermal energy were determined with the aid of the Monte Carlo method. A brief description of the mechanics involved in the use of this method was given in Sec. 4.4d. Again, we will not enter into a discussion of the method but will confine our attention to the use of the results which were obtained.

The Monte Carlo calculations carried out for the infinite-medium model described above were used to determine the distribution of neutrons around thermal energy. These distributions were obtained by tracing the individual life histories of many neutrons. The density of neutrons and the various collision densities at each energy were computed from the statistics compiled from these individual histories. Some of the

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results obtained by this method are presented in Fig. 4.28. As in the collision-density calculation described in Sec. 4.4d, these results are given in terms of the speed ratio $x \equiv \delta v/v_T$, where v is the neutron speed, δ a scale factor, and $v_T \equiv (3kT_N/m)^{\frac{1}{2}}$. The curves shown in Fig. 4.28 give the neutron flux $\phi(x) \equiv n(x)x$ for media of various mass number A and absorption cross section (the parameter κ is proportional to Σ_a, see Sec. 4.4d). The solid-line curves in the figure are the neutron fluxes obtained from the Monte Carlo calculation. The broken-line curves are the Maxwell-Boltzmann fluxes that would be obtained in infinite media of the same mass number and temperature but which contained no sources or sinks. The systems described by the two curves in each of the three cases

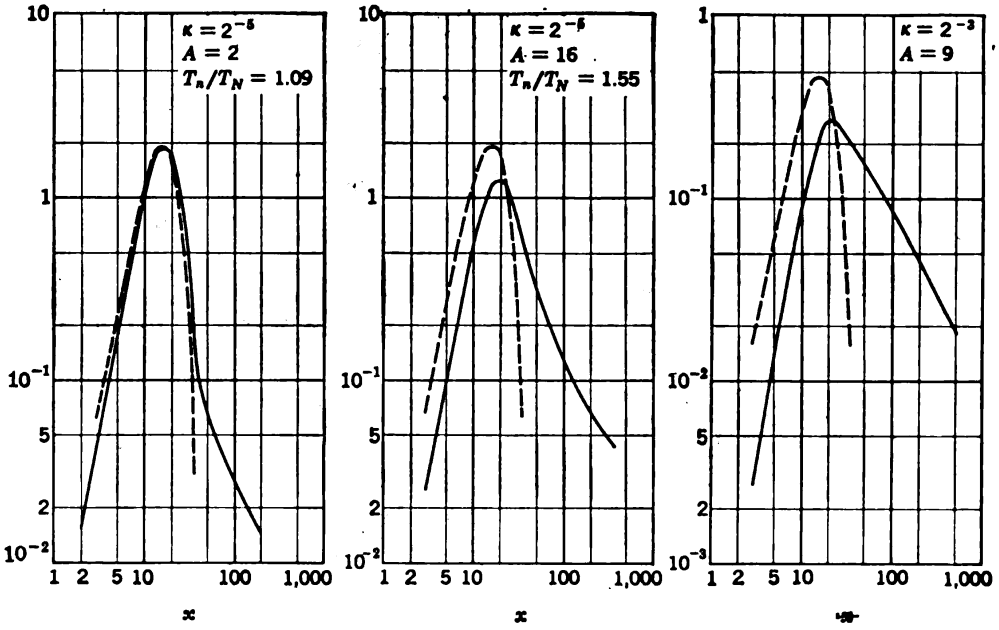

FIG. 4.28 Neutron flux in the thermal range.

presented contain the same total number of neutrons per unit volume; thus, in the thermal range, the distortion of the solid-line curves relative to the broken-line curves is a measure of the effect of the $(1/v)$ absorber on the neutron speed distribution.

A useful measure of the flux distortion is provided by the concept of the effective neutron temperature T_n. We define this temperature as that number which when used in (4.197) gives the best least-squares fit of a Maxwell-Boltzmann distribution to the computed flux (solid-line curve) in the range[1] $0 \leq x \leq 35$. The ratio of the effective neutron temperature to the moderator temperature T_n/T_N is indicated in the figure for the first two cases. The ratio has been omitted from the last case ($A = 9$, $\kappa = 2^{-3}$) because the flux was so severely distorted from a

[1] The width of this range was selected fairly arbitrarily. The intent was to exclude the $(1/E)$ region of the flux.

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Maxwell-Boltzmann distribution that the concept of an effective neutron temperature was meaningless. It was found that the linear relation

$$T_n = T_N(1 + 1.11A\kappa) \tag{4.211}\star$$

correlated fairly well[1] all the cases reported in the reference paper. This relation provides then a crude interpolative device for estimating the parameter T_n from a knowledge of the moderator temperature T_N, its mass number A, and its absorption properties κ. This result may be used in the relations (4.197) and (4.200). A more accurate estimate of the neutron distribution than can be obtained from this adjusted Maxwell-Boltzmann relation would require a detailed calculation of the Monte Carlo type or a solution of the Boltzmann equation.

e. Temperature Corrections to Cross-section Data. In many practical cases of interest the operating temperature of the reactor will be above the temperature at which the available cross-section data were measured. For these situations it is possible to develop a procedure for estimating the reaction rates at the higher temperature from the measured data at the lower temperature. In the analysis which follows[2] we derive first a general relationship between the reaction rates for neutrons of a given speed with a medium at two different temperatures. This is followed by an approximate treatment which is valid when the two temperatures of interest differ only slightly.

We begin by stating the integral relation for the reaction rate in the more general notation previously mentioned; thus, in place of (4.210), we introduce

$$R(v;T_N) = \int_{\mathbf{V}} \sigma^{\infty}(|\mathbf{v} - \mathbf{V}|)|\mathbf{v} - \mathbf{V}|\, \mathfrak{M}(\mathbf{V})\, d\mathbf{V} \tag{4.212}\star$$

where $\mathfrak{M}$ is given by (4.198). For convenience in notation we define also

$$\mathfrak{M}(\mathbf{V},\theta) \equiv \frac{e^{-V^2/4\theta}}{(4\pi\theta)^{\frac{3}{2}}} \qquad \theta \equiv \frac{kT}{2M} \tag{4.213}$$

then with

$$S(\rho) \equiv \rho\sigma^{\infty}(\rho) \qquad \rho \equiv |\boldsymbol{\rho}| = |\mathbf{v} - \mathbf{V}| \tag{4.214}$$

(4.212) takes the form

$$\begin{aligned} R(v;\theta_N) &= \int_{\boldsymbol{\rho}} S(\rho)\, \mathfrak{M}(\mathbf{v} - \boldsymbol{\rho}, \theta_N)\, d\boldsymbol{\rho} \\ &= \frac{1}{(4\pi\theta_N)^{\frac{3}{2}}} \int_{\boldsymbol{\rho}} S(\rho)\, e^{-|\mathbf{v}-\boldsymbol{\rho}|^2/4\theta_N}\, d\boldsymbol{\rho} \end{aligned} \tag{4.215}$$

In deriving the relationship between the reaction rate at one temperature $R(v;\theta_N)$ with that at a higher temperature $R(v;\theta'_N)$, where $\theta'_N > \theta_N$,

[1] To within 5 per cent, for the cases considered. This relation may be used for systems wherein $A\kappa < 0.2$.

[2] This relationship was pointed out to the authors by R. R. Coveyou.

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we will work with the Fourier transforms of these functions rather than with the actual functions. For this treatment we require the three-dimensional Fourier transform of $R(v;\theta_N)$ with respect to the variable $\mathbf{v}$. We define the one-dimensional Fourier transform of any function $f(x)$ by the relation

$$\mathfrak{F}\{f(x)\} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i\xi x}\, dx \tag{4.216}$$

where ξ is the transform variable. If $\mathbf{s}$ is the three-dimensional transform variable, where the components of $\mathbf{s}$ are given by

$$\mathbf{s} \equiv \mathbf{s}(\xi,\eta,\zeta) \tag{4.217}$$

then the three-dimensional transform of R is

$$\mathfrak{F}\{R(v;\theta_N)\} \equiv \mathfrak{R}(s;\theta_N) = \frac{1}{(2\pi)^{\frac{3}{2}}} \int_{\mathbf{v}} R(v;\theta_N) e^{i\mathbf{s}\cdot\mathbf{v}}\, d\mathbf{v} \tag{4.218}$$

The substitution of R from (4.215) into this relation yields

$$\mathfrak{R}(s;\theta_N) = \frac{1}{(8\pi^2\theta_N)^{\frac{3}{2}}} \int_{\boldsymbol{\varrho}} S(\rho)\, d\boldsymbol{\varrho} \int_{\mathbf{v}} \exp\left[-\frac{1}{4\theta_N} |\mathbf{v} - \boldsymbol{\varrho}|^2 + i\mathbf{s}\cdot\mathbf{v} \right] d\mathbf{v} \tag{4.219}$$

If we define the components of the velocity vectors $\mathbf{v}$ and $\boldsymbol{\varrho}$ by

$$\mathbf{v} \equiv \mathbf{v}(x,y,z) \qquad \boldsymbol{\varrho} \equiv \boldsymbol{\varrho}(a,b,c) \tag{4.220}$$

then the exponent in (4.219) may be expanded into the form

$$\begin{aligned} -\frac{1}{4\theta_N} |\mathbf{v} - \boldsymbol{\varrho}|^2 + i\mathbf{s}\cdot\mathbf{v} = &\left[-\frac{1}{4\theta_N} (x-a)^2 + i\xi x \right] \\ &+ \left[-\frac{1}{4\theta_N} (y-b)^2 + i\eta y \right] + \left[-\frac{1}{4\theta_N} (z-c)^2 + i\zeta z \right] \end{aligned} \tag{4.221}$$

Now, the x, y, and z integrations implied in $d\mathbf{v}$ are identical, mathematically. Consider, therefore, the x integration.

$$\begin{aligned} \int_{-\infty}^{\infty} \exp\left[-\frac{1}{4\theta_N} (x-a)^2 + i\xi x \right] dx \\ = (4\pi\theta_N)^{\frac{1}{2}} \exp\left[-\frac{a^2}{4\theta_N} + \theta_N \left(\frac{a}{2\theta_N} + i\xi \right)^2 \right] \end{aligned}$$

which is readily obtained by completing the square in the exponent and applying the definition of the error integral. If this operation is repeated for the y and z integrations, it follows immediately that the $\mathbf{v}$ integral of (4.219) reduces to

$$\int_{\mathbf{v}} \exp\left[-\frac{1}{4\theta_N} |\mathbf{v} - \boldsymbol{\varrho}|^2 + i\mathbf{s}\cdot\mathbf{v} \right] d\mathbf{v} = (4\pi\theta_N)^{\frac{3}{2}} e^{i\mathbf{s}\cdot\boldsymbol{\varrho} - s^2\theta_N} \tag{4.222}$$

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The substitution of this result into (4.219) yields

$$\mathfrak{R}(s;\theta_N) = \frac{e^{-s^2\theta_N}}{(2\pi)^{\frac{3}{2}}} \int_{\varrho} S(\rho)\, e^{i\mathbf{s}\cdot\varrho}\, d\varrho \qquad (4.223)\star$$

It is of interest to note that the above integral is in fact the Fourier transform of $\rho\sigma^{\infty}(\rho)$, the reaction rate corresponding to the fundamental interaction function.

The procedure outlined above may be repeated for the reaction rate at the higher temperature $R(v;\theta'_N)$. The Fourier transform of this function is, obviously,

$$\mathfrak{R}(s;\theta'_N) = \frac{e^{-s^2\theta'_N}}{(2\pi)^{\frac{3}{2}}} \int_{\varrho} S(\rho)\, e^{i\mathbf{s}\cdot\varrho}\, d\varrho \qquad (4.224)$$

If we compare this result with (4.223), we find that

$$\mathfrak{R}(s;\theta'_N) = \mathfrak{R}(s;\theta_N)\, e^{-s^2(\theta'_N-\theta_N)} \qquad (4.225)\star$$

Thus we have obtained a simple relationship between the Fourier transforms of the reaction rates at two different temperatures; in order to obtain the relationship between the actual functions, we require only an inversion of Eq. (4.225).

The inversion operation for the Fourier transform is given the symbol $\mathfrak{F}^{-1}\{\ \}$, where, for example,

$$\mathfrak{F}^{-1}\{f(\xi)\} \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)\, e^{-ix\xi}\, d\xi \qquad (4.226)$$

If we apply this operation (in three dimensions) to (4.225), we obtain

$$R(v;\theta'_N) = \mathfrak{F}^{-1}\{\mathfrak{R}(s;\theta_N)\, e^{-s^2(\theta'_N-\theta_N)}\} \qquad (4.227)$$

The right-hand side of this expression involves the p duct of two Fourier transforms. The first of these $\mathfrak{R}(s;\theta_N)$ is of course the transform of $R(v;\theta_N)$. The second is the transform of the Gaussian function in $\mathbf{v}$ space, i.e.,

$$\mathfrak{F}\left\{\frac{\exp\left[-(x^2+y^2+z^2)/4(\theta'_N-\theta_N)\right]}{[4\pi(\theta'_N-\theta_N)]^{\frac{3}{2}}}\right\} = \frac{\exp\left[-(\xi^2+\eta^2+\zeta^2)(\theta'_N-\theta_N)\right]}{(2\pi)^{\frac{3}{2}}} \qquad (4.228)$$

where $v^2 \equiv x^2+y^2+z^2$ and $s^2 \equiv \xi^2+\eta^2+\zeta^2$. This result is easily demonstrated if one notes that

$$\mathfrak{F}^{-1}\{e^{-\beta\xi^2}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\beta\xi^2-i\xi x}\, d\xi = \frac{e^{-x^2/4\beta}}{(2\beta)^{\frac{1}{2}}} \qquad (4.229)$$

Since the relation (4.227) is given as the product of two known transforms, the inversion may be carried out with the aid of the convolution

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theorem:[1] In terms of the present definition of the Fourier transform, this theorem states that

$$\mathfrak{F}^{-1}\{g(\xi)\,h(\xi)\} \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\xi)\,h(\xi)\,e^{-ix\xi}\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(x')\,H(x-x')\,dx' \quad (4.230)$$

where $g(\xi)$ and $h(\xi)$ are the transforms of $G(x)$ and $H(x)$. We now apply this operation to (4.227). Let

$$\mathfrak{M}(s, \theta'_N - \theta_N) \equiv \frac{e^{-s^2(\theta'_N-\theta_N)}}{(2\pi)^{\frac{1}{2}}} \quad (4.231)$$

then by (4.228),

$$\mathfrak{F}^{-1}\{\mathfrak{M}(s, \theta'_N - \theta_N)\} = \mathfrak{M}(v, \theta'_N - \theta_N) \quad (4.232)$$

where $\mathfrak{M}$ is the function (4.213). The convolution theorem yields, then,

$$R(v;\theta'_N) = \int_{\varrho} R(\rho;\theta_N)\,\mathfrak{M}(\mathbf{v}-\boldsymbol{\varrho}, \theta'_N - \theta_N)\,d\boldsymbol{\varrho}$$

and the application of (4.213) gives

$$R(v;\theta'_N) = \int_{\varrho} \frac{R(\rho;\theta_N)\,e^{-|\mathbf{v}-\boldsymbol{\varrho}|^2/4(\theta'_N-\theta_N)}}{[4\pi(\theta'_N-\theta_N)]^{\frac{3}{2}}}\,d\boldsymbol{\varrho} \quad (4.233)\star$$

This equation relates the reaction rate (cross section) of a material at the temperature T'_N to its cross section at a *lower* temperature T_N. As in all the preceding analyses, it is assumed that the nuclei are distributed according to the Maxwell-Boltzmann relation (4.198). The above equation may be used then to compute the cross-section curves for a reactor operating at any temperature T'_N from the known cross-section data of the reactor material which have been determined at some temperature T_N. Note that if $T'_N \equiv T_N$ then Eq. (4.233) reduces to an identity [see also (4.225)]. In general, the indicated integration must be carried out in detail. There is one special case, however, which leads to an especially simple result. If the measured cross-section curve varies as $1/v$, which is the case for many absorbers in the low-energy range, then it is easily shown that the reaction rate is independent of the moderator temperature. For example, if we take $\sigma_a(v) \equiv c_0/v$, where c_0 is some constant, then from (4.233)

$$R(v;\theta'_N) \equiv v\sigma_a(v;\theta'_N) = \int_{\varrho} \frac{\rho(c_0/\rho)\,e^{-|\mathbf{v}-\boldsymbol{\varrho}|^2/4(\theta'_N-\theta_N)}}{[4\pi(\theta'_N-\theta_N)]^{\frac{3}{2}}}\,d\boldsymbol{\varrho} = c_0 \quad (4.234)$$

wherein we have used the fact that $\mathfrak{M}$ is a normalized function. Thus in a system in which the absorption is $1/v$, the *absorption reaction rate is*

[1] See, for example, I. N. Sneddon, "Fourier Transforms," pp. 23–27, McGraw-Hill Book Company, Inc., New York, 1951.

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independent of speed and temperature and can be determined from any single point on the cross-section curve.

Another interesting property of Eq. (4.233) is that it is identical in form, mathematically, to the integral solution of the *heat-conduction* equation.[1] Thus the reaction-rate function $R(v;\theta_N)$ must also satisfy the differential equation

$$\nabla^2 R(v;\theta_N) = \frac{\partial}{\partial \theta_N} R(v;\theta_N) \tag{4.235}$$

where it is now understood that the Laplacian operator ∇^2 refers to the $\mathbf{v}$ space, the coordinates of which are x, y, and z. A further analogy of some interest is the fact that the reaction rate R has the same mathematical properties as the slowing-down function $q(\mathbf{r},\tau)$, which is developed in Chap. 6 (in particular, compare with the Fermi age model of Sec. 6.2a).

Finally, it should be pointed out that the general result (4.233) may be used to compute the temperature coefficient (i.e., temperature dependence) of a Breit-Wigner resonance (see Sec. 6.5g). This has considerable practical significance from the standpoint of reactor control.

In the event that the reactor system of interest is to operate at a temperature slightly above the temperature at which the cross-section data were obtained, it is possible to derive a simple relationship between $R(v;\theta_N)$ and $R(v;\theta'_N)$, which does not require the evaluation of the integral in (4.233). Let us consider therefore the situation wherein $\theta'_N = \theta_N + \Delta\theta_N$ and $\Delta\theta_N/\theta_N \ll 1$. In this case we can generate a solution for $R(v;\theta'_N)$ by making a Taylor series expansion about θ_N. Thus,

$$R(v;\theta'_N) \simeq R(v;\theta_N) + \Delta\theta_N \frac{\partial}{\partial \theta_N} R(v;\theta_N) \tag{4.236}$$

We compute the derivative of R with respect to θ_N by utilizing (4.235). Therefore, in general,

$$R(v;\theta'_N) \simeq R(v;\theta_N) + \Delta\theta_N \nabla^2 R(v;\theta_N)$$

But, R is a spherically symmetric function in velocity space; hence,

$$R(v;\theta'_N) \simeq R(v;\theta_N) + \Delta\theta_N \left\{ \frac{1}{v} \frac{\partial^2}{\partial v^2} [vR(v;\theta_N)] \right\} \tag{4.237}$$

where the derivative term may be obtained from the cross-section data at the temperature corresponding to θ_N. In actual practice,[2] the applica-

[1] See, for example, H. S. Carslaw and J. C. Jaeger, "Conduction of Heat in Solids," chap. XIII, Oxford University Press, New York, 1948.

[2] A Monte Carlo reactor code is available for carrying out this computation. See R. R. Coveyou and W. E. Kinney, "Neutron Reaction Rate Codes," Neutron Physics Division Annual Progress Report for Period Ending Sept. 1, 1959, Oak Ridge National Laboratory, ORNL-2842, pp. 144-145, Nov. 9, 1959.

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tion of (4.237) may offer some computational difficulties, especially if the cross-section curve is a rapidly varying function of v, since the indicated operation involves the determination of the curvature of $\sigma(v)$. Thus the integral relation (4.233), which involves a "smoothing" type mathematical operation, may be more satisfactory from this standpoint.

f. Thermal-group Cross Section for $1/v$ Absorber. As has been frequently noted, many materials of practical importance to reactor technology are $1/v$ absorbers in the thermal-energy range. The calculation of the thermal-group absorption cross sections for materials of this type is therefore of special interest; moreover, it is fortuitous that, in these cases, the general relation (4.209) or (4.207) may be evaluated analytically.

The appropriate form for the thermal-group absorption cross section in the notation of Eq. (4.209) is given by

$$\sigma_a^{\text{th}} \equiv \frac{\int_0^\infty v\sigma_a(v)\,\mathrm{m}(v)\,dv}{\int_0^\infty v\,\mathrm{m}(v)\,dv} \tag{4.238}$$

where $\mathrm{m}(v)$ is defined by (4.200). If we take for the present case

$$\sigma_a(v) \equiv \frac{c_0}{v} \tag{4.239}$$

then (4.238) reduces to

$$\sigma_a^{\text{th}} = \frac{c_0 \int_0^\infty \mathrm{m}(v)\,dv}{\int_0^\infty v\,\mathrm{m}(v)\,dv} = \frac{c_0}{\int_0^\infty v\,\mathrm{m}(v)\,dv} \tag{4.240}$$

since m is a normalized function. The denominator of this result is clearly the expression for the average value of the speed ($\bar{v}$) corresponding to the Maxwell-Boltzmann distribution. That is

$$\int_0^\infty v\,\mathrm{m}(v)\,dv \equiv \bar{v} \tag{4.241}$$

If the indicated integration is carried out, the expression for $\bar{v}$ is found to be

$$\bar{v} = 2\left(\frac{2kT_n}{\pi m}\right)^{\frac{1}{2}} \tag{4.242}$$

We evaluate the constant c_0 in (4.240) in terms of a measured value of σ_a at some reference temperature (for example, the values given in Table 2.2). First we write (4.239) in terms of the neutron energy E.

$$\sigma_a(E) = c_0\left(\frac{m}{2E}\right)^{\frac{1}{2}} \tag{4.243}$$

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If we denote by T_* the reference energy at which the σ_a has been measured, then for $E = kT_*$ we have

$$\sigma_a(kT_*) = c_0\left(\frac{m}{2kT_*}\right)^{\frac{1}{2}} \tag{4.244}$$

which provides a relation for computing c_0 in terms of known constants. The substitution of this expression for c_0, along with $\bar{v}$ from (4.242), into (4.240) yields

$$\sigma_a^{\text{th}} = \sigma_a(kT_*)\left(\frac{\pi T_*}{4T_n}\right)^{\frac{1}{2}} = \sigma_a(kT_n)\left(\frac{\pi}{4}\right)^{\frac{1}{2}} \tag{4.245}\star$$

Note, also, that

$$\Sigma_a^{\text{th}} = N\sigma_a^{\text{th}} \tag{4.246}$$

g. Thermal-group Cross Section for Non-$1/v$ Absorber. In the case that the absorption cross section of a substance is not $1/v$, the thermal-group cross section may be computed from the general relation (4.209). This calculation requires, of course, a detailed knowledge of the cross-section curve in the thermal range. In the absence of such information, it is frequently assumed in practice that the absorption (or fission) cross section is $1/v$, and the result (4.245) is used directly as a first estimate of the thermal-group properties. For these estimates the constant of proportionality c_0 [see Eq. (4.239)] may be obtained from the more generally available 2,200 m/sec value of the cross section[1] [see (4.244)].

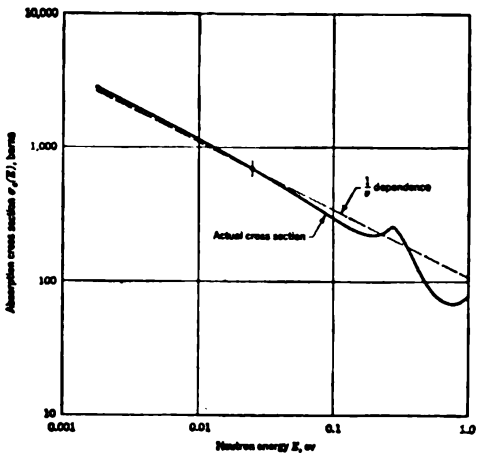

FIG. 4.29 Absorption cross section of U^{235}.

Detailed cross-section curves in the low-energy range are presently available for a number of the more important reactor materials (see, for example, Fig. 4.29), and the general scheme described in Sec. 4.7b may be used to compute the thermal-group values. As a rule the integration indicated in (4.209) must be carried out numerically; however, in a few instances, notably in the case of the fissionable materials U^{233}, U^{235}, and Pu^{239}, some of the preliminary computation has already been performed. Thus for these fuels data are available which permit one to compute the thermal-group cross sections by applying a suitable correction factor to the values obtained directly from the cross-section curves. One definition of such a factor has been

[1] See, for example, Donald J. Hughes and John A. Harvey, in U.S. Atomic Energy Commission, "Neutron Cross Sections," McGraw-Hill Book Company, Inc., New York, 1955.

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given by Harvey and Sanders,[1] the so-called f factor (here represented by the Greek symbol $\mathcal{f}$). This quantity is defined so that

$$\sigma^{\text{th}} \equiv \left(\frac{\pi}{4}\right)^{\frac{1}{2}} \sigma(kT_n)\ \mathcal{f}(T_n) \tag{4.247}$$

where $\sigma(kT_n)$ denotes the value to be read off from the cross-section curve at the neutron energy corresponding to the temperature T_n. The function $\mathcal{f}(T_n)$ defined by (4.247) may be written in the form

$$\mathcal{f}(T_n) = \frac{\sigma^{\text{th}}}{(\pi/4)^{\frac{1}{2}}\sigma(kT_n)} = \frac{1}{(\pi/4)^{\frac{1}{2}}\sigma(kT_n)} \frac{\int_0^\infty \sigma(v)\ \mathrm{m}(v) v\, dv}{\int_0^\infty \mathrm{m}(v) v\, dv} \tag{4.248}$$

The application of (4.241) yields

$$\mathcal{f}(T_n) = \frac{1}{(\pi/4)^{\frac{1}{2}}\bar{v}\,\sigma(kT_n)} \int_0^\infty \sigma(v)\ \mathrm{m}(v) v\, dv \tag{4.249}$$

which is in the form used by Harvey and Sanders. The argument T_n has been indicated in the correction factor $\mathcal{f}$ in recognition of the fact that $\mathrm{m}(v)$ is a function of T_n, the effective neutron temperature [refer to Eq. (4.211)]. This correction accounts, then, for the non-$1/v$ part of the cross section and is based on a Maxwell-Boltzmann distribution for the temperature T_n. Curves of this function have been computed for the principal fuels and are available in Hughes and Harvey[2] and in the reference paper. The data for U^{233}, U^{235}, and Pu^{239} have been reproduced in Fig. 4.30.

An alternative formulation of a suitable correction factor for non-$1/v$ behavior is given by Westcott.[3] This is the g factor; it is related to the $\mathcal{f}$ factor given above by

$$g(T_n) = \mathcal{f}(T_n) \frac{\sigma(kT_n)}{\sigma(kT_*)} \left(\frac{T_n}{T_*}\right)^{\frac{1}{2}} \tag{4.250}$$

Westcott defines also a second correction which may be applied to the effective thermal cross section in order to account for the lower portion of the $1/E$ tail from the slowing-down distribution. As mentioned previously, the total neutron population may be represented by a thermal group, described by a Maxwell-Boltzmann distribution, plus a slowing-down portion, approximated by a $1/E$ distribution. In practice, the use of these models usually yields an abrupt intersection somewhere in the

[1] J. A. Harvey and J. E. Sanders, in R. A. Charpie et al. (eds.), "Progress in Nuclear Energy," ser. 1, vol. 1, Pergamon Press, New York, 1956.

[2] *Ibid.*

[3] C. H. Westcott, "Effective Cross Section Values for Well-moderated Thermal Reactor Spectra," Atomic Energy of Canada Limited, Chalk River, Ontario, CRRP-680, Jan. 25, 1957.

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kT range where the $1/E$ portion can be significantly larger than the "fast tail" of the thermal group (refer to Sec. 4.8). Values of the $1/E$ correction factor have been obtained for several important elements and are tabulated in the reference report.

h. Scattering from Molecules. The purpose of the present discussion is to present in a very elementary way some of the essential features of the scattering process between neutrons and molecules. The scattering of neutrons from bound nuclei (nuclei in molecules) is an extremely complex interaction phenomenon, and a detailed study of the physics involved is beyond the scope of this work. The primary interest in this process from the viewpoint of the reactor physicist is the determination of slow-neutron cross sections of various compounds of practical importance in

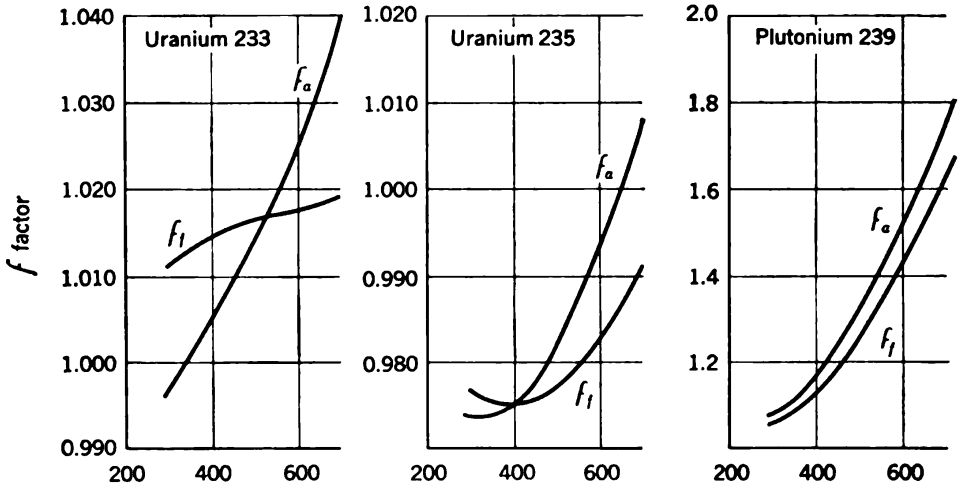

FIG. 4.30 f factors for fission and absorption cross sections.

reactor design. Since these data are usually best obtained from cross-section measurements using the actual materials, the analytical treatment of the process is of little immediate concern to the reactor physicist. However, a discussion of the principal phenomena involved in this interaction may be of some value in achieving an accurate interpretation of the experimental data. Moreover, the process itself is of considerable interest to the nuclear physicist since the diffraction of slow neutrons by gases has become an important research tool. Much of the theoretical work on this problem has been carried out in this connection.[1]

The difficulty in describing the scattering of neutrons by molecules arises from the fact that such interactions are the combined result of

[1] See, for example, E. Fermi, *Ricerca sci.*, **7**, 13 (1936); W. Lamb, *Phys. Rev.*, **55**, 190 (1939); R. G. Sacks and E. Teller, *Phys. Rev.*, **60**, 18 (1941); A. Messiah, *Phys. Rev.*, **84**, 204 (1951); A. C. Zemach and R. J. Glauber, *Phys. Rev.*, **101**, 118 (1956), and **102**, 129 (1956).

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nuclear and *molecular* effects. At high neutron energies (above the electron-volt range) the molecular effects are not important in these collisions, and the process is effectively a nuclear one. Below the electron-volt range, however, and especially in the thermal range, molecular binding forces are comparable in magnitude to the "impact" forces from the incident neutron, and molecular phenomena contribute appreciably to the over-all collision process. Thus, the scattering cross section of molecules for thermal neutrons is determined by several different kinds of particle interactions. The principal phenomena involved are: (1) the nuclear scattering process per se, (2) the natural frequencies of the chemical bonds which hold the target nucleus in the molecule and the various vibrational states of the molecule, (3) the rotational energy states of the molecule, and (4) the interference effects from the multiple scattering centers (nuclei) in the molecule.

The extent to which each of these factors influences the interaction is determined by the neutron energy. At very high energies (between 1 and 100 ev), the scattering process is essentially a nuclear phenomenon. In these cases, the neutron energy is so great that the average recoil energy transferred to the struck nucleus[1] is sufficient to rupture the chemical bonds, and the nucleus behaves as if it were unbound (an atomic gas). As the neutron energy is reduced to the 0.1-ev range, items 2 and 3 become important. In this range the excitation energy of the highest internal quantum states of the molecule are comparable to the translational energy of the neutron. Depending upon the temperature of the medium, a certain number of these rotational and vibrational states of the molecule will already be excited, and when a collision occurs, some fraction of the neutron energy is imparted to the molecule. This energy transfer may result in the transition of the molecule to higher rotational states and/or the excitation of various vibrational modes. In some cases, depending on the neutron energy, the initial internal-energy state of the molecule, and the mass of the struck nucleus, the energy transfer may be in the opposite direction, and the neutron may emerge from the collision with a speed greater than its initial value. These considerations become increasingly important as the neutron energy approaches the thermal range. Finally, there are the interference effects (4) noted above. These are generally very complex, and arise when the neutron wavelength is comparable to atomic spacing.

The influence of these several factors on neutron-nucleus interactions is well displayed in the case of neutron scattering from hydrogen nuclei bound in molecules. This process has been studied in detail and experimental measurements of the scattering cross section of hydrogen in the water molecule show that there is a substantial change in the cross section

[1] Clearly, the energy transferred depends on the neutron energy and the mass of the struck nucleus.

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from the high- to the low-energy range. The cross-section data are shown in Fig. 4.31. The steady rise in σ_s from the 20.36-barn value in the electron-volt range to around 80 barns at 0.004 ev is due to the *combined* effects of the various phenomena mentioned above.

Blatt and Weisskopf[1] give an analytical treatment of the effect of chemical binding on the scattering cross section. They show that for

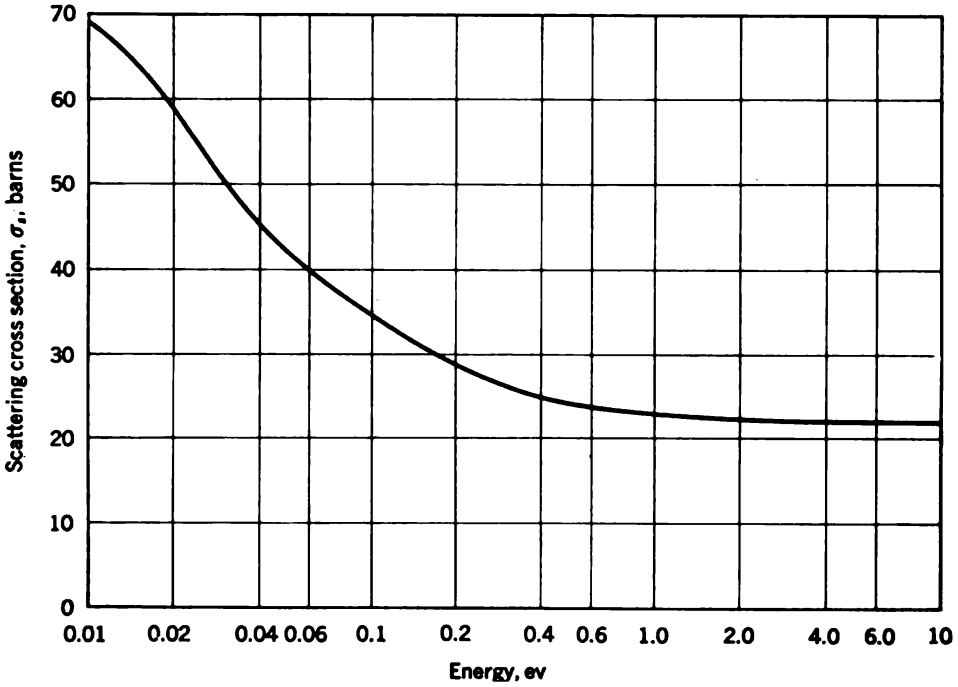

FIG. 4.31 Scattering cross section of hydrogen measured in water.

very slow neutrons the scattering cross section $(\sigma_s)_B$ of a hydrogen nucleus bound to a molecule of mass A_m (in units of neutron mass) is given by

$$(\sigma_s)_B = 4\left(\frac{A_m}{1 + A_m}\right)^2 (\sigma_s)_F$$

where $(\sigma_s)_F$ is the cross section of the hydrogen nucleus in the free (atomic) state. The generalization of this result to the case of a bound nucleus of mass A is

$$(\sigma_s)_B = \left(\frac{1 + A}{A}\right)^2 \left(\frac{A_m}{1 + A_m}\right)^2 (\sigma_s)_F \tag{4.251}$$

It is interesting to note that, for many molecules, the factor involving A_m will be very near unity and the principal contribution to the chemical binding effect stems from the factor involving the mass of the struck nucleus. Even then, the effect of the chemical bond is of importance primarily in connection with the very light nuclei.

[1] Blatt and Weisskopf, *op. cit.*, pp. 71–80.

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4.8 Four-factor Formula

a. Thermal Group. The analysis of Sec. 4.6 dealt with the energy distribution of neutrons in an infinite multiplying medium wherein neutrons could slow down to zero energy. We found in Sec. 4.7, however, that in all media at temperatures above absolute zero the neutron population at the very low energies acquired a peaked distribution around some energy of the order of kT which was determined by the thermal motion (temperature) of the moderating material. We showed also that this distribution could be well approximated by a Maxwell-Boltzmann distribution if the medium was not too highly absorbing. In fact, we were able to specify an effective neutron temperature T_n for this purpose which was dependent upon the actual moderator temperature and its absorption properties [see Eq. (4.211)]. This temperature could be used to define the neutron distribution [see Eqs. (4.197) and (4.200)] in the thermal range.

In the present section we combine these results, the neutron distribution in the high-energy range according to Sec. 4.6 and the distribution in the low-energy range according to Sec. 4.7. This gives a complete description of the neutron population in energy space from the source energy down to, and including, the thermal range. We will use this picture to carry out a more realistic analysis of the neutron physics in an infinite multiplying medium. For this calculation we propose the following model: (1) Medium is infinite, isotropic, and homogeneous. (2) System is at steady state. (3) Neutrons appear in the system according to some source distribution $S(u)$. (4) These source neutrons slow down to the thermal range as if they were in a medium in which there was no thermal motion of the nuclei. (5) At some energy $E = E_{th}$, the slowing-down process ends, and all neutrons (not captured while slowing down) enter a one-velocity diffusion process; they wander through the medium at this energy and suffer no change in speed as a result of scattering collisions. (6) These *thermal neutrons* are eventually lost by absorption.

Thus our model replaces the actual neutron distribution by a slowing-down distribution plus a *thermal group*. The thermal group consists of neutrons of one speed, this speed to be defined by the effective neutron temperature T_n. The actual thermal distribution, which was approximated by the Maxwell-Boltzmann relation (4.200), is replaced then by a "spike" or "group" at $E = E_{th}$ (see Fig. 4.32). The selection of this thermal energy E_{th} is discussed shortly.

The mathematical statement of the model described above is (cf. Sec. 4.6a)

$$\frac{d}{du}\, q(u) = -\Sigma_a(u)\, \phi(u) + S(u) \tag{4.145}$$

$$q(u) = \xi \Sigma_t(u)\, \phi(u) \qquad 0 \leq u < u_{th} \tag{4.112}$$

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where $q(0) = 0$ and u_{th} corresponds to E_{th}. For the thermal group we use

$$\Sigma_a^{th}\phi_{th} = \lim_{u \to u_{th}} q(u) \equiv q(u_{th}) \tag{4.252}$$

which follows from the physical requirement that, if the system is to be at steady state, the rate at which neutrons enter the thermal group $q(u_{th})$

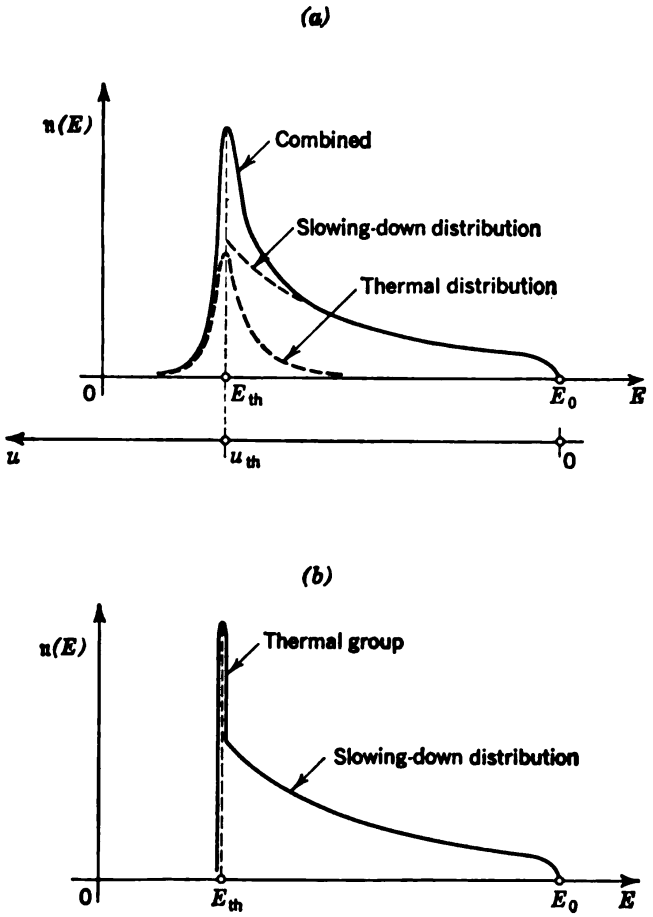

FIG. 4.32 Distribution of thermal neutrons: (*a*) actual distribution; (*b*) model.

must be balanced by the rate at which they are removed $\Sigma_a^{th}\phi_{th}$. In the above relations we have used the definitions

Σ_a^{th} = absorption cross section of thermal group
$\phi_{th} = n_{th}v_{th}$ = thermal neutron flux
n_{th} = number of thermal neutrons per unit volume
v_{th} = speed of thermal neutrons

The absorption cross section for the thermal group is computed by means of Eq. (4.238). As already demonstrated, in the event that the absorber has a $1/v$ cross section, the integral yields an especially simple form; this result is given in Eqs. (4.245) and (4.246). The absorption cross section

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of many materials has this $1/v$ dependence, fortunately, and these relations may be used directly to obtain the Σ_a^{th}.

An appropriate definition of the thermal flux which is consistent with the thermal-cross section formula is given by [cf. (4.204)]

$$\phi_{th} \equiv \int_0^\infty n(v)v\,dv = n_{th}\bar{v} \tag{4.253}$$

where the average speed $\bar{v}$ is defined by (4.242). According to this definition the appropriate value for the speed of the thermal group is $v_{th} = \bar{v}$; however, the most probable speed v_0 is sometimes used to describe the speed of thermal neutrons. In terms of the Maxwell-Boltzmann distribution, v_0 is found to be[1]

$$v_0 = v(kT_n) = \left(\frac{2kT_n}{m}\right)^{\frac{1}{2}} \tag{4.254}$$

This number represents the position of the maximum point in the Maxwell-Boltzmann distribution $\mathfrak{m}(v)$ [see Eq. (4.200)]. Note that v_0 corresponds to the energy kT_n so that we immediately identify E_{th} as

$$E_{th} = \tfrac{1}{2}mv_0^2 = kT_n \tag{4.255}\star$$

It must be recognized that this model, which treats the neutron distribution by two separate regions (one for the high-energy range and one for the thermal group), only approximately represents the actual distribution, since the slowing-down process just above the thermal range is incorrectly represented. We have also neglected certain crystalline effects which may play an important role in the low-energy range; neutrons may scatter from several nuclei simultaneously in what amounts to a diffraction phenomenon. This effect changes the effective scattering cross section (see Sec. 4.7h) and lowers the average logarithmic energy loss per collision.

The solution to Eqs. (4.145) and (4.112) was previously obtained and is given in Eq. (4.147). If we compute $q(u_{th})$ using this relation and substitute the result into (4.252), we obtain for the thermal flux

$$\phi_{th} = \frac{1}{\Sigma_a^{th}} \int_0^{u_{th}} S(u)\, p(u_{th};u)\, du \tag{4.256}$$

where $p(u_{th};u)$ is given by (4.148). The neutron flux in the range $0 \leq u < u_{th}$ is computed from (4.112) using (4.147).

$$\phi(u) = \frac{1}{\xi\Sigma_t(u)} \int_0^u S(u')\, p(u;u')\, du' \tag{4.257}$$

[1] See D. K. Holmes and R. V. Meghreblian, "Notes on Reactor Analysis," part II, ORNL CF 54-7-88, Oak Ridge National Laboratory, Aug. 2, 1955.

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If all the neutrons are introduced at lethargy $u = 0$, Eqs. (4.256) and (4.257) reduce to

$$\phi_{th} = \frac{S_0}{\Sigma_a^{th}} p(u_{th};0) \equiv \frac{S_0 p_{th}}{\Sigma_a^{th}} \tag{4.258}$$

$$\phi(u) = \frac{S_0 p(u;0)}{\xi \Sigma_t(u)} \equiv \frac{S_0 p(u)}{\xi \Sigma_t(u)} \tag{4.259}$$

where $S_0 \equiv$ number of neutrons introduced per unit volume per unit time at $u = 0$
$p_{th} \equiv p(u_{th};0) =$ resonance-escape probability from $u = 0$ to thermal energy
$p(u) = p(u;0) =$ resonance-escape probability from $u = 0$ to u (4.260)

It should be noted that Eqs. (4.258) and (4.259) are not accurate. It is imagined that the S_0 neutrons appear with lethargy zero and emerge with directions of motion isotropically distributed about the point of birth. As these neutrons move away from this point, there are certain probabilities per unit distance of travel that they will make both absorption and scattering collisions. If the neutrons are to contribute to the slowing-down density for lethargies $u > 0$, their first collisions must be scattering collisions; thus in reality, only the fraction $\Sigma_s(0)/\Sigma_t(0)$ of the S_0 neutrons should be counted as a source for the slowing-down process. We will continue to neglect this effect (*first-flight* losses from sources), since the absorption cross sections are often very small at higher energies. It is understood, then, that at any time it is felt to be important the source strength may be reduced by the ratio Σ_s/Σ_t.

b. Criticality Equation. Suppose that the infinite homogeneous medium contains fissionable material and that the only sources of neutrons present are the fission reactions. Then, as above, if the system is to operate in steady state, the number of neutrons produced from fissions must equal the number of neutrons introduced into the medium, namely, $S(u)$. For this case the source function has the form

$$S(u) = \nu \chi(u) \left[\int_0^{u_{th}} \Sigma_f(u')\, \phi(u')\, du' + \Sigma_f^{th} \phi_{th} \right] \tag{4.261}$$

The fission spectrum $\chi(u)$ has the shape shown in Fig. 4.25, but for the present calculation we set the origin of the u axis at the far left so that $\chi(0)$ may be taken as zero. Note that the integration in (4.261) extends up to but does not include the thermal group. The last term $\Sigma_f^{th}\phi_{th}$ accounts for the thermal fissions.

We choose to introduce one neutron per unit volume per unit time by requiring that

$$\int_0^{u_{th}} S(u)\, du \equiv 1 \tag{4.262}$$

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Therefore, if we integrate (4.261) from $u = 0$ to u_{th}, we obtain

$$1 = \nu \int_0^{u_{th}} \chi(u)\, du \left[\int_0^{u_{th}} \Sigma_f(u')\, \phi(u')\, du' + \Sigma_f^{th} \phi_{th} \right]$$

But, χ is a normalized function [see Eq. (4.152)]; hence

$$1 = \nu \left[\int_0^{u_{th}} \Sigma_f(u')\, \phi(u')\, du' + \Sigma_f^{th} \phi_{th} \right] \tag{4.263}$$

We expand this expression by introducing the solutions (4.256) and (4.257) and noting that in this case $S(u) \equiv \chi(u)$.

$$1 = \nu \left[\int_0^{u_{th}} \gamma(u)\, du \int_0^{u} \chi(u')\, p(u;u')\, du' + \frac{\Sigma_f^{th}}{\Sigma_a^{th}} \int_0^{u_{th}} \chi(u)\, p(u_{th};u)\, du \right] \tag{4.264}$$

where the function $\gamma(u)$ is given by Eq. (4.161). The above relation is the criticality condition for the infinite homogeneous reactor, using the asymptotic solution for the slowing-down distribution and the concept of a thermal group. Equation (4.264) must be satisfied if the system is to operate in steady state. Now, if the system is not critical, the multiplication constant k_∞ may be computed analogously to the reasoning leading to Eq. (4.166). Thus we employ again the concept of a fictitious neutron yield per fission ν_c and the relation

$$k_\infty = \frac{\nu}{\nu_c} \tag{4.167}$$

We compute ν_c from the equation

$$1 = \nu_c \left[\int_0^{u_{th}} \gamma(u)\, du \int_0^{u} \chi(u')\, p(u;u')\, du' + \frac{\Sigma_f^{th}}{\Sigma_a^{th}} \int_0^{u_{th}} \chi(u)\, p(u_{th};u)\, du \right] \tag{4.265}$$

Or, by substituting this expression for ν_c into (4.167),

$$k_\infty = \nu \left[\int_0^{u_{th}} \gamma(u)\, du \int_0^{u} \chi(u')\, p(u;u')\, du' + \frac{\Sigma_f^{th}}{\Sigma_a^{th}} \int_0^{u_{th}} \chi(u)\, p(u_{th};u)\, du \right] \tag{4.266}$$

It is convenient to write this statement in the form

$$k_\infty = \begin{bmatrix} \text{average number} \\ \text{of neutrons} \\ \text{per fission} \end{bmatrix} \left[\begin{pmatrix} \text{number of} \\ \text{fast fissions} \end{pmatrix} + \begin{pmatrix} \text{number of} \\ \text{thermal fissions} \end{pmatrix} \right] \tag{4.267}$$

where we use the expression *fast fissions* to denote the number of fissions

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per unit volume per unit time produced by neutrons with energies above E_{th}, and *thermal fissions* to denote those produced by neutrons in the thermal group represented by the energy $E = E_{th}$. Thus we may write

$$k_\infty = \nu \left[\begin{matrix}\text{thermal} \\ \text{fissions}\end{matrix}\right] \left[\frac{\text{fast fissions + thermal fissions}}{\text{thermal fissions}}\right] \tag{4.268}$$

If we define ϵ the *fast fission factor* by

$$\epsilon \equiv \frac{\text{total fissions}}{\text{thermal fissions}} \tag{4.269}\star$$

then (4.266) takes the form

$$k_\infty = \frac{\nu \Sigma_f^{th} \epsilon}{\Sigma_a^{th}} \int_0^{u_{th}} \chi(u)\, p(u_{th};u)\, du \tag{4.270}\star$$

This result may be written in a more compact manner by introducing an *average resonance escape probability* $\overline{p_{th}}$ for neutrons released according to the fission spectrum; thus,

$$\overline{p_{th}} \equiv \int_0^{u_{th}} \chi(u)\, p(u_{th};u)\, du \tag{4.271}$$

and Eq. (4.270) reduces to

$$k_\infty = \frac{\nu \Sigma_f^{th}\, \overline{p_{th}} \epsilon}{\Sigma_a^{th}} \tag{4.272}$$

This relation for computing k_∞ is useful only if ϵ may be calculated or estimated on bases other than those given here. An especially simple situation occurs in the case of thermal reactors (see Sec. 1.2). In these systems, $\epsilon \simeq 1$ and for many purposes may be taken as equal to unity. The deviation of ϵ from unity may also be estimated, without introducing any large error, on the basis of known values for similar reactors. However, in general (4.270) is to be used in calculating k_∞.

If it is a good approximation that all neutrons from fission are introduced at one energy (chosen at some appropriate energy near the peak of the fission spectrum), then by setting the zero of lethargy to correspond to this energy,

$$\chi(u) = \delta(u) \tag{4.163}$$

The expression for the multiplication constant in the infinite medium (4.266) becomes

$$k_\infty = \nu \left[\int_0^{u_{th}} \gamma(u)\, p(u)\, du + \frac{\Sigma_f^{th} p_{th}}{\Sigma_a^{th}}\right] \tag{4.273}$$

or in the form of (4.272),

$$k_\infty = \frac{\nu \Sigma_f^{th} \epsilon p_{th}}{\Sigma_a^{th}} \tag{4.274}$$

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Following the treatment of Eqs. (3.19) and (3.22), we write

$$k_\infty = \frac{\nu\Sigma_f^{th}}{(\Sigma_a^{th})_F} \frac{(\Sigma_a^{th})_F}{\Sigma_a^{th}} p_{th}\epsilon$$

where the subscript F refers to the fuel. By our usual definitions for η and f [Eq. (3.23)]

$$k_\infty = \eta f p_{th}\epsilon \tag{4.275}\star$$

This equation is referred to as the "four-factor formula" and is the extension of (3.22) to the *multivelocity* reactor. Again it must be mentioned that (4.275) is useful only if ϵ can be easily estimated; in general (4.273) or (4.266) is to be applied. To summarize, then, we use the relation

$$\epsilon = \frac{\displaystyle\int_0^{u_{th}} \gamma(u)\,du \int_0^u \chi(u')\,p(u;u')\,du' + \frac{\Sigma_f^{th}}{\Sigma_a^{th}} \int_0^{u_{th}} \chi(u)\,p(u_{th};u)\,du}{\displaystyle\frac{\Sigma_f^{th}}{\Sigma_a^{th}} \int_0^{u_{th}} \chi(u)\,p(u_{th};u)\,du} \tag{4.276}$$

for the fast effect in the case of the general fission spectrum $\chi(u)$. If on the other hand (4.163) is an acceptable approximation for $\chi(u)$, then (4.276) reduces to

$$\epsilon = \frac{\displaystyle\int_0^{u_{th}} \gamma(u)\,p(u)\,du + \Sigma_f^{th}p_{th}/\Sigma_a^{th}}{\Sigma_f^{th}p_{th}/\Sigma_a^{th}} \tag{4.277}$$

The four-factor formula (4.275) may be given a simple physical interpretation. First we note that in the case of a critical system $k_\infty \equiv 1$, so that

$$1 = \eta f p_{th}\epsilon \tag{4.278}$$

This relation may be used to follow the neutron production cycle at steady state. If we begin the cycle by introducing one thermal neutron into the multiplying medium, then a fraction $(\Sigma_a^{th})_F/\Sigma_a^{th}$ is captured by the fissionable material. Of these, a fraction $\Sigma_f^{th}/(\Sigma_a^{th})_F$ causes fission. Thus the single thermal neutron produces $f\Sigma_f^{th}/(\Sigma_a^{th})_F$ fissions. However, if we consider that in general the fuel material will also possess some nonzero fission cross section in the fast range, then for each thermal fission there will be ϵ thermal fissions plus fast fissions. The total number of fissions in the system is, therefore, $\epsilon f\Sigma_f^{th}/(\Sigma_a^{th})_F$. Each fission produces ν neutrons on the average. Thus $\eta\epsilon f$ fast neutrons appear for each thermal neutron added to the system. These fast neutrons are eventually slowed down to thermal energy by the various scattering collisions with nuclei. However, only a certain fraction survive the entire slowing-down process since some neutrons are removed from the system by fast captures. The fraction reaching thermal is p_{th}; therefore, $\eta f\epsilon p_{th}$ fast neutrons are eventually thermalized. Clearly, if the system is to remain at steady state,

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$\eta f \epsilon p_{\text{th}} \equiv 1$, so as to reproduce the original thermal neutron which started the cycle; thence, (4.278).

c. Application of Four-factor Formula. As an illustration of the use of the four-factor formula (4.275), consider an infinite medium which consists of a homogeneous mixture of two elementary substances, one acting as the moderator and the other as the fuel material. The problem is to find the multiplication constant of the medium, given the concentrations of the fuel and moderator, the neutron cross sections of each as a function of energy, and the neutron production properties of the fuel. In many cases of practical interest, particularly in thermal reactors, the fuel concentration will be very small in comparison to the moderator concentration. Under these circumstances the volume-fraction effects discussed in Sec. 3.3 may be neglected and the moderator concentration N_M may be taken to be that of the pure moderator material [see Eqs. (3.33)]. In the calculations which follow, we will assume that this is a valid assumption for the system in question. Further, we will assume that the fission spectrum may be well approximated by the monoenergetic source (4.163).

The appropriate formula for k_∞ in the present case is seen to be (4.273). From the form of this equation it is evident that both the fast and thermal fissions must be calculated; the computation of the fast fissions involves integrals which are usually performed numerically over the cross sections and which are not readily presentable in this discussion. However, it should be observed that in many cases the fraction of the fissions caused by thermal neutrons increases as the fuel concentration decreases; therefore, it is possible to find a range of fuel concentrations for which the fast fissions are negligible (although the multiplication constant may not be greater than unity for this range of fuel concentrations). Let us consider, then, the ratio of fast fissions to thermal fissions as the fuel concentration is decreased. Now,

$$\frac{\text{Fast fissions}}{\text{Thermal fissions}} = \frac{\int_0^{u_{\text{th}}} \Sigma_f(u)\, \phi(u)\, du}{\Sigma_f^{\text{th}} \phi_{\text{th}}} = \frac{\int_0^{u_{\text{th}}} \sigma_f(u)\, \phi(u)\, du}{\sigma_f^{\text{th}} \phi_{\text{th}}} \tag{4.279}$$

where the fluxes are obtained from (4.112) and (4.252). These may be written

$$\phi(u) = \frac{q(u)}{\xi \Sigma_t(u)} \qquad \phi_{\text{th}} = \frac{q(u_{\text{th}})}{\Sigma_a^{\text{th}}} \tag{4.280}$$

For a given source strength (at some high energy) both $\phi(u)$ and ϕ_{th} increase as the fuel concentration is lowered. With decreasing fuel concentration, the slowing-down density will be increased at every value of u; however, the greatest increase will be in $q(u_{\text{th}})$ since $q(u_{\text{th}})$ records the cumulative effect from the entire fast range. Further, and more impor-

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tant, is the fact that both $\Sigma_t(u)$ and Σ_a^{th} decrease with decreasing fuel concentration. Again, this gives an increase in both the fast and thermal fluxes; moreover, the thermal flux will in most cases gain relative to the fast flux since the fuel-absorption cross section will usually be a larger part of the thermal-absorption cross section than that of the total fast cross section. (Note that in the fast range the total cross section is due primarily to the scattering cross section.) The combined effect of these variations is to decrease the ratio (4.279) as N_F decreases.

Now consider a moderator-fuel mixture in which the ratio N_F/N_M is so low that the fast fissions may be completely neglected, that is, in Eq. (4.275), $\epsilon \simeq 1$. Then

$$k_\infty \simeq \eta f p_{th} \tag{4.281}$$

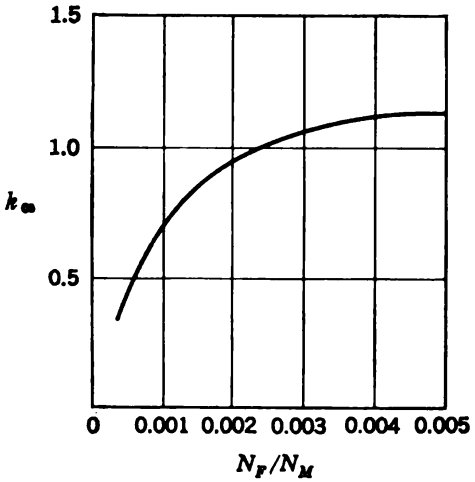

FIG. 4.33 Infinite-medium multiplication as a function of the atomic ratio of fuel to moderator.

The quantity η is fixed once the fuel is selected (for U^{235}, the value 2.08). For this case the thermal utilization is

$$f = \left[\frac{N_F\sigma_a^{(F)}}{N_F\sigma_a^{(F)} + N_M\sigma_a^{(M)}}\right]_{th} = \left\{1 + \frac{N_M}{N_F}\left[\frac{\sigma_a^{(M)}}{\sigma_a^{(F)}}\right]_{th}\right\}^{-1} \tag{4.282}$$

The ratio of the thermal-absorption cross sections of carbon and U^{235}, for example, is

$$\left[\frac{\sigma_a^{(M)}}{\sigma_a^{(F)}}\right]_{th} \simeq 0.0047$$

The calculation of the resonance-escape probability involves an integral over the fast cross sections; thus

$$p_{th} = \exp\left[-\int_0^{u_{th}} \frac{\Sigma_a(u)\,du}{\xi\Sigma_t(u)}\right] \tag{4.283}$$

Because the absorption cross section is high and the ξ is low (high mass of the fuel nuclei) for the fuel, it may be valid to make the following approximation for all energies:

$$\frac{\Sigma_a(u)}{\xi\Sigma_t(u)} \simeq \frac{N_F\sigma_a^{(F)}(u)}{\xi_M N_M \sigma_s^{(M)}(u)}$$

Further, the scattering cross section of the moderator will often have so little variation over the fast range that some representative value may be taken over the entire range; then, Eq. (4.283) may be approximated by

$$p_{th} \simeq \exp\left[-\frac{N_F}{\xi_M N_M \sigma_s^{(M)}}\int_0^{u_{th}} \sigma_a^{(F)}(u)\,du\right] \tag{4.284}$$

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The integral which appears in this expression is called the *resonance integral:*

$$\text{Resonance integral} \equiv \int_0^{u_{th}} \sigma_a(u)\, du = \int_{E_{th}}^{E_0} \sigma_a(E) \frac{dE}{E} \qquad (4.285)$$

Values of this integral have been obtained for various fuel materials and are given in the literature.[1]

Equation (4.281) may be written in terms of these results; thus,

$$k_\infty = \eta \left\{1 + \frac{N_M}{N_F}\left[\frac{\sigma_a^{(M)}}{\sigma_a^{(F)}}\right]_{th}\right\}^{-1} \exp\left[-\frac{N_F}{\xi_M N_M \sigma_s^{(M)}} \int_0^{u_{th}} \sigma_a^{(F)}(u)\, du\right] \qquad (4.286)$$

Then, with given values for η, $[\sigma_a^{(M)}/\sigma_a^{(F)}]_{th}$, $\xi_M\sigma_s^{(M)}$, and the resonance integral, k_∞ may be computed as a function of N_F/N_M. Such a calculation might yield a curve of the type shown in Fig. 4.33. Of course, it would be necessary to determine whether the assumption of no fast fissions ($\epsilon \simeq 1$) is acceptable up to concentrations as high as $N_F/N_M = 0.005$.

PROBLEMS

4.1 Let the frequency function for the cosine of the scattering angle in the center-of-mass system be $f(\eta)$, as given below, for the case in which a particle of mass M_1 collides with a particle of mass M_2 which is initially at rest in the laboratory system:

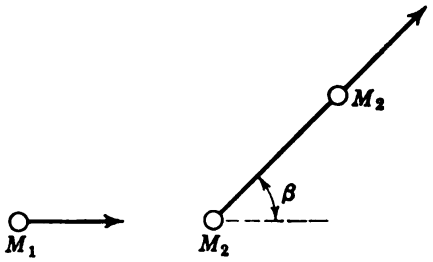
$$\begin{aligned} f(\eta) &= 0 && \eta < -1 \\ &= \tfrac{1}{2}(1 + a\eta) && -1 \le \eta \le 1 \qquad 0 \le a \le 1 \\ &= 0 && \eta > 1 \end{aligned}$$

As compared to the frequency function for the case of isotropic scattering in the center-of-mass system, $f(\eta)$ has an added term $\frac{1}{2}a\eta$. This situation is called p scattering. Note that the relation between the neutron energies before and after collision E_0 and E, respectively, is unaffected however by the addition of the p component; that is,

$$E = \frac{M_1^2 + M_2^2 + 2M_1M_2\eta}{(M_1 + M_2)^2} E_0$$

Compute $\xi \equiv \overline{\ln E_0/E}$, and express the results in terms of ξ_0, where

$$\xi_0 \equiv 1 + \frac{\alpha \ln \alpha}{1 - \alpha} \qquad \eta \equiv \cos\beta$$
$$\alpha \equiv \left(\frac{M_1 - M_2}{M_1 + M_2}\right)^2$$



Collision in center-of-mass system

4.2 If the probability that a neutron of kinetic energy E_0 emerging from a scattering collision with energy in dE about E is given by

$$\begin{aligned} p(E;E_0)\, dE &= 0 && E > E_0 \\ &= \frac{dE}{(1 - \alpha)E_0} && \alpha E_0 \le E \le E_0 \\ &= 0 && E \le \alpha E_0 \end{aligned}$$

[1] See, for example, L. Dresner, The Effective Resonance Integrals of U-238 and Th-232, *J. Am. Nuclear Soc.*, **1**, 1, 68–79 (1956).

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where $\alpha \equiv [(A - 1)/(A + 1)]^2$ and A = nuclear mass number, show that the average kinetic energy of the neutron after two collisions is

$$\overline{E_2} = \left(\frac{1 + \alpha}{2}\right)^2 E_0$$

It is possible to show a more general relation of this type, namely,

$$\overline{E_n} = \left(\frac{1 + \alpha}{2}\right)^n E_0$$

4.3 Calculate the average number of neutron-nucleus collisions needed to reduce the kinetic energy of a neutron from 1 Mev to 1 ev on three different bases. Treat all neutrons as if they suffered their average variable change at each collision on the three scales energy, lethargy, and speed. Take the value 0.1 for the α of the medium in which the neutrons are slowing down. (Note that the average speed change per collision can be calculated from the frequency function given in Prob. 4.2.)

4.4 Consider the slowing down of neutrons in an infinite, nonabsorbing homogeneous medium of a single nuclear species with thermal motion neglected. Use the variable $w \equiv E_0/E$, where E_0 is the kinetic energy with which the neutrons are introduced.

a. Find the distribution function and the frequency function for w after a single scattering collision (assume that the initial value of $w = w_*$).

b. Find the average gain in w per collision.

c. By considering the balance of neutrons in an interval dw, find the (asymptotic) integral equation for the scattering-collision density $F(w)$, where $F(w)\,dw \equiv \Sigma_s(w)\,\phi(w)\,dw$.

d. Compute the slowing-down density $q(w)$ as an integral over the scattering-collision density.

4.5 Neutrons of energy E_0 are produced uniformly throughout an infinite medium. Subsequently, they slow down by elastic collisions until they reach E_{th} and there enter a one-velocity diffusion process. The medium is characterized by the macroscopic cross sections Σ_s, Σ_a, Σ_f, and Σ_t, which are all inversely proportional to the neutron speed at *all* speeds. Find the fraction of fissions caused by thermal neutrons. Assume the following slowing-down density and flux relation: $q(E) = \xi\Sigma_t(E)\,\phi(E)E$.

4.6 Consider a homogeneous infinite medium for which the macroscopic scattering and absorption cross sections may be taken to be constants and for which the resonance-escape probability is given by

$$p(u;u_0) \equiv \exp\left[-\int_{u_0}^{u} \frac{\Sigma_a(u')\,du'}{\bar{\xi}\Sigma_t(u')}\right]$$

where $\bar{\xi}$ is the appropriate average gain in lethargy per collision.

a. Find the expression for the slowing-down density $q(u)$ at some u much larger than the source value in the two cases: (1) all neutrons are introduced with lethargy u_1, and (2) neutrons are introduced uniformly in lethargy over the interval $0 \leq u \leq u_*$.

b. Find u_1, as a function of u_*, such that the expressions for $q(u)$ obtained in the two cases above are identical for $u > u_*$ (in the first approximation $u_1 \simeq u_*/2$, for $\Sigma_a u_*/\bar{\xi}\Sigma_t \ll 1$).

4.7 Consider an infinite medium of uniformly distributed fission sources and moderating material. Neutrons appear at all energies from these sources and subsequently slow down by elastic collision. A good approximation to the fission spectrum is given by the normalized function

$$z(E)\,dE = \frac{Ee^{-E/E_m}\,dE}{E_m^2}$$

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a. Determine the corresponding function $\chi(u)\,du$ in terms of the lethargy variable $u \equiv \ln E_*/E$, and $h(\zeta)\,d\zeta$ in terms of ζ, where $\zeta \equiv u - u_m$ and $u_m \equiv \ln E_*/E_m$.

b. Obtain an approximation for $h(\zeta)$ by expanding its exponent in a Taylor series about its minimum point ζ_0 (use only the first three terms of the series).

c. Compute and plot the exact and approximate expressions for the function $h(\zeta)$, using $E_* = 10$ Mev and $E_m = 1$ Mev.

d. If $\Sigma_a(u)/\xi\Sigma_t(u) = \beta = \text{constant}$, find the slowing-down density $q(u)$, using the approximate solution for $\chi(u)$ based on the results of part *b* above.

e. Show that this result (part *d*) varies like the slowing-down density due to a monoenergetic source at $u_0 \equiv \zeta_0 + u_m$ of the *same total strength*[1] when u is far removed from u_0.

4.8 Consider a homogeneous infinite medium consisting of a mixture of a fuel material and a moderator material into which neutrons are introduced with a high energy E_0. Suppose that each of the macroscopic cross sections $\Sigma_a(E)$, $\Sigma_s(E)$, and $\Sigma_f(E)$ has the following energy dependence:

$$\begin{aligned} \Sigma(E) &= \Sigma^{(0)} = \text{constant} \qquad E_1 \leq E \leq E_0 \\ &= \Sigma^{(1)} = \text{constant} \qquad E_{\text{th}} \leq E \leq E_1 \\ &= \Sigma^{\text{th}} = \text{constant} \qquad E = E_{\text{th}} \end{aligned}$$

Assume also that $q(E) = \xi\Sigma_t(E)\,\phi(E)E$ and

$$p(E;E_0) = \exp\left[-\int_E^{E_0} \frac{\Sigma_a(E')\,dE'}{\xi\Sigma_t(E')E'}\right]$$

Define

$$\beta_i \equiv \frac{\Sigma_a^{(i)}}{\xi\Sigma_t^{(i)}} \qquad \gamma_i \equiv \frac{\Sigma_f^{(i)}}{\xi\Sigma_t^{(i)}}$$

where $i = 0$, 1, or "th" according to the above notation and $\gamma_{\text{th}}/\beta_{\text{th}} \equiv \Sigma_f^{\text{th}}/\Sigma_a^{\text{th}}$.

a. Find the general expression for the multiplication constant of this medium in terms of the parameters: β_0, β_1, γ_0, γ_1, $\gamma_{\text{th}}/\beta_{\text{th}}$, E_0, E_1, and E_{th}.

b. Compute the numerical value of the multiplication constant using

$$\begin{aligned} \nu = 2.46 \qquad A_M = 12 \qquad A_F = 235 \\ E_0 = 10^6 \text{ ev} \qquad E_1 = 10 \text{ ev} \qquad E_{\text{th}} = 0.025 \text{ ev} \end{aligned}$$

and an atomic ratio of moderator to fuel of $N_M/N_F = 500$. Take the average microscopic cross sections (in barns) to be as follows:

Energy range, ev	Fuel, F			Moderator, M	
	σ_a	σ_f	σ_s	σ_a	σ_s
$10 \rightarrow 10^6$	1.0	0.5	5.0	0.001	5.0
$0.025 \rightarrow 10$	10.	2.0	5.0	0.003	5.0
0.025	650	550	5.0	0.010	5.0

c. What percentage of the fissions is caused by thermal neutrons?

[1] Note that the integral of the approximation for $\chi(u)$ is not unity.

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CHAPTER 5

DIFFUSION THEORY: THE HOMOGENEOUS ONE-VELOCITY REACTOR

5.1 One-velocity Diffusion Equation

a. One-velocity Model. The most general problem of interest in reactor analysis is the determination of the neutron density in a finite, inhomogeneous anisotropic medium as a function of space, neutron velocity, and time coordinates. For the present, the general problem will be avoided, and attention will be confined to uniform, homogeneous, and isotropic media in steady state. Even with this idealization, the problem of computing the neutron flux at all space points and velocities is sufficiently difficult that it is worth while to lead up to the general problem through a series of simpler cases. The first of these we have already considered in the analysis of Chap. 4 which dealt with the energy dependence of the neutron flux. In this case, the spatial dependence was removed by considering only homogeneous infinite media. The directional dependence was removed by limiting the description to an integrated flux $\phi(v)$ which specified only the neutron speed.[1]

It is the purpose of the present chapter to study the second model in the series. This model limits the description of the neutron population to its dependence on the spatial coordinates. Thus for the time being we remove the energy dependence entirely by assuming that *all* the neutrons in a system (including neutrons from sources) have the same speed. Later, in Chap. 6, we will consider a third, more complex, model[2] which gives a description of the neutron population in terms of the neutron energy and the spatial coordinates simultaneously. Further generalization is achieved in Chap. 7 wherein the flux is defined in terms of not only spatial coordinates and energy but also the direction of motion of the neutrons (the angular dependence of the flux). This is the complete transport-theory formulation.

The basis of the following analysis, then, is the *one-velocity* approximation. This model was introduced in Chap. 3, but our attention there was confined to the infinite medium. In the present treatment we will

[1] The calculation of the total track length $\phi(v)$ took into account the contributions of *all* neutrons of speed v irrespective of their direction of motion.

[2] The Fermi age model.

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examine in detail the spatial distribution of neutrons in finite systems of various shapes; moreover, we will consider both multiplying and nonmultiplying media. It is recognized that the one-velocity model, like the infinite-medium "multivelocity" model of Chap. 4, gives a very limited picture of the neutron distribution, inasmuch as each of these models focuses attention on only one feature of the problem. Thus while neither of these models is very realistic in itself, each presents, in a simplified form, concepts and general results which remain valid in the more general case.

b. Neutron-balance Equation. The immediate purpose of this calculation is the formulation of a neutron balance (or continuity) relation, analogous to the infinite-medium equation (3.8), which accounts for all the neutrons in an elementary volume of the system in question. The present development of the neutron-balance condition, however, is not mathematically rigorous and appeals in main part to physical intuition. Even though the analysis is relatively crude and does not bring out clearly the nature and extent of the approximations involved, it is expected nevertheless to be instructive as an introductory calculation. A rigorous derivation of the result obtained below can be developed only from the more accurate statement of the transport equation. We will use this approach later in Chap. 7 in connection with the general transport-theory methods.

The model we use for the present calculation has the following features: (1) the system is a homogeneous and isotropic medium of finite extent; (2) all neutron sources and sinks are uniformly distributed throughout the system; and (3) the one-velocity model applies. Thus all neutrons in the system, including those which appear from sources, have the same speed v. We specify the neutron population in terms of the density $n(\mathbf{r},t)$, where

$$n(\mathbf{r},t) \equiv \begin{array}{l}\text{number of neutrons of speed } v \text{ per unit} \\ \text{volume around space point } \mathbf{r} \text{ at time } t\end{array} \tag{5.1}$$

We will also use the neutron flux $\phi(\mathbf{r},t)$:

$$\phi(\mathbf{r},t) \equiv \begin{array}{l}\text{total track length of neutrons of speed } v \text{ per unit} \\ \text{volume per unit time around space point } \mathbf{r} \text{ at time } t\end{array} \tag{5.2}$$

By the usual definition

$$\phi(\mathbf{r},t) = v\, n(\mathbf{r},t) \tag{5.3}$$

Note that these functions refer to neutrons of a single speed v according to our model. In all the analyses of this chapter the speed v will be implied in our definitions of n and ϕ, and therefore the number v will not appear as an argument of these functions. Also note that the argument $\mathbf{r}$ denotes a general position vector in some appropriate coordinate system set up in

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the medium. The position $\mathbf{r}$ will be specified by means of one, two, or three space coordinates as required for a specific calculation.

We now generalize the balance condition (3.8) to the present case. For the finite medium, the number of neutrons lost from a given volume element in unit time by passage across the boundaries of the element will not, in general, be compensated by a corresponding gain. That is, there will be in general a net "leakage" (gain or loss) from the volume element. An appropriate term must be introduced into the neutron-balance relation to account for these net losses or gains through the boundaries. A verbal statement of a balance condition for the differential volume element $d\mathbf{r}$ which includes such a term is:

$$\begin{bmatrix}\text{Time rate of increase of the}\\ \text{neutron density in } d\mathbf{r} \text{ about } \mathbf{r}\end{bmatrix} = \begin{bmatrix}\text{number of neutrons pro-}\\ \text{duced in } d\mathbf{r} \text{ per unit time}\end{bmatrix} - \begin{bmatrix}\text{number of neutrons absorbed}\\ \text{in } d\mathbf{r} \text{ per unit time}\end{bmatrix} - \begin{bmatrix}\text{net number of neutrons}\\ \text{escaping from } d\mathbf{r} \text{ per}\\ \text{unit time}\end{bmatrix} \tag{5.4}$$

If we introduce the absorption cross section Σ_a for neutrons of speed v and the source function $S(\mathbf{r},t)$, where

$$S(\mathbf{r},t) \equiv \begin{matrix}\text{number of neutrons of speed } v \text{ produced per unit}\\ \text{volume per unit time around } \mathbf{r} \text{ at time } t\end{matrix} \tag{5.5}$$

the mathematical statement of Eq. (5.4) is

$$\frac{\partial}{\partial t} n(\mathbf{r},t)\, d\mathbf{r} = S(\mathbf{r},t)\, d\mathbf{r} - \Sigma_a \phi(\mathbf{r},t)\, d\mathbf{r} - \mathfrak{L}(\mathbf{r},t)\, d\mathbf{r} \tag{5.6}$$

The function $\mathfrak{L}(\mathbf{r},t)\, d\mathbf{r}$ denotes the net losses from the volume element $d\mathbf{r}$ due to transport of neutrons through the boundaries.

$$\mathfrak{L}(\mathbf{r},t) \equiv \begin{matrix}\text{net number of neutrons of speed } v \text{ lost per unit}\\ \text{volume around point } \mathbf{r} \text{ per unit time at time } t \text{ due to}\\ \text{passage through boundaries of the volume}\end{matrix} \tag{5.7}$$

The primary interest in this chapter is the derivation of solutions to Eq. (5.6) for various situations of importance to reactor physics. In each case we require also a suitable set of boundary conditions and an analytical expression for the leakage term $\mathfrak{L}(\mathbf{r},t)$. The remainder of the present section is devoted to the derivation of useful relations for defining these quantities.

c. Fick's Law Approximation for Net Neutron Current. The calculation of the leakage term $\mathfrak{L}(\mathbf{r},t)$ in Eq. (5.6) is somewhat involved, and a proper estimate of this quantity will not be considered until we have introduced the machinery of the transport equation. For our present needs it will suffice to use instead a convenient approximation which

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has given good results. What is really desired in making an estimate of the leakage is a knowledge of the net current of neutrons at any point, since, if this were available, it would be a simple matter to sum up the contributions from the net outward currents across all boundaries of the volume element in question. In order to introduce the approximating expression for the current, a simplified case is used.

Consider then a portion of a medium in which the neutron density is a function of only one space variable x and is independent of time (i.e., assume that steady state has been reached). Suppose a test area dA is placed in the medium at the station x and so oriented that its surface is normal to the direction of the axis (see Fig. 5.1). At any instant, neutrons are passing through this test surface from both sides and at all angles. We describe the net effect of these passages by means of the net current which we define

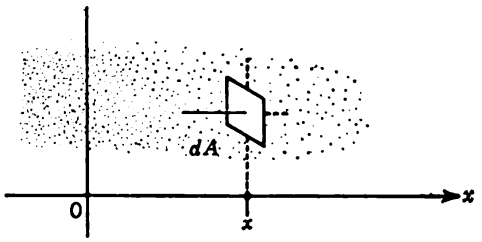

FIG. 5.1 Differential area dA.

$$J_x(x)\,dA \equiv \begin{array}{l}\text{net number of neutrons which pass through test area per unit time, with neutrons moving positively in } x \text{ counted as positive and those moving negatively in } x \text{ counted as negative}\end{array} \tag{5.8}$$

The approximating form for this function is chosen by the analogy of the over-all neutron movement to a diffusion process.[1] In the case of particles wandering through a medium in such a manner that they suffer many collisions in traveling over distances of interest, and emerge from each collision with a nearly isotropic distribution in direction, it has been found that the assumption that the net current be proportional to the negative gradient of the particle concentration leads to a correct description of many aspects of the process. In the one-dimensional neutron problem under consideration, then, we might describe the behavior by the assumption

$$J_x(x) = -Dv\frac{d}{dx}n(x) \tag{5.9}$$

This is the form of *Fick's law* for the diffusion of gases. The form of the proportionality constant in (5.9) is chosen for convenience in introducing the neutron flux; thus,

$$J_x(x) = -D\frac{d}{dx}\phi(x) \tag{5.10}$$

[1] P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," Part I, p. 173, McGraw-Hill Book Company, Inc., New York, 1953.

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since $\phi \equiv vn$. Note that the use of (5.9) or (5.10) implies that in a region in which the neutron density is spatially constant there is no net current, i.e.,

$$J_x(x) = 0 \qquad \text{for } n(x) = n_0, \text{ a constant} \tag{5.11}$$

This result is expected physically, since as many neutrons move through an element of area dA from the right as from the left.

It must be emphasized that (5.10) is not rigorously correct for the neutron case, and it will be inapplicable to some cases of practical interest. On the other hand, there are many situations in which it may be used profitably. In any event, the neutron application is intuitively appealing because of the analogous situation in gases; however, in the gas there is a pressure, due to molecule-molecule collisions, which is higher in the more densely populated region, and this pressure causes a net driving force toward the low-density regions. In the case of neutrons there are virtually no neutron-neutron collisions; the drift from the high-density regions to the low-density regions is a purely statistical effect (which is also present in the gas case) and is due to the fact that there are more neutrons available for the movement from high density to low than for the reverse movement.

The constant of proportionality D which appears in (5.9) and (5.10) is called the *diffusion coefficient*. In general it will be a function of the nuclear properties of the medium (and therefore a spatially dependent function in nonhomogeneous systems) and of the neutron speed. If we define the net neutron current in the units "neutrons per unit area per unit time," it follows that the diffusion coefficient D has the units of *length*. This definition differs from the one customarily used in gas-diffusion problems. In the gas problems the coefficient $\mathfrak{D}$ has the units "length squared divided by time." Thus if the current of gas particles is defined analogously to the neutron problem, $\mathfrak{D} \equiv vD$. Evidently the particle speed v is absorbed into the definition of the proportionality constant in the gas problem.

A detailed development of the concept of neutron current given in Chap. 7 shows that

$$D = \frac{\lambda_{tr}}{3} \tag{5.12}$$

where λ_{tr} is called the *transport mean free path*. In the usual way, we define a macroscopic cross section to correspond to λ_{tr}; thus

$$\Sigma_{tr} \equiv \frac{1}{\lambda_{tr}} \equiv \Sigma_t - \Sigma_s\overline{\mu_0} \tag{5.13}$$

where $\overline{\mu_0}$ is given by (4.30). This expression for Σ_{tr} is derived in Chap. 7. It is of interest to note that in moderator-type materials wherein $\Sigma_t \simeq \Sigma_s$,

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$\Sigma_{tr} \simeq \Sigma_s(1 - \overline{\mu_0})$; or, in terms of the mean free paths,

$$\lambda_{tr} \simeq \frac{\lambda_s}{1 - \overline{\mu_0}} \geq \lambda_s \tag{5.14}$$

which tells us that the transport mean free path is at least as large as the scattering mean free path and increases with increasing values of $\overline{\mu_0}$, the average cosine of the scattering angle in the laboratory system. Since large values of $\overline{\mu_0}$ correspond to small scattering angles, we find that the flow of neutrons is aided by forwardness in the scattering process. Thus one would expect, in very general terms, that low-nuclear-mass materials would support larger currents for a given neutron flux (density) gradient [see (4.30)].

Some values[1] of the diffusion coefficient for various materials of interest to reactor technology are listed in Table 5.1.

TABLE 5.1 THERMAL DIFFUSION LENGTHS AND DIFFUSION COEFFICIENTS[a]

Material	Density, g/cm³	Diffusion length L, cm	Diffusion coefficient D, cm	Absorption cross section, σ_a
Water	1.00	2.85	0.16	0.66 barns
Heavy water[b]	1.10			
Pure	....	171	0.800	
0.16% H_2O	....	116	0.883	0.93 mb
Beryllium	1.85	20.8	0.477	9 mb
Beryllium oxide	2.69	29.0	0.300	
Graphite	1.60			
GBF	....	54.4	1.043	4.4 mb
AGOT	....	52.0	1.040	4.8 mb

[a] These data were obtained from U.S. Atomic Energy Commission, "Reactor Handbook," vol. I, Table 1.5.4, p. 487, Document AECD-3645, March, 1955.

[b] It should be noted that L for D_2O is a very sensitive function of the H_2O content.

The concept of net current introduced above may be used to obtain an analytical expression for the leakage term $\mathfrak{L}$ from an elementary volume in the medium in question. Consider the differential volume $d\mathbf{r} \equiv dx\,dy\,dz$ shown in Fig. 5.2 which has faces normal to the x, y, and z axes. If we assume as before that $n = n(x)$, then clearly there will be no net leakage of neutrons across the faces $dx\,dy$ and $dx\,dz$ by the argument preceding Eq. (5.11). Thus,

$$\begin{bmatrix}\text{Net loss of}\\ \text{neutrons in}\\ \text{unit time}\\ \text{from } d\mathbf{r}\end{bmatrix} = \begin{bmatrix}\text{net current in positive } x \text{ direction}\\ \text{through face}\\ dy\,dz \text{ at } x + dx\end{bmatrix} - \begin{bmatrix}\text{net current in positive } x \text{ direction}\\ \text{through face}\\ dy\,dz \text{ at } x\end{bmatrix} \tag{5.15}$$

[1] Further discussion of these quantities will be given in Sec. 5.5 in connection with the diffusion-length concept.

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Or, in differential notation

$$\begin{aligned}\text{Net loss of neutrons per unit volume per unit time} &= \frac{J_x(x + dx)\, dy\, dz - J_x(x)\, dy\, dz}{d\mathbf{r}} \\ &\to \frac{d}{dx} J_x(x) \quad (5.16)\end{aligned}$$

In terms of the flux [using (5.10) and assuming that D is a constant]

$$\text{Net loss of neutrons per unit volume per unit time} = -D \frac{d^2}{dx^2} \phi(x) \quad (5.17)$$

Equation (5.16) shows that, in order that there be a net leakage different from zero, there must exist a spatial variation of the net neutron current.

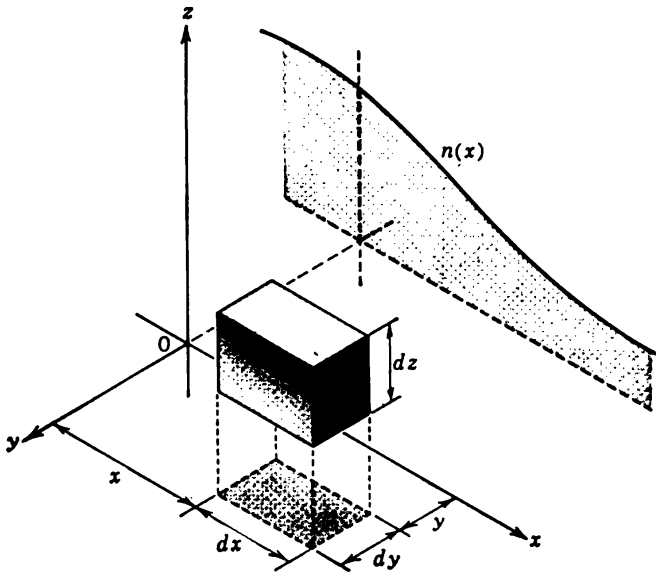

FIG. 5.2 Differential volume $d\mathbf{r} = dx\, dy\, dz$.

In the general problem the neutron density would be a function of all three space coordinates; thus, for a Cartesian system, we would define a density function $n = n(x,y,z)$. In this situation there would be a net movement of neutrons along each of the three coordinate axes. As before, we would compute a net current in the x direction $J_x(\mathbf{r})$; but now there will also be, in general, components in the y and z directions. These we compute in the same way as we did the x component. If $J_y(\mathbf{r})$ and $J_z(\mathbf{r})$ denote these components, and $d\mathbf{r}$ is a differential volume at $\mathbf{r}$, then the functions J_x, J_y, and J_z represent the flow rate of neutrons through the three sets of faces of this volume. In dealing with these quantities it is convenient to combine them into a single vector function $\mathbf{J}(\mathbf{r})$ (see

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Fig. 5.3). Thus the *vector net current* $\mathbf{J}$ is defined

$$\mathbf{J}(\mathbf{r}) \equiv J_x(\mathbf{r})\mathbf{i} + J_y(\mathbf{r})\mathbf{j} + J_z(\mathbf{r})\mathbf{k} \tag{5.18}$$

The symbols $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ denote unit vectors in the x, y, and z directions, respectively. This relation may also be written in terms of the neutron flux $\phi(\mathbf{r})$ by writing the components J_x, J_y, and J_z in the form of (5.10); then

$$\mathbf{J}(\mathbf{r}) = -D\left[\mathbf{i}\frac{\partial}{\partial x}\phi(\mathbf{r}) + \mathbf{j}\frac{\partial}{\partial y}\phi(\mathbf{r}) + \mathbf{k}\frac{\partial}{\partial z}\phi(\mathbf{r})\right] = -D\nabla\phi(\mathbf{r}) \tag{5.19}$$

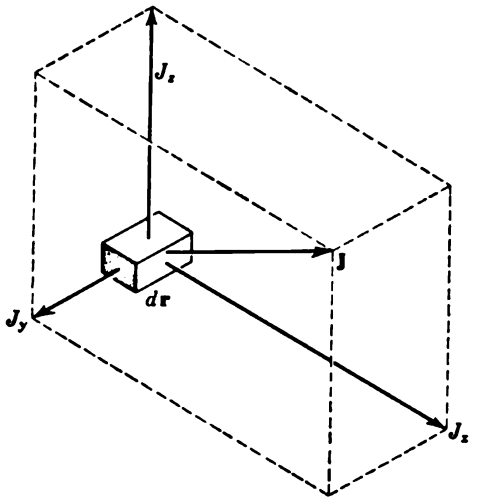

FIG. 5.3 Net neutron current.

We compute the leakage $\mathfrak{L}$ from the unit volume at $\mathbf{r}$ for this general case wherein $n = n(\mathbf{r})$ by applying the operation (5.17) to each of the three sets of faces of the volume. In this way we obtain the net leakage per unit volume at $\mathbf{r}$ per unit time.

$$\mathfrak{L} = \nabla \cdot \mathbf{J} = -D\nabla^2\phi \tag{5.20}$$

This expression is the generalization of the form (5.17) to a neutron flux which is a function of all three space coordinates. We will use this analytical expression for the net leakage in (5.6). The neutron-balance equation is then written

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},t) = S(\mathbf{r},t) - \Sigma_a\phi(\mathbf{r},t) + D\nabla^2\phi(\mathbf{r},t) \tag{5.21}\star$$

where we have used also (5.3). In the steady-state condition, this equation reduces to

$$-D\nabla^2\phi(\mathbf{r}) + \Sigma_a\phi(\mathbf{r}) = S(\mathbf{r}) \tag{5.22}\star$$

The time-dependent function $\phi(\mathbf{r},t)$ is here replaced by the steady-state function $\phi(\mathbf{r})$. The remainder of this chapter is concerned with the solution of Eqs. (5.21) and (5.22) for various specifications of the source S and of the geometric configuration of the medium in which the one-velocity neutrons are diffusing.

d. Neutron Partial Currents. In much of the analysis which follows it will be convenient to separate any given component of the net current into two parts. According to our original definition [see Eq. (5.8)] the net current at any point gives the net flow rate of neutrons through a unit area; thus, J_x is the excess of neutrons moving to the right (in the

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direction of positive x) over those moving to the left. An appropriate description of J_x, then, would be in terms of these two currents, one toward positive x, the other toward negative x. Therefore we define

$$J_x(\mathbf{r}) \equiv j_+(\mathbf{r}) - j_-(\mathbf{r}) \tag{5.23}$$

where $J_x(\mathbf{r})$ is the net current toward positive x at space point $\mathbf{r}$. The functions j_+ and j_- are called the *partial currents* along the x axis at $\mathbf{r}$. Specifically,

$$j_+(\mathbf{r}) \equiv \begin{array}{l}\text{number of neutrons which pass per unit time from} \\ \text{left to right (irrespective of the incident angle)} \\ \text{through a unit area normal to } x \text{ axis at } \mathbf{r}\end{array} \tag{5.24}$$

It is important to note that every neutron that passes through the unit area is counted if its velocity vector has a component in the positive x direction, regardless of the angle which its trajectory makes with this area. The partial current $j_-(\mathbf{r})$ defines the movement toward negative x in an analogous way. The two functions j_+ and j_- account for all the neutrons which pass through a unit area perpendicular to the x axis, and J_x is the measure of the net movement toward positive x. We now compute the partial current $j_-(\mathbf{r})$.

Consider a differential area dA in an infinite medium supporting a one-velocity neutron flux. Let the center of dA be the origin of a Cartesian coordinate system such that the x axis is normal to dA. Define also a polar coordinate system such that the co-latitude angle θ is measured from the positive x axis and the azimuth angle ψ from the positive y axis (see Fig. 5.4). Then, by definition, $j_-(0)\,dA$ is the number of neutrons which pass through dA per unit time having made their last scattering collision in the upper half-space in which x is positive. We can compute this number by summing all the contributions from such collisions in the half-space. Consider then an elementary volume $d\mathbf{r}$ at space point $\mathbf{r} = \mathbf{r}(r,\theta,\psi)$ (see Fig. 5.4). There are $\Sigma_s\phi(\mathbf{r})\,d\mathbf{r}$ scattering collisions per unit time in this volume. Of the neutrons leaving $d\mathbf{r}$ from these collisions, a certain fraction is directed toward dA. We will estimate this fraction on the assumption that the neutrons scattered in $d\mathbf{r}$ emerge with an isotropic distribution in angle; i.e., for the medium in question elastic scatterings are isotropic in the laboratory system.

It is difficult to improve upon the isotropic assumption with the present method of approach, and it is exactly at this point that the information about the angular distribution of the neutrons is lost. In general, the neutron scattering is not isotropic in (L), and the neutrons scattered in $d\mathbf{r}$ will emerge isotropically only if they impinged upon $d\mathbf{r}$ isotropically. Thus, if there is a net transport of neutrons in the region of $d\mathbf{r}$, the assumption of isotropic scattering is surely invalid, and the error made cannot be estimated by the theory (in this form). Suppose,

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however, that we accept it for the purpose of completing the calculation and correct the theory later by using the results of more exact models. We compute then the number of neutrons directed toward dA on the assumption that they emerge from $d\mathbf{r}$ uniformly in 4π solid angle. The probability that a neutron leaving $d\mathbf{r}$ isotropically is headed toward dA is simply the projected area of dA visible at $d\mathbf{r}$ divided by the surface area of a sphere of radius r. The projected area of dA is seen to be $\cos\theta\, dA$;

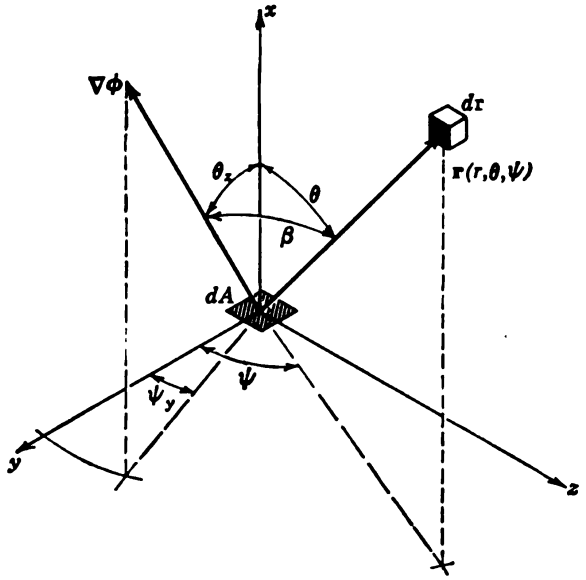

FIG. 5.4 Orientation of volume $d\mathbf{r}$ and area dA.

thus the fraction of neutrons leaving $d\mathbf{r}$ which are directed toward dA is $\cos\theta\, dA/4\pi r^2$.

Not all the neutrons headed for dA actually reach dA, however. Some of these neutrons suffer further collisions along the path and are thereby removed from the stream either by being absorbed or by being scattered into a new direction which does not pass through dA. We estimate this number on the assumption that the principal source of removals is scattering collisions; i.e., we neglect absorption losses. Thus we are implying that the medium in question is only slightly absorbing. We will show later that this is in fact the only condition under which diffusion theory yields a good approximation of the spatial distribution of neutrons. If we assume that scattering collisions are the only mechanism by which neutrons can be removed from the group directed toward dA, then by the concepts developed in Sec. 2.2 we have [cf. Eq. (2.17)]

$$P(\xi > r) \equiv 1 - P(\xi \leq r) = 1 - (1 - e^{-\Sigma_s r}) = e^{-\Sigma_s r} \qquad (5.25)$$

Thus $e^{-\Sigma_s r}$ is the probability that a neutron headed for dA traverses the distance r without suffering a scattering collision along the way. We multiply this result with $\cos\theta\, dA/4\pi r^2$ to obtain the fraction of neutrons

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leaving $d\mathbf{r}$ which reach dA. If we define a current

$$j_-(0;r,\theta,\psi)\,d\mathbf{r}\,dA \equiv \begin{array}{l}\text{number of neutrons which pass through}\\ dA \text{ per unit time due to last scattering}\\ \text{collisions in } d\mathbf{r} \text{ about } \mathbf{r}(r,\theta,\psi)\end{array} \tag{5.26}$$

then for $\mu \equiv \cos\theta$ and $d\mathbf{r} = r^2\,dr\,d\mu\,d\psi$, we have[1]

$$j_-(0;r,\mu,\psi)\,d\mathbf{r}\,dA = \frac{\mu e^{-\Sigma_s r}\Sigma_s \phi(r,\mu,\psi)\,d\mathbf{r}\,dA}{4\pi r^2} \tag{5.27}$$

The integral of this quantity over all r yields the contribution to the current through dA from all neutrons which made their last scattering collision somewhere in the differential solid angle $d\mu\,d\psi$ (see Fig. 5.5). We define a second function

$$j_-(0;\mu,\psi)\,d\mu\,d\psi\,dA \equiv \begin{array}{l}\text{number of neutrons which pass through } dA\\ \text{per unit time having made their last}\\ \text{scattering collision somewhere in the solid}\\ \text{angle } d\mu\,d\psi \text{ about the direction } (\mu,\psi)\end{array} \tag{5.28}$$

and obtain

$$j_-(0;\mu,\psi)\,d\mu\,d\psi\,dA \equiv \frac{\mu\,d\mu\,d\psi\Sigma_s\,dA}{4\pi}\int_0^\infty e^{-\Sigma_s r}\phi(r,\mu,\psi)\,dr \tag{5.29}$$

In order to proceed we must have some additional information as to the functional form of ϕ over the positive x half-space. Since we are dealing with a general situation, such specific information is not available; however, the presence of the exponential factor $e^{-\Sigma_s r}$ in (5.29) allows us to make an approximation of the integral which does not require so detailed a knowledge of ϕ. It is evident that because of the exponential in the integrand, the principal contribution to the integral comes from the region of a few mean-free-path lengths about the origin, i.e., about the elementary area dA. Thus, if the flux is sufficiently slowly varying, approximate results may be obtained by expanding ϕ in a Taylor series about the origin. (Note that this requirement is consistent with the previous assumption that the medium is slightly absorbing.) Let us use, then,

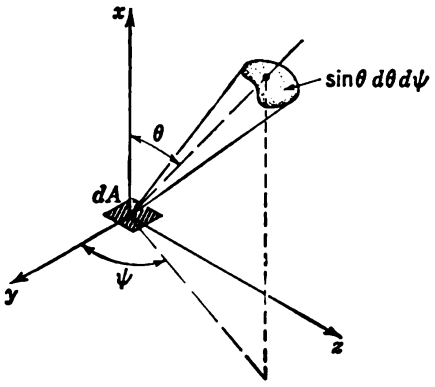

FIG. 5.5 Scatterings from direction (θ,ψ).

$$\phi(\mathbf{r}) = \phi(0) + \mathbf{r}\cdot\nabla\phi(0) + \tfrac{1}{2}\mathbf{r}^2\nabla^2\phi(0) + \cdots \tag{5.30}$$

For the present calculation the first two terms will suffice. The first term

[1] The parameters r, μ, and ψ indicate the source of the current in question.

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of (5.30) is the flux at dA; the second term is the first derivative correction for the displacement r. If the angle between the vectors $\mathbf{r}$ and $\nabla\phi$ is designated β, then

$$\mathbf{r} \cdot \nabla\phi(0) = r|\nabla\phi(0)| \cos\beta \tag{5.31}$$

(see Fig. 5.4). The substitution of the first two terms of (5.30) into (5.29) along with the expression (5.31) yields

$$\begin{aligned} j_-(0;\mu,\psi)\, d\mu\, d\psi\, dA &\cong \frac{\mu\, d\mu\, d\psi \Sigma_s\, dA}{4\pi} \\ &\quad \times \int_0^\infty [\phi(0)e^{-\Sigma_s r} + |\nabla\phi(0)| \cos\beta\, re^{-\Sigma_s r}]\, dr \\ &= \frac{\mu\, d\mu\, d\psi\, dA}{4\pi}\left[\phi(0) + \frac{|\nabla\phi(0)| \cos\beta}{\Sigma_s}\right] \end{aligned} \tag{5.32}$$

The integral of this quantity over all ψ is the contribution to the current at dA (in direction of negative x) due to all neutrons which made their last scattering collision in the conical shell $d\mu$ (see Fig. 5.6). We define then

$$j_-(0;\mu)\, d\mu\, dA \equiv \begin{array}{l}\text{number of neutrons passing through } dA \\ \text{per unit time which} \\ \text{have made their last} \\ \text{scattering collision in} \\ \text{the shell } d\mu \text{ about } \mu\end{array} \tag{5.33}$$

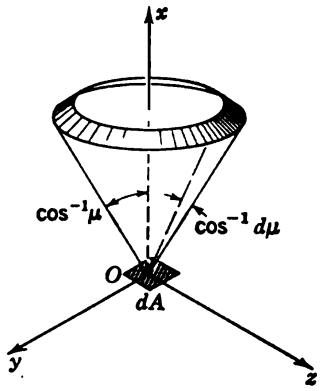

FIG. 5.6 Scatterings from conical shell $d\mu$.

and

$$\begin{aligned} j_-(0;\mu)\, d\mu &= \int_0^{2\pi} j_-(0;\mu,\psi)\, d\psi \\ &= \frac{\mu\, d\mu}{4\pi}\left[2\pi\phi(0) + \frac{|\nabla\phi(0)|}{\Sigma_s}\int_0^{2\pi} \cos\beta\, d\psi\right] \end{aligned} \tag{5.34}$$

We can write $\cos\beta$ in terms of the direction angles of the two vectors $\mathbf{r}$ and $\nabla\phi$. If we denote by θ_x and ψ_y the co-latitude and azimuth angles of $\nabla\phi$ (see Fig. 5.4), then

$$\cos\beta = \cos\theta_x \cos\theta + \sin\theta_x \sin\theta \cos(\psi_y - \psi) \tag{5.35}$$

Substitution into (5.34) yields

$$j_-(0;\mu)\, d\mu = \frac{1}{2}\, \mu\, d\mu\left[\phi(0) + \frac{|\nabla\phi(0)|\mu \cos\theta_x}{\Sigma_s}\right] \tag{5.36}$$

Equation (5.36) gives the current through unit area normal to the x axis (located at the origin) due to all last scatterings in a cone of width $d\mu$ about the x axis. This result will be very useful for computing the current in systems in which the neutron density is azimuthally symmetric.

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A final integration of (5.36) over all μ in the upper half-space gives the partial current through unit area in the direction of negative x. Thus,

$$j_-(0) = \int_0^1 j_-(0;\mu)\, d\mu = \frac{1}{4}\phi(0) + \frac{1}{6\Sigma_s}|\nabla\phi(0)|\cos\theta_x \qquad (5.37)$$

The computation of the partial current $j_+(0)$ may be carried out in the same way. The procedure is straightforward; however, some care must

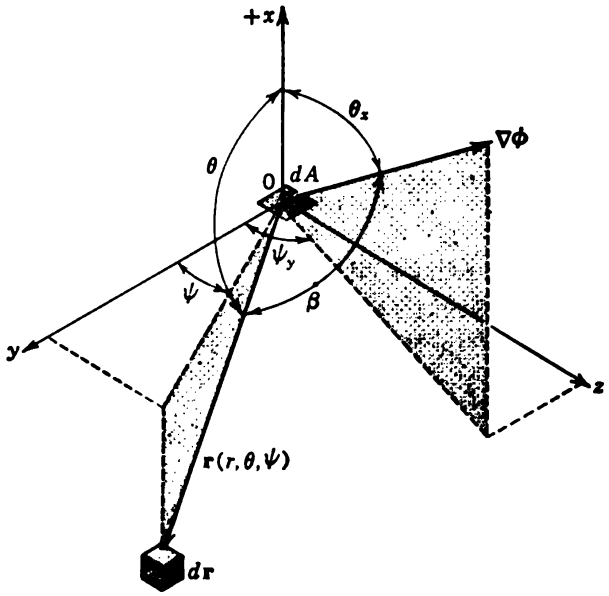

FIG. 5.7 Coordinate system for negative x half-space.

be taken in the definition of the direction angles of the vector quantities involved. Thus, we take

$$j_+(0;r,\mu,\psi)\, dA \equiv \Sigma_s\phi(r,\mu,\psi)e^{-\Sigma_s r}\left|\frac{\mu\, dA}{4\pi r^2}\right| \qquad (5.38)$$

where the absolute magnitude sign is introduced in order to account for the fact that, throughout the negative x half-space, μ takes on the values -1 to 0 (see Fig. 5.7). The contributions from the solid angle $d\mu\, d\psi$ are

$$j_+(0;\mu,\psi)\, d\mu\, d\psi = \frac{|\mu|\, d\mu\, d\psi}{4\pi}\left[\phi(0) + \frac{|\nabla\phi(0)|\cos\beta}{\Sigma_s}\right] \qquad (5.39)$$

and from the conical shell $d\mu$

$$j_+(0;\mu)\, d\mu = \frac{1}{2}|\mu|\, d\mu\left[\phi(0) + \frac{|\nabla\phi(0)|\mu\cos\theta_x}{\Sigma_s}\right] \qquad (5.40)$$

Finally

$$\begin{aligned} j_+(0) &= \frac{1}{2}\left[\phi(0)\int_{-1}^0 |\mu|\, d\mu + \frac{\cos\theta_x|\nabla\phi(0)|}{\Sigma_s}\int_{-1}^0 |\mu|\mu\, d\mu\right] \\ &= \frac{1}{4}\phi(0) - \frac{|\nabla\phi(0)|\cos\theta_x}{6\Sigma_s} \end{aligned} \qquad (5.41)$$

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These expressions for the partial currents (5.37) and (5.41) may be used to compute the net current according to the statement (5.23). Thus, for an arbitrary space point $\mathbf{r}$ in an infinite medium, we obtain

$$J_x(\mathbf{r}) = -\frac{1}{3\Sigma_s}|\nabla\phi(\mathbf{r})|\cos\theta_x \tag{5.42}$$

where θ_x is the angle which the vector $\nabla\phi$ makes with the positive x axis. If the flux is a function of only one space coordinate, say x, the angle $\theta_x = 0$, and $\phi(\mathbf{r}) = \phi(x)$; moreover, $|\nabla\phi(\mathbf{r})| = \phi'(x)$, and

$$J_x(\mathbf{r}) = J_x(x) = -\frac{1}{3\Sigma_s}\phi'(x) \tag{5.43}$$

A comparison of this result with (5.10) shows that the elementary model used in the calculation of (5.37) and (5.41) yields an expression for the net current which differs from that obtained by a more elaborate theory through the constant of proportionality. Whereas the transport theory gives for the diffusion coefficient [see Eqs. (5.12) and (5.13)]

$$D = \frac{1}{3(\Sigma_t - \Sigma_s\overline{\mu_0})} \tag{5.44}$$

diffusion theory gives

$$D = \frac{1}{3\Sigma_s} \tag{5.45}$$

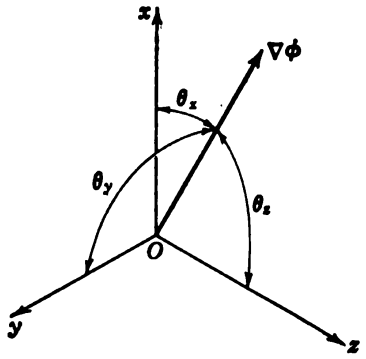

FIG. 5.8 Gradient of neutron flux.

Thus although diffusion theory is able to predict the functional form of the net current, it fails to give a correct estimate of the diffusion coefficient. However, the estimate (5.45) is fairly good for media of large mass number and small absorption cross section; if both Σ_a/Σ_s and $1/A$ are much less than unity, then by (5.44)

$$D \simeq \frac{1}{3\Sigma_s(1 - \overline{\mu_0})} = \frac{1}{3\Sigma_s(1 - 2/3A)} \simeq \frac{1}{3\Sigma_s} \tag{5.46}$$

In all subsequent work we will use the relation (5.44) for calculating the diffusion coefficient.

The result (5.42) for the net current can be used to construct the vector net current at point $\mathbf{r}$. If θ_y and θ_z denote the angles which $\nabla\phi$ makes with the y and z axes, respectively (see Fig. 5.8), then

$$\begin{aligned} J_x(\mathbf{r}) &= -D|\nabla\phi(\mathbf{r})|\cos\theta_x \\ J_y(\mathbf{r}) &= -D|\nabla\phi(\mathbf{r})|\cos\theta_y \\ J_z(\mathbf{r}) &= -D|\nabla\phi(\mathbf{r})|\cos\theta_z \end{aligned} \tag{5.47}$$

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where we have used (5.44) for the constant of proportionality. The expressions (5.47) may be combined according to (5.18)

$$\begin{aligned}\mathbf{J}(\mathbf{r}) &= J_x(\mathbf{r})\mathbf{i} + J_y(\mathbf{r})\mathbf{j} + J_z(\mathbf{r})\mathbf{k} \\ &= -D|\nabla\phi(\mathbf{r})|(\mathbf{i}\cos\theta_x + \mathbf{j}\cos\theta_y + \mathbf{k}\cos\theta_z) = -D\nabla\phi(\mathbf{r})\end{aligned}$$

as is to be expected. The partial currents may also be written in terms of D; thus in place of (5.37) and (5.41) we will use

$$\begin{aligned}j_+(\mathbf{r}) &= \frac{1}{4}\phi(\mathbf{r}) - \frac{D}{2}|\nabla\phi(\mathbf{r})|\cos\theta_x \\ j_-(\mathbf{r}) &= \frac{1}{4}\phi(\mathbf{r}) + \frac{D}{2}|\nabla\phi(\mathbf{r})|\cos\theta_x\end{aligned} \tag{5.48}$$

If $\phi(\mathbf{r}) \equiv \phi(x)$, we have

$$\begin{aligned}j_+(x) &= \frac{1}{4}\phi(x) - \frac{D}{2}\phi'(x) \\ j_-(x) &= \frac{1}{4}\phi(x) + \frac{D}{2}\phi'(x)\end{aligned} \tag{5.49}$$

It is of interest to note that if there is no net current at a given space point, i.e., if $|\nabla\phi(\mathbf{r})| \equiv 0$, then

$$j_+(\mathbf{r}) = j_-(\mathbf{r}) = \tfrac{1}{4}\phi(\mathbf{r}) \tag{5.50}$$

which states that, if as many neutrons are moving through a unit area at $\mathbf{r}$ along a given direction as are moving in the opposite direction, then the neutron flow rate is $\phi/4$. This case was previously mentioned in Sec. 3.5.

For the remainder of this chapter we will use the following set of equations in discussing the one-velocity approximation:

$$\begin{aligned}-D\nabla^2\phi(\mathbf{r}) + \Sigma_a\phi(\mathbf{r}) &= S(\mathbf{r}) && (a) \\ \mathbf{J}(\mathbf{r}) &= -D\nabla\phi(\mathbf{r}) && (b) \\ D = \frac{1}{3\Sigma_{tr}} &= \frac{1}{3(\Sigma_t - 2\Sigma_s/3A)} && (c)\end{aligned} \tag{5.51}\star$$

We will require in addition to these equations a set of boundary conditions which define the specific problem. A general discussion of these conditions is given below.

e. General Statement of Boundary Conditions. A complete statement of the boundary conditions which are to be used in the application of the one-velocity model is not possible on the basis of the physical concepts so far developed. These considerations[1] must be deferred until we have a more detailed description of the neutron flux than presently available. It is possible nevertheless to establish an appropriate set of

[1] A detailed description of the boundary conditions to be used in the one-velocity model involves the methods and results of transport theory (see Chap. 7).

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boundary conditions from the machinery of the diffusion theory which, although lacking in details, will not be inconsistent with the results of more accurate models.

On the basis of physical arguments we must require that the neutron flux satisfy the following conditions:

(1) The neutron flux must be finite and nonnegative in all regions of a medium except possibly with certain source distributions (5.52)

(2) At an interface between two media with different diffusion properties, the density (or flux) of neutrons moving in any given direction must be a continuous function through the interface (5.53)

(3) At a vacuum interface (the outermost boundary of a system) there must be no neutrons returning to the medium from any direction in the vacuum (5.54)

These conditions are perfectly general and must be met, at least in approximate form, by whatever function is specified for the neutron-density distribution. The neutron flux defined by the elementary diffusion theory cannot meet all these conditions. The function $\phi(\mathbf{r})$ is much too coarse a description of the neutron population in that it gives the track length per unit volume per unit time of *all* neutrons of a single speed. Conditions such as (2) above require, however, a much more detailed statement, and so must be replaced by less exacting requirements. Before inquiring into the nature of these approximations, let us first examine the implications of the three conditions as stated.

Consider condition (1): it is clear from this requirement that all regions of a medium which are in the vicinity of an elementary source (such as a point source) are to be excluded. Equation (5.51*a*) does not apply then in the neighborhood of a singularity; everywhere else, however, the flux $\phi(\mathbf{r})$ is defined and must be bounded. Condition (2) requires that at an interface between two dissimilar media the density of neutrons moving in a given direction must be identical on either side of the interface. This statement implies, of course, that there are no neutron sources or sinks at the interface. Thus (2) states that neutrons are not to "pile up" at such boundaries. Condition (3) is a special case of (2) and implies that, once a neutron has left the outer boundary of the system, it will not return[1] to the system. This requirement is stated in detail, namely, that no neutrons return from *any* given direction.

That conditions (2) and (3) cannot be satisfied by the function $\phi(\mathbf{r})$ is entirely clear; $\phi(\mathbf{r})$ gives the total track length, whereas what is needed is the track length (or density) of neutrons which are traveling in a par-

[1] This is of course true only if the outer boundary of the system is a simply connected and nonreentrant surface.

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ticular direction. Such a function which specifies the angular dependence of the neutron flux can be obtained only from the transport equation. For our present needs, then, conditions (2) and (3) are far too detailed and must be replaced by suitable approximations which deal with the bulk behavior of the neutron population rather than with the detailed behavior of specific directional components. The approximations to the exact boundary conditions which we will use in connection with Eqs. (5.51) are:

(1) The neutron flux must be finite and nonnegative in all regions in which the diffusion equation (5.51*a*) applies (5.55)

(2′) The partial currents must be continuous through an interface between two dissimilar media (5.56)

(3) At a vacuum interface, the neutron flux must satisfy one of the following two conditions: (*a*) The partial current from the vacuum must be zero, or (*b*) the function which describes the flux must vanish at some finite distance outside the actual geometric boundary of the system (5.57)

Boundary condition (1) is essentially as before. Boundary condition (2′) may be illustrated by deriving the appropriate relations for the interface condition in a system in which the flux is a function of only one space coordinate. Consider, therefore, two semi-infinite media with different diffusion properties and with a common boundary at $x = 0$. Denote the medium to the left by the index $m = 1$ and that to the right by $m = 2$. Further to generalize the problem, assume that at the interface there is a plane of isotropic sources with strength q_0 neutrons per unit area per unit time. Assume also that the presence of this source does not alter the diffusion properties of the media. If $j_+^{(m)}(x)$ denotes the partial neutron current in medium m at point x in the positive x direction, then by condition (2′)

$$\begin{aligned} j_+^{(2)}(0) &= \tfrac{1}{2}q_0 + j_+^{(1)}(0) \\ j_-^{(1)}(0) &= \tfrac{1}{2}q_0 + j_-^{(2)}(0) \end{aligned} \tag{5.58}$$

If we add these equations and introduce the expressions (5.49) for the partial currents, we obtain

$$J_2(0) - J_1(0) = q_0 \tag{5.59}$$

Thus if the source is removed ($q_0 \equiv 0$), the net current through the interface must be a continuous function, and it is discontinuous only in the presence of a neutron source $q_0 \neq 0$ (see Fig. 5.9). The subtraction of Eqs. (5.58) yields

$$\phi_2(0) = \phi_1(0) \tag{5.60}$$

Therefore, independent of the source strength q_0, condition (2′) requires that the neutron flux be continuous through the interface. Since $\phi(x)$

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satisfies an ordinary second-order differential equation, the two conditions (5.59) and (5.60) can be specified independently. An alternate statement of (2′) is

(2) The neutron flux and net current must be continuous at an interface (in the absence of a source) (5.61)

It is important to recognize that condition (2′) or (2) is arbitrary and serves only as an approximation to the original condition (5.53). Although the requirement that the net current be continuous at an interface is satisfactory both from a physical standpoint and analytically, the requirement that the flux be continuous is not entirely meaningful. The

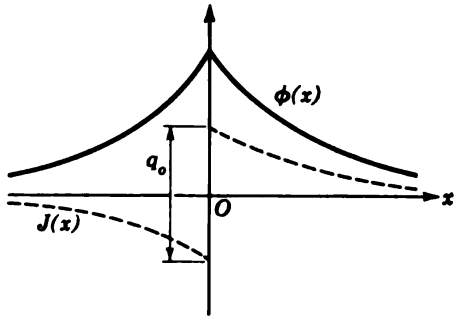

FIG. 5.9 Flux and net current around interface.

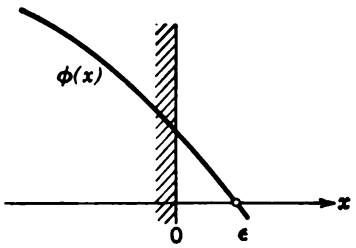

FIG. 5.10 Vacuum interface.

difficulty arises from the fact that the flux $\phi(\mathbf{r})$ as defined by the diffusion equation (5.51*a*) is not a good representation of the neutron population in the vicinity of a material discontinuity. In fact, as will be shown later, this is precisely one of the situations under which diffusion theory breaks down. To place a condition on a function in a region in which it most poorly represents the physics is certainly questionable. A more satisfactory interface condition for the flux, called the Serber condition, has been suggested,[1] but this condition is not useful for dealing with arbitrary geometries. Thus condition (5.60) is generally used for want of a more accurate and easily applied relation.

Boundary condition (3) is the approximation to the physical requirement (5.54) that no neutrons return to the system from the vacuum along any direction. This requirement cannot be rigorously satisfied in the present case in which we wish to apply the diffusion approximations (5.51). Therefore we impose the less stringent condition that the *partial* current from the vacuum be identically zero. Hence

$$j_-(0) \equiv 0 \tag{5.62}$$

where the origin of the coordinate system is placed at the physical boundary of the medium (see Fig. 5.10). This condition may be written in

[1] The Serber condition is frequently used in spherically symmetric systems (see Sec. 8.3).

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terms of the flux and its gradient at the boundary. We have by (5.49)

$$\frac{\phi'(0)}{\phi(0)} = -\frac{1}{2D} \qquad (5.63)\star$$

Equation (5.62) gives an acceptable boundary condition at a medium-vacuum interface for use with the diffusion equations. However, it should be realized that this relation places a condition on a function which was developed on the basis of the diffusion approximation. It will be recalled that, in the derivation of Eq. (5.37) for the partial current, it was assumed that the flux was a slowly varying function at the point in question and therefore could be represented by two terms of a Taylor series. This requirement is hardly satisfied at any interface, least likely at a vacuum interface. In such regions higher-order terms in the series would be required, and the use of only the first two terms, i.e., the diffusion approximation, is at best a first estimate.

While Eq. (5.63) is quite easily applied in most problems, it has been shown that there is an even simpler condition which affords a very good approximation in a great many useful cases. This approximating condition may be developed on the basis of the following observation. Within the interior of a medium supporting a neutron gas, for example, a reactor, the neutron density (flux) is relatively high, since in general the neutron sources (fissionable material) are located in this region. Neutrons produced by these sources are eventually lost by capture collisions or escapes from the system. Thus a close inspection of the neutron population in the medium would reveal that there is a continuous net movement of neutrons toward the outer boundaries of the system. Moreover, this net outward flow becomes more pronounced as the physical boundary (vacuum interface) is approached. In the limit, at the outer surface, all the neutron velocity vectors will be directed out into the surrounding vacuum. Thus at each station, progressing outward from the interior, there is a net flow of neutrons away from the center. Whereas the neutron population in the central regions is augmented by neutrons returning from scattering collisions in regions beyond, the outermost portions of the system are supplied only by the neutron sources in the center. Thus the neutron density falls off rapidly as the surface of the system is approached; however, it does not vanish entirely inasmuch as there is always a finite number of neutrons passing through the outer layers of the system.

It would appear then that one could establish a reasonable approximation to the vacuum interface condition by requiring that the function representing the neutron density vanish at some appropriate point *near* the physical boundary. Note that our requirement is that the flux vanish in the vicinity of the boundary and not *at* the boundary. The problem, of course, is to determine the distance of this point from the

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surface such that the resulting statement of the boundary condition best approximates the exact requirement. Let us call this distance ϵ, and suppose that in a typical problem the general solution to the diffusion equation is known and that one of the arbitrary constants has been eliminated by an appropriate boundary condition at some interior point. The vacuum interface condition may then be applied to eliminate the remaining constant. If the solution mentioned above is $\phi(x)$, then we will require that

$$\phi(\epsilon) = 0 \tag{5.64}$$

where for convenience we have placed the origin at the surface (see Fig. 5.10). If the approximating form of the flux $\phi(x)$, which is now determined everywhere in the medium, is to be a good estimate of the actual flux, then it is necessary to satisfy condition (5.63) as closely as possible at $x = 0$. Now consider $\phi(0)$; if ϵ is sufficiently small, we can write

$$\phi(\epsilon) = \phi(0) + \phi'(0)\epsilon + \cdots \tag{5.65}$$

But since $\phi(\epsilon)$ has been set equal to zero, (5.65) gives

$$\frac{\phi'(0)}{\phi(0)} = -\frac{1}{\epsilon} \tag{5.66}$$

By comparison with (5.63) it is seen that the best value of ϵ, under the condition that the expansion (5.65) be valid, is

$$\epsilon = 2D = \tfrac{2}{3}\lambda_{tr} \tag{5.67}$$

in which the last step follows from (5.12). We may summarize as follows: to the extent that the equation

$$\phi(x_0 + \tfrac{2}{3}\lambda_{tr}) = \phi(x_0) + \phi'(x_0)\tfrac{2}{3}\lambda_{tr}$$

holds (here we take the coordinate of the boundary to be x_0), the more accurate boundary condition

$$j_-(x_0) = 0 \tag{5.68}$$

may be replaced by the condition

$$\phi(x_0 + \tfrac{2}{3}\lambda_{tr}) = 0 \qquad \text{or} \qquad \phi(x_0 + \epsilon) = 0 \tag{5.69}$$

It is important to note that (5.69) and (5.68) are in fact different conditions and, if applied separately to the same system, will yield different solutions for the flux ϕ. Although these solutions will differ only slightly in those situations wherein the diffusion theory is applicable, they will by no means be identical.[1]

The quantity ϵ is often called the *extrapolation distance*, or *extrapolation length*, because the analytical solution to the flux is extrapolated beyond the physical boundary of the medium into a region in which the diffusion

[1] An illustration is given in Sec. 5.3a.

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equation does not apply. Since (5.65) includes terms only up to the first order, this process is sometimes referred to as "linear extrapolation." Note that the use of (5.69) implies that the medium extends out to point $(x_0 + \frac{2}{3}\lambda_{tr})$, since the diffusion equation is applied out to that point; however, this is merely a device for obtaining a good approximation to the flux *inside* the actual medium.

For many cases, particularly large thermal reactors, the use of the extrapolation distance, which is small in these cases in comparison with the dimensions of the reactor, has given very good results; the approximating condition (5.69) will be consistently used throughout this work to replace the more accurate condition (5.68). For convenience we shall also use the notation

$$x_0 \equiv x_0 + \epsilon \tag{5.70}$$

The estimate obtained above for ϵ (5.67) is based on the diffusion theory. Other estimates can be obtained with the application of more accurate models, such as transport theory.[1] A somewhat better number than (5.67) is

$$\epsilon = 0.7104\lambda_{tr} \tag{5.71}\star$$

Further, Eqs. (5.67) and (5.71) are each specifically calculated for planar boundaries; but they are approximately correct for boundaries of small curvature. For boundaries of increasingly greater curvature, ϵ monotonically approaches a limiting value of $\frac{2}{3}\lambda_{tr}$.

We conclude the present section with a summary of the diffusion-theory boundary conditions which we will generally apply throughout the remainder of this work. These conditions are the statements (5.55), (5.61), and (5.69):

(*a*) The neutron flux must be finite and nonnegative in all regions in which the diffusion equation applies
(*b*) The flux and net current must be continuous at an interface between two different media
(*c*) The flux must vanish at the extrapolated outer boundary of the system

(5.72)★

In using these conditions in connection with the diffusion theory, it must be borne in mind that they are only approximations within the limitations of the theory to the precise statements (5.52), (5.53), and (5.54).

5.2 Elementary Source in Infinite Media

a. Plane Source. We introduce the application of the diffusion equations to the determination of the spatial distribution of neutrons by con-

[1] S. Frankel and E. Nelson, "Methods of Treatment of Displacement Integral Equations," AECD-3497, September, 1953.

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sidering infinite media which contain various elementary sources. The results obtained in these calculations are applied later to neutron-diffusion problems which involve more complex source distributions and to systems of finite extent. Initially, for the sake of simplicity, we consider only steady-state problems and systems in which the spatial distribution of the flux can be expressed in terms of a single space coordinate. Thus, we begin with the diffusion of neutrons from isotropic plane, point, and line sources. In the case of the plane source, we have a simple rectilinear system described by a single coordinate x; in the case of the point source, we have a spherically symmetric system; and in the case of the line source, the problem has axial symmetry. These analyses are followed by an elementary development of the method of Green's function. The section is concluded with some applications of the method to the simple singular sources and, finally, to the general problem with arbitrary source distribution.

The appropriate differential equations for the neutron-diffusion problem in rectilinear coordinates is obtained from (5.51) and takes the form

$$\begin{aligned} -D\nabla^2\phi(x,y,z) + \Sigma_a\phi(x,y,z) &= S(x,y,z) \qquad (a) \\ \mathbf{J}(x,y,z) &= -D\nabla\phi(x,y,z) \qquad (b) \end{aligned} \tag{5.73}$$

We specialize this system to the case of a plane source in an infinite medium. Consider then a plane of isotropic neutron sources of strength q_0 neutrons per unit area per unit time (i.e., the sources emit neutrons with equal probability in all directions from all points on the source plane). We assume that, once a neutron is released from the source, the source plane is no longer a special region in the medium. The problem is to determine the neutron-flux distribution throughout the infinite medium in the steady state, that is, long after the source has begun emitting neutrons. The concept of a plane distribution of isotropic sources reduces the problem to finding the spatial dependence of the flux in terms of a single coordinate, in the time-independent case. If x denotes the coordinate normal to the infinite-source plane (which is placed at $x = 0$), then clearly the flux must be independent of y and z. Thus $\phi(x,y,z) \to \phi(x)$, and for $x \neq 0$, (5.73a) reduces to

$$\phi''(x) - \frac{1}{L^2}\phi(x) = 0 \qquad x \neq 0 \tag{5.74}$$

where we define the *diffusion length* L by

$$L^2 \equiv \frac{D}{\Sigma_a} \tag{5.75}\star$$

The general solution to (5.74) may be written in the exponential form

$$\phi(x) = Ae^{x/L} + Be^{-x/L} \tag{5.76}$$

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and applies only to regions away from the origin. We note however that the system is symmetric about $x = 0$; thus we may consider the solution for $x > 0$ alone. Boundary condition (5.72a) requires that $\phi(x)$ remain bounded for all x. In order that this condition be satisfied for arbitrarily large x, we must take $A \equiv 0$. We have, then, $\phi(x) = Be^{-x/L}$ for $x > 0$. A more general form which is applicable to $x \lesseqgtr 0$ is

$$\phi(x) = Be^{-|x|/L}$$

The constant B is evaluated by applying the special boundary condition (5.59). In the present situation, the source is isotropic; therefore, exactly one-half the neutrons must be released to each side. Thus,

$$\lim_{|x|\to 0} J(x) = \tfrac{1}{2}q_0 \tag{5.77}$$

We compute $J(x)$ from (5.10) and note that this is the specialization of (5.51b) to the present problem. Substitution of this result into (5.77) yields, in taking the limit as $|x| \to 0$, $B = Lq_0/2D$. The solution to (5.74) is, therefore,

$$\phi(x) = \frac{q_0 L e^{-|x|/L}}{2D} \tag{5.78}\star$$

This expression gives the spatial distribution of neutrons in an infinite medium with a plane isotropic source (see Fig. 5.11).

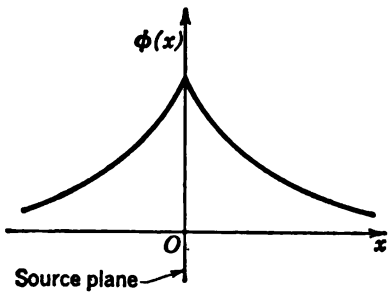

FIG. 5.11 Flux distribution due to plane source.

b. Point Source. If the neutron sources are concentrated at a point, and the emission is isotropic, then the problem is spherically symmetric. It is convenient in this case to describe the spatial distribution in terms of polar coordinates with the origin at the source; therefore, we take $\phi(\mathbf{r}) \equiv \phi(r,\theta,\psi)$, and (5.51$a$) may be written

$$-D\left[\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\phi) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\psi^2}\right] + \Sigma_a\phi = 0 \tag{5.79}$$

for $r > 0$. For the present case, ϕ is independent of the angles θ and ψ; so $\phi \equiv \phi(r)$, and (5.79) reduces to

$$\frac{1}{r}\frac{d^2}{dr^2}[r\phi(r)] - \frac{1}{L^2}\phi(r) = 0 \qquad r > 0 \tag{5.80}$$

In the usual way, we define a function $F(r) \equiv r\phi(r)$ and obtain from (5.80)

$$F''(r) - \frac{1}{L^2}F(r) = 0 \tag{5.81}$$

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which is in the form of (5.74). The general solution to (5.80) is, therefore,

$$\phi(r) = \frac{1}{r}\left(Ae^{r/L} + Be^{-r/L}\right) \tag{5.82}$$

As before, it is required that $A \equiv 0$ so as to satisfy boundary condition (5.72*a*). The constant B is determined by the condition on the source strength. If q_0 denotes the rate of neutron emission from the origin (neutrons per unit time), then the total net outflow of neutrons from the source is given by the limit of the product of the net current in the radial direction and the area of a sphere as the radius $r \to 0$. Thus

$$\lim_{r\to 0} 4\pi r^2 J(r) = q_0 \tag{5.83}$$

where $J(r)$ is obtained from (5.51*b*),

$$J(r) = -D\phi'(r) = \frac{DB}{r^2}(1 + r/L)e^{-r/L}$$

In the limit as $r \to 0$, we obtain from (5.83) the relation $4\pi DB = q_0$. The neutron flux due to an isotropic point source is, therefore,

$$\phi(r) = \frac{q_0 e^{-r/L}}{4\pi Dr} \tag{5.84}$$

There is an alternate boundary condition to (5.83) which gives the same results. Since we are dealing with the steady-state condition, the neutrons supplied by the source during unit time must balance the total loss of neutrons throughout the medium during unit time. The only mechanism of neutron loss in the infinite-medium one-velocity system is absorption; therefore,

$$q_0 = \int \Sigma_a \phi(\mathbf{r})\, d\mathbf{r} = \int_0^\infty \Sigma_a \phi(r) 4\pi r^2\, dr \tag{5.85}$$

For the chosen flux $\phi(r) = Be^{-r/L}/r$, we obtain $q_0 = 4\pi\Sigma_a L^2 B = 4\pi DB$, which is in agreement with the previous result. The analogous condition could have been used in the planar problem discussed in Sec. 5.2a or in the cylindrically symmetric problem to follow.

c. Line Source. A third form of the one-dimensional problem of neutron-flux distribution may be derived by considering a line distribution of isotropic sources. For convenience we take the diffusion equation (5.51*a*) in cylindrical coordinates and define $\phi(\mathbf{r}) \equiv \phi(\rho,\varphi,z)$.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\phi}{L^2} = 0 \tag{5.86}$$

For an infinite line of uniform sources and axial symmetry, (5.86) reduces to

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}\phi(\rho)\right] - \frac{1}{L^2}\phi(\rho) = 0 \qquad \rho > 0 \tag{5.87}$$

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where ρ denotes the radial coordinate measured from the line of sources at the origin. If we define a new variable $x \equiv \rho/L$, this equation takes the standard form of the modified Bessel equation[1]

$$\phi''(x) + \frac{1}{x}\phi'(x) - \phi(x) = 0 \tag{5.88}$$

The general solution is

$$\phi(x) = AI_0(x) + BK_0(x) \tag{5.89}$$

where $I_0(x)$ and $K_0(x)$ are the modified Bessel functions of order zero of the first and second kind, respectively. Complete tabulations of these functions are available in the standard sources.[2] For convenience we

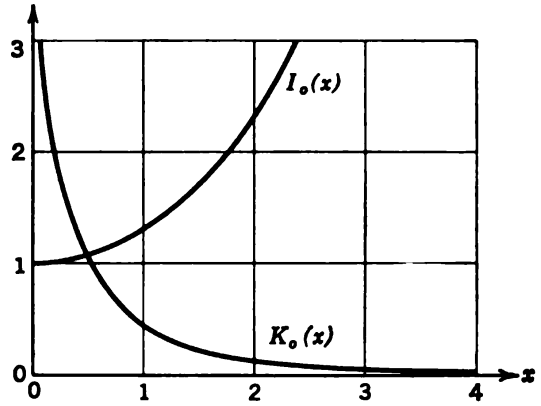

FIG. 5.12 Zero-order modified Bessel functions.

give below the relation between these functions and the regular functions J and Y of the first and second kind. In general, for the nth-order functions,

$$\begin{aligned} I_n(t) &= i^{-n}J_n(it) \\ K_n(t) &= \frac{\pi}{2}i^{n+1}[J_n(it) + iY_n(it)] \equiv \frac{\pi}{2}i^{n+1}H_n^{(1)}(it) \end{aligned} \tag{5.90}$$

where $H_n^{(1)}(it)$ is the Hankel function of the first kind and t is real.

The application of the boundary condition (5.72a) shows that the arbitrary constant A must be taken identically zero, since $I_0(t)$ becomes arbitrarily large as $t \to \infty$ (see Fig. 5.12). The value of B is obtained by imposing the source condition [cf. Eq. (5.83)].

$$\lim_{\rho \to 0} 2\pi\rho J(\rho) = q_0 \tag{5.91}$$

[1] See, for example, F. B. Hildebrand, "Advanced Calculus for Engineers," pp. 160–161, Prentice-Hall, Englewood Cliffs, N.J., 1950; H. Bateman, "Partial Differential Equations," pp. 398–403, Dover Publications, New York, 1944.

[2] See, for example, E. Jahnke and F. Emde, "Tables of Functions," Dover Publications, New York, 1945.

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where the net current $J(\rho)$ is given by

$$J(\rho) = -D\phi'(\rho) = \frac{DB}{L} K_1\left(\frac{\rho}{L}\right)$$

It can be shown that

$$\lim_{\rho \to 0} \rho K_1\left(\frac{\rho}{L}\right) = L$$

and it follows that $B = q_0/2\pi D$. The flux distribution due to an isotropic line source in an infinite medium is therefore

$$\phi(\rho) = \frac{q_0 K_0(\rho/L)}{2\pi D} \tag{5.92}\star$$

d. Kernel Method. In the foregoing discussion, the steady-state neutron flux in an infinite medium was obtained for three different source distributions, namely, plane source, point source, and line source. With these results it is possible to determine the steady-state flux from any distribution of sources in an infinite medium. As an introduction to the technique for calculating the flux for more general source distributions, consider the case of two point sources located at the space points $\mathbf{r}_1$ and $\mathbf{r}_2$ and of strengths q_1 and q_2 (see Fig. 5.13). The neutron flux at the field point $\mathbf{r}$ due to the point source at $\mathbf{r}_1$ may be determined from (5.84). Note that (5.84) is given in terms of the distance between the field point and the source; thus, if the source q_i *alone* ($i = 1, 2$) were present in the medium, the steady-state flux $\phi_i(\mathbf{r})$ would be given by

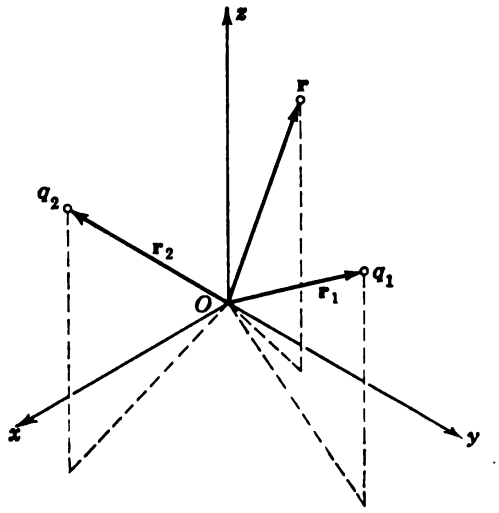

FIG. 5.13 Medium with two point sources.

$$\phi_i(\mathbf{r}) = \frac{q_i e^{-|\mathbf{r}-\mathbf{r}_i|/L}}{4\pi D|\mathbf{r} - \mathbf{r}_i|} \tag{5.93}$$

If the neutron-neutron interactions are neglected and if it is assumed that the presence of a point source does not affect the diffusion properties of the medium in its immediate neighborhood, the behavior of the neutron flux from either source will be unaffected by the presence of the other. Thus the total flux at $\mathbf{r}$ is given by the sum

$$\phi(\mathbf{r}) = \phi_1(\mathbf{r}) + \phi_2(\mathbf{r}) \tag{5.94}$$

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Formally, this result follows from the fact that the diffusion equation, of which ϕ, ϕ_1, and ϕ_2 are solutions, is linear; therefore, the solutions may be superimposed.

Since the above line of reasoning for two point sources would not be altered for three, four, or any number of point sources, it is concluded that, if N point sources located at points $\mathbf{r}_i (i = 1, 2, 3, \ldots N)$ and of strengths $q_i (i = 1, 2, 3, \ldots N)$ are present in the infinite medium, the total flux at a field point $\mathbf{r}$ due to all N would be given by

$$\phi(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i e^{-|\mathbf{r}-\mathbf{r}_i|/L}}{4\pi D|\mathbf{r} - \mathbf{r}_i|} \tag{5.95}$$

This general result due to a discrete distribution of sources may be generalized to a continuous distribution. Thus if $S(\mathbf{r}')\, d\mathbf{r}'$ is the total number of neutrons produced isotropically per unit time in volume $d\mathbf{r}'$ about space point $\mathbf{r}'$, then we define

$$\begin{aligned} \phi(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' &\equiv \frac{S(\mathbf{r}')\, d\mathbf{r}' e^{-|\mathbf{r}-\mathbf{r}'|/L}}{4\pi D|\mathbf{r} - \mathbf{r}'|} \\ &\equiv \begin{array}{l}\text{track length per unit volume per unit time at } \mathbf{r} \text{ due} \\ \text{to the } S(\mathbf{r}')\, d\mathbf{r}' \text{ neutrons released per unit time} \\ \text{from the volume } d\mathbf{r}' \text{ about } \mathbf{r}'\end{array} \end{aligned} \tag{5.96}$$

By the arguments given above for the discrete distribution of sources, we can compute the total track length per unit volume per unit time at $\mathbf{r}$ due to all sources in the medium by integrating (5.96) over all space.

$$\phi(\mathbf{r}) = \int \phi(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' = \int \frac{S(\mathbf{r}')\, e^{-|\mathbf{r}-\mathbf{r}'|/L}\, d\mathbf{r}'}{4\pi D|\mathbf{r} - \mathbf{r}'|} \tag{5.97}\star$$

The method introduced here is known as the method of Green's functions or the kernel method.[1] The function (5.96) is in fact the Green's function, or elementary solution, to the diffusion equation. The general solution to the diffusion equation for a system with an arbitrary source distribution can be constructed from these elementary solutions by applying (5.97).

In order to demonstrate the application of this general method to neutron diffusion in infinite media, the plane-source and line-source distribution problems (Secs. 5.2a and 5.2c) will be treated using (5.97).

Plane-source Distribution. The plane source of neutrons may be regarded as a uniform distribution of point sources over an infinite plane. Consider then the case in which isotropic point sources are distributed uniformly over a plane at $x = 0$, such that q_0 neutrons are released per unit time per unit area from all points of this plane. The flux at any

[1] See, for example, Morse and Feshbach, *op. cit.*, Part I, chap. 7.

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point x may be obtained by summing the contributions from every point on the plane (see Fig. 5.14). The contribution to the flux at x from an

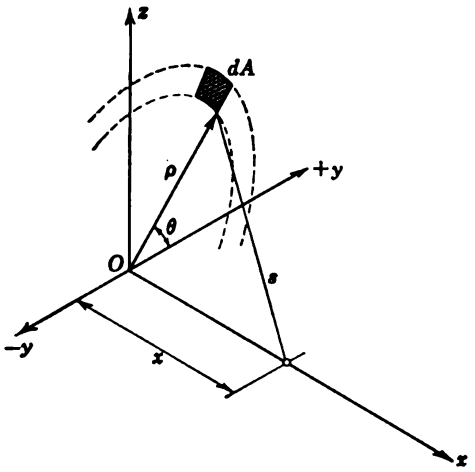

FIG. 5.14 Plane-source distribution.

area dA whose polar coordinates are (ρ,θ) in the $x = 0$ plane and which is a distance s from the field point is, from (5.84),

$$\begin{aligned}\phi(x;s)\,dA &\equiv \text{track length per unit volume per unit time at } x \text{ due}\\ &\qquad \text{to sources in } dA \text{ at distance } s \text{ from } x\\ &\equiv \frac{q_0\,dA\,e^{-s/L}}{4\pi Ds}\end{aligned} \tag{5.98}$$

The total flux $\phi(x)$ is the integral of this quantity over the entire plane $x = 0$.

$$\phi(x) = \int \phi(x;s)\,dA$$

If we write $dA = \rho\,d\rho\,d\theta$ and note that $s^2 = \rho^2 + x^2$, then this integral reduces to

$$\phi(x) = \frac{q_0}{4\pi D}\int_0^{2\pi}\int_0^{\infty}\frac{e^{-s/L}\rho\,d\rho\,d\theta}{s} = \frac{q_0}{2D}\int_{|x|}^{\infty} e^{-s/L}\,ds = \frac{Lq_0e^{-|x|/L}}{2D}$$

which is in agreement with (5.78).

The integral relation (5.97) may be used more directly in achieving this result for the planar distribution if we specify the source strength in terms of the delta function (cf. Sec. 4.6d). We choose for the source function $S(\mathbf{r})$, which appears in (5.97), the form

$$S(\mathbf{r}) \equiv A\delta(x) \tag{5.99}$$

where $\delta(x)$ is the Dirac delta function; the constant A we compute by requiring that the integral of the source distribution (5.99) over a unit area of source plane and over all x be q_0. Thus,

$$A\int_{-\infty}^{\infty}\delta(x)\,dx\int_{-1}^{1}dy\int_{-1}^{1}dz \equiv q_0$$

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and $A = q_0$. The substitution of $S(\mathbf{r}) = q_0\,\delta(x)$ into (5.97) yields

$$\phi(\mathbf{r}) = \frac{q_0}{4\pi D}\int \frac{\delta(x')\,e^{-|\mathbf{r}-\mathbf{r}'|/L}\,dx'\,dy'\,dz'}{|\mathbf{r}-\mathbf{r}'|}$$

where we have defined $d\mathbf{r}' \equiv dx'\,dy'\,dz'$. If we write the distance $|\mathbf{r}-\mathbf{r}'|$ in terms of the coordinates of the two points $\mathbf{r}$ and $\mathbf{r}'$, then

$$\phi(x,y,z)$$

$$= \frac{q_0}{4\pi D}\int_{-\infty}^{\infty}\int\int \frac{\delta(x')\exp\left\{-\frac{1}{L}[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{1}{2}}\right\}dx'\,dy'\,dz'}{[(x-x')^2+(y-y')^2+(z-z')^2]^{\frac{1}{2}}}$$

$$= \frac{q_0}{4\pi D}\int_{-\infty}^{\infty}\int \frac{\exp\left\{-\frac{1}{L}[x^2+(y-y')^2+(z-z')^2]^{\frac{1}{2}}\right\}dy'\,dz'}{[x^2+(y-y')^2+(z-z')^2]^{\frac{1}{2}}}$$

By transforming the double integral in this relation to polar coordinates in the $y'z'$ plane (see Fig. 5.14), the result (5.78) is obtained directly.

Line-source Distribution. In the case of the line source in the infinite medium, the distribution of point sources may be chosen as

$$S(\mathbf{r}) \equiv A\,\delta(x)\,\delta(y) = \frac{A\,\delta(\rho)}{2\pi\rho} \tag{5.100}$$

As before, we evaluate the constant A by requiring that the integral of S over a unit dimension of source be equal to the specified strength. Thus if q_0 is the number of neutrons released per unit time from unit length of line, then

$$q_0 = \int S(\mathbf{r})\,d\mathbf{r} = A\int_{-\infty}^{\infty}\delta(x)\,dx\int_{-\infty}^{\infty}\delta(y)\,dy\int_0^1 dz$$

$$= A\int_0^{\infty}\frac{\delta(\rho)\rho\,d\rho}{2\pi\rho}\int_0^{2\pi}d\theta\int_0^1 dz \equiv A$$

The substitution of (5.100) into (5.97), using $A = q_0$, yields

$$\phi(\rho,\theta,z) = \frac{q_0}{8\pi^2 D}\int_{-\infty}^{\infty}dz'\int_0^{\infty}\delta(\rho')\,d\rho'$$

$$\int_0^{2\pi}\frac{\exp\left\{-\frac{1}{L}[(\rho\cos\theta-\rho'\cos\theta')^2+(\rho\sin\theta-\rho'\sin\theta')^2+(z-z')^2]^{\frac{1}{2}}\right\}d\theta'}{[(\rho\cos\theta-\rho'\cos\theta')^2+(\rho\sin\theta-\rho'\sin\theta')^2+(z-z')^2]^{\frac{1}{2}}}$$

$$= \frac{q_0}{4\pi D}\int_{-\infty}^{\infty}\frac{\exp\left\{-\frac{1}{L}[\rho^2+(z-z')^2]^{\frac{1}{2}}\right\}dz'}{[\rho^2+(z-z')^2]^{\frac{1}{2}}}$$

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If we define $\sinh \xi \equiv (z - z')/\rho$, it follows that

$$\phi(\rho,\theta,z) \rightarrow \phi(\rho) = \frac{q_0}{4\pi D} \int_{-\infty}^{\infty} \exp\left(-\frac{\rho}{L}\cosh \xi\right) d\xi$$

This form may be recognized as the Hankel function of the first kind of order zero:[1]

$$H_0^{(1)}(z) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \exp\,(iz \cosh t)\, dt$$

If we identify $-\rho/L \equiv iz$, then

$$\phi(\rho) = \frac{iq_0}{4D} H_0^{(1)}\left(\frac{i\rho}{L}\right)$$

However, by (5.90),

$$H_0^{(1)}(ix) = \frac{2}{i\pi} K_0(x)$$

Therefore

$$\phi(\rho) = \frac{q_0 K_0(\rho/L)}{2\pi D}$$

which is in agreement with the solution (5.92).

5.3 Diffusion in Finite Media

a. The Vacuum-interface Condition. The purpose of the present section is to display some of the features of the approximate boundary conditions developed in Sec. 5.1 and to show how these approximations may be utilized to specify the situation at the outer surface of a system and at interfaces within. We begin with a simple calculation which involves the application of the vacuum-interface condition. In the analysis which follows we examine the steady-state flux distribution in an infinite-slab geometry due to a plane source placed on one face. This calculation is performed for two different outside boundary conditions on the source-free face of the slab, namely, condition (5.57*a*), which requires that the partial current of neutrons from the vacuum be zero, and condition (5.57*b*), which requires that the flux vanish at the extrapolated boundary [see also (5.69)]. The results from these two calculations are compared so as to show some of the implications of the diffusion theory as applied to the boundary conditions at an outside face of a system.

Consider, then, a homogeneous slab of width a in the x direction and

[1] E. T. Copson, "Theory of Functions of a Complex Variable," p. 344, example 22, Oxford University Press, London, 1935.

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infinite in the y and z directions, as shown in Fig. 5.15. At the plane $x = 0$, there is specified a uniform distribution of isotropic sources of strength q_0 neutrons per unit area per unit time. Because of the finite extent of the x dimension and the nonsymmetry of the problem, there will be a nonuniform distribution of the neutron population along the x direction; the dependence on the y and z directions for given x, however, will be uniform. Thus, the flux is given by the function $\phi(x)$, and the appropriate differential equation is [see (5.75) for L^2]

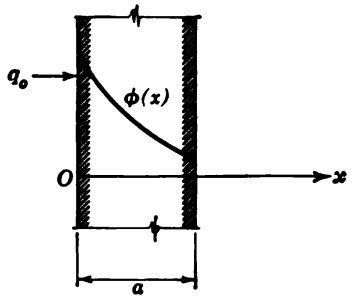

FIG. 5.15 Infinite slab of width a.

$$\phi''(x) - \frac{1}{L^2}\phi(x) = 0$$

The general solution may be written

$$\phi(x) = A_1 e^{x/L} + B_1 e^{-x/L} \qquad \text{or} \qquad \phi(x) = A_2 \sinh \frac{x}{L} + B_2 \cosh \frac{x}{L}$$

We seek two specific solutions to this problem, one based on boundary condition (5.57a) and the other on (5.57b). In each case we require also a condition on the source plane which, when used with either (5.57a) or (5.57b), yields the arbitrary constants A_1 and B_1 or A_2 and B_2.

The boundary conditions to be used in the first case, namely, that the return current from the vacuum be zero at the source-free face of the slab, are

$$(1)\ j_-(a) = 0 \qquad (2)\ \lim_{x\to 0} j_+(x) = \tfrac{1}{2}q_0 \tag{5.101}$$

We use for the partial currents the expressions (5.49). It is easily shown that the application of these conditions to the second of the general solutions given above yields the flux $\phi_1(x)$:

$$\phi_1(x) = 4q_0\left[\frac{\sinh \varkappa(a-x) + 2D\varkappa \cosh \varkappa(a-x)}{(2D\varkappa+1)^2 e^{a\varkappa} - (2D\varkappa-1)^2 e^{-a\varkappa}}\right] \tag{5.102}$$

where we have introduced the symbol

$$\varkappa \equiv \frac{1}{L} \tag{5.103}\star$$

The extrapolated boundary-condition approximation (5.57b) to the condition (5.57a) yields the set of conditions

$$(1)\ \phi(\tilde{a}) = 0 \qquad (2)\ \lim_{x\to 0} j_+(x) = \tfrac{1}{2}q_0 \tag{5.104}$$

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where $\tilde{a} \equiv a + 0.7104\lambda_{tr}$ according to (5.70). The only change here is, of course, in condition (1), where the current condition has been replaced by the requirement that the flux vanish at the extrapolated boundary $x = \tilde{a}$. Let the solution of the diffusion equation given above which corresponds to this set of conditions be $\phi_{II}(x)$; then, one obtains, upon evaluating the constants A_2 and B_2, the result

$$\phi_{II}(x) = 2q_0\left[\frac{\sinh \varkappa(\tilde{a} - x)}{\sinh \varkappa\tilde{a} + 2D\varkappa \cosh \varkappa\tilde{a}}\right] \tag{5.105}$$

We can obtain some insight into the physical requirements which must be satisfied if the diffusion theory is to be valid in a given situation by comparing the results ϕ_I and ϕ_{II} obtained above. Specifically, let us compare the expressions (5.102) and (5.105) in the case when $2D\varkappa \ll 1$ and $2D/a \ll 1$. It is readily shown that, under this restriction, both of these solutions reduce to the same form, namely,

$$\frac{2q_0 \sinh \varkappa(\tilde{a} - x)}{\sinh \varkappa\tilde{a}} \simeq \phi_I(x),\ \phi_{II}(x) \tag{5.106}$$

Thus the solutions of the present problem based on the accurate and approximate statements of the vacuum-interface condition are in good agreement provided that $2D\varkappa \ll 1$ and $2D/a \ll 1$. Let us examine now the consequences of these requirements. We have from (5.12) and (5.75)

$$\begin{aligned} 2D\varkappa &= \frac{2D}{L} = 2\sqrt{D\Sigma_a} = 2\sqrt{\frac{\Sigma_a}{3\Sigma_{tr}}} \\ \frac{2D}{a} &= \frac{2}{3\Sigma_{tr}a} \end{aligned} \tag{5.107}$$

Therefore

$$\begin{aligned} &(1)\ 2D\varkappa \ll 1 \quad \text{implies} \quad 2\sqrt{\frac{\Sigma_a}{3\Sigma_{tr}}} \ll 1 \\ &(2)\ \frac{2D}{a} \ll 1 \quad \text{implies} \quad \frac{2}{3\Sigma_{tr}a} \ll 1 \end{aligned} \tag{5.108}$$

The first requirement, then, is that the absorption cross section be small relative to the transport cross section. The second requirement is that the transport mean free path be small in comparison to a, the characteristic dimension (size) of the system. We find therefore that, in order for the results based on the extrapolated boundary condition to match well the results based on the no-return current condition, the absorption cross section must be small as compared with the scattering cross section [note that for most nuclei $\Sigma_{tr} \simeq \Sigma_s$ by (5.46)]. Moreover, the scattering mean free path (again using $\lambda_{tr} \simeq \lambda_s$) must be appreciably smaller than the size of the system. This condition is in fact a prerequisite if diffusion theory is to be valid in a given situation. We conclude then that the

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approximate and accurate statements of the vacuum-interface condition are equivalent only if the medium is slightly absorbing and if the system is many mean free paths wide (i.e., diffusion theory must be valid).

The approximate result (5.106) may be used to illustrate the effect of the diffusion of neutrons on neutron-migration problems. Suppose that, as in Fig. 2.6, n_0 neutrons per unit area per unit time (of the one-velocity of diffusion) are incident upon one face of an infinite slab of thickness a.

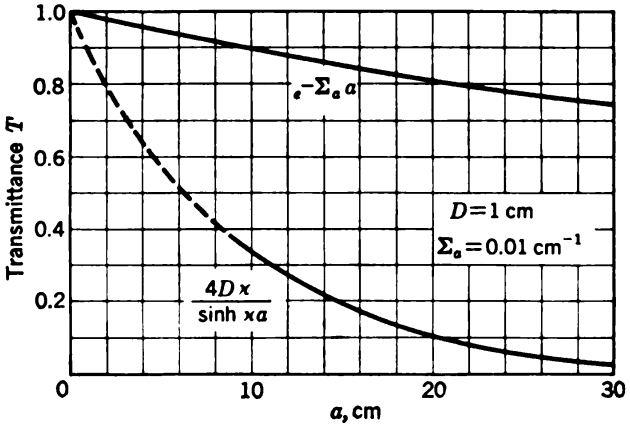

FIG. 5.16 Transmittance of slab.

Let us compute the transmittance[1] of this slab on the basis of the following two physical models: (1) Neutrons are not scattered in the slab but move straight through, unless they experience a capture collision, in which case they are removed from the beam. (2) Neutrons enter a diffusion process the instant they strike the slab.

For case 2, the results of the above analysis apply with a source strength n_0 to replace the factor $q_0/2$ which appears in (5.106) [see (2) in (5.101)]. If the number of neutrons transmitted is counted as those received by a noncollimated counter on the far side of the slab, then for case 2 this is precisely the net current from the back face. By assuming that the extrapolation distance is small in comparison with the slab width [which is implied in the inequality (2) of (5.108)], the net current calculated at the extrapolated back face will very closely approximate the actual current. Then, in the two cases, the transmittance T is computed as follows:

Case 1:
$$T_1 = \frac{n_0 e^{-\Sigma_a a}}{n_0} = e^{-\Sigma_a a}$$

Case 2:
$$T_2 = \frac{-D\phi'(\tilde{a})}{n_0} = \frac{4Dx}{\sinh x\tilde{a}} \simeq \frac{4Dx}{\sinh xa}$$

A comparison of these results is given in Fig. 5.16 for the case $D = 1$ cm and $\Sigma_a = 0.01$ cm^{-1} as a function of slab width. The rapid decrease in

[1] The fraction of incident neutrons transmitted through the slab.

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the transmittance for the diffusion case may be understood when it is recalled that a diffusing neutron travels a long tortuous path in passing through the slab and is therefore exposed to capture collisions over a distance much greater than the actual slab width. The nonscattering medium of case 1, on the other hand, has the distance a in which to capture the neutron. Note that the graph of the transmittance for the diffusion medium is broken at the lower limit for small slab widths in as much as diffusion theory does not apply for $a/L \ll 1$ [see (5.108)].

b. Elementary Sources in Finite Geometry. The treatment above illustrates the general method for finding the spatial variation of the steady-state neutron flux in finite media into which neutrons are introduced from localized sources. The mathematical problem is the solution of the following boundary-value problem:

Differential equation:

$$-D\nabla^2\phi(\mathbf{r}) + \Sigma_a\phi(\mathbf{r}) = 0 \qquad \text{(to apply everywhere except at source points)} \tag{5.109}$$

Boundary conditions:

(1) $\phi = 0$ everywhere on the extrapolated surface of the medium (i.e., the medium-vacuum interface) except at points where there are localized sources

(2) Net number of neutrons moving away from each source point per unit time (in terms of the neutron flux in that region) must be equal to the source strength at that point

(5.110)

Table 5.2 gives the results of the application of these conditions to a few systems. Each of these single-medium systems is taken to be composed of a material whose diffusion length is L. The geometries of Table 5.2 involve only one space coordinate; the boundary-value problem described by the equations (5.109) and (5.110) may, of course, be considered for more complex geometries (e.g., a finite block of material with a point source at the geometric center); these involve the technique of the separation of variables. One such problem is treated in Sec. 5.5, which deals with the measurement of the diffusion length.

c. Albedo Concept. It was shown in Sec. 5.1e that the interface boundary condition between two dissimilar media could be expressed in terms of the net current and flux [see (5.61)]. This approach is generally useful only if the regions on either side of the interface are only slightly absorbing and several scattering mean free paths in width; i.e., each region must satisfy the requirements of diffusion theory. It is to be recognized, however, that even in this case the diffusion theory does not yield an accurate representation of the neutron flux in the vicinity of the interface (within a distance of a few mean free paths). In the case that

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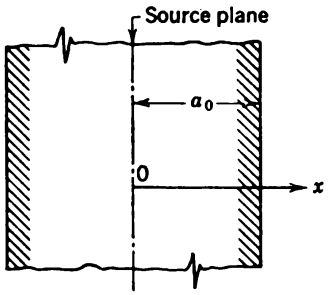
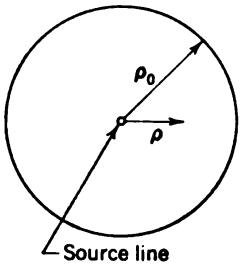
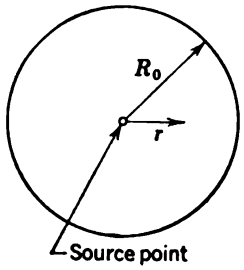
one of the media is highly absorbing or narrow in width, the application of the diffusion equations is even more questionable. This model is certainly invalid in the narrow (or highly absorbing) medium; moreover, its use in the "diffusion" medium near the common boundary is also

TABLE 5.2 SOLUTIONS FOR THE ELEMENTARY SOURCES IN FINITE GEOMETRY

	Geometry		
	Slab	Cylinder	Sphere
Description of system	Slab with planar source on the midplane	Infinitely long cylinder with an axially placed source line	Sphere with point source at the center
Source strength	q_0 neutrons per unit area of source plane per unit time	q_0 neutrons per unit length of source line per unit time	q_0 neutrons per unit time
Dimension	Half-width of slab is a_0	Radius of cylinder is ρ_0	Radius of sphere is R_0
			
Explicit statement of boundary conditions	(1) $\phi(\pm \tilde{a}_0) = 0$ $\phi(x) = \phi(-x)$	$\phi(\tilde{\rho}_0) = 0$	$\phi(\tilde{R}_0) = 0$
	(2) $\lim_{x \to 0^+} [-D\phi'(x)] = \frac{q_0}{2}$	$\lim_{\rho \to 0} [-2\pi\rho D\phi'(\rho)] = q_0$	$\lim_{r \to 0} [-4\pi r^2 D\phi'(r)] = q_0$
General solution to differential equation	$\phi(x) = A \sinh \frac{x}{L} + B \cosh \frac{x}{L}$	$\phi(\rho) = AI_0(\rho/L) + BK_0(\rho/L)$	$\phi(r) = \frac{1}{r}\left[A \sinh \frac{r}{L} + B \cosh \frac{r}{L}\right]$

The complete solutions are:

Slab: $$\phi(x) = \frac{q_0 L}{2D \cosh \tilde{a}_0/L} \sinh\left(\frac{\tilde{a}_0 - x}{L}\right) \qquad x \geq 0$$

Cylinder: $$\phi(\rho) = \frac{q_0}{2\pi D I_0(\tilde{\rho}_0/L)} [I_0(\tilde{\rho}_0/L)\, K_0(\rho/L) - K_0(\tilde{\rho}_0/L)\, I_0(\rho/L)]$$

Sphere: $$\phi(r) = \frac{q_0}{r 4\pi D \sinh \tilde{R}_0/L} \sinh\left(\frac{\tilde{R}_0 - r}{L}\right)$$

limited.[1] It would appear then that the present machinery of the diffusion theory is insufficient for coping with interface situations of this type.

We find, however, that this is not exactly the case. Granted that the

[1] Under these circumstances the flux near the boundary is highly "directional," and the two-term approximation of diffusion theory is inadequate [see (5.30)].

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theory is inapplicable to the narrow or highly absorbing region, it may still be used as a first approximation for describing the flux in the diffusion medium. The approximation is not so good as in the case of an interface between two diffusion media, but it is still useful for our present purpose. The remaining problem, then, is the specification of the effect of the "nondiffusion" region on the neutron population in the diffusion region. That is, we should like to find a way to specify the boundary condition on the diffusion medium in terms of the nuclear properties of the nondiffusion region. This can be done very conveniently by means of the so-called *albedo* concept. We define the albedo of a given material as

$$\text{Albedo} \equiv \frac{\text{return current from a region}}{\text{incident current into the region}} \tag{5.111}$$

Thus the albedo is a measure of the reflectivity of a medium for neutrons of a given speed, i.e., its ability to return a neutron back to the space from which it was incident on the medium in question. It is clear that, if the albedo of a nondiffusion medium is known, one could apply this information to the adjoining diffusion region by a simple statement on the current condition at the interface. In practice, one can either measure the albedo of various materials (and geometric shapes) or compute it by means of more sophisticated models such as the transport theory. In any event such a number can be obtained[1] and applied directly to the remainder of the system as a boundary condition. This approach is especially useful for treating very thin regions, such as plates, foils, or shells. Thus, for example, one might treat containers and pressure vessels in a reactor. A knowledge of the albedo of these components permits one to use more elementary theories for other major portions of the system. Further, the albedo concept is extremely effective for describing the properties of thin, highly absorbing plates or rods of the type used as nuclear-fuel elements.

If the adjoining regions are both diffusion materials, an estimate can be obtained for the albedo through the use of the partial-current relations. Consider, as an introductory example, the case of a plane source of strength q_0 neutrons per unit area per unit time placed on the face of a semi-infinite slab. If the origin of the coordinate system is at the source plane, the general solution of the diffusion equation has the form of Eq. (5.78), namely,

$$\phi(x) = Ce^{-x/L} \tag{5.112}$$

We evaluate the constant C by means of the partial-current condition [see (5.49)]

$$j_+(0) = \frac{1}{4}\phi(0) - \frac{D}{2}\phi'(0) = \frac{1}{2}q_0 \tag{5.113}$$

[1] An elementary method for "thin" regions will be discussed in Sec. 5.8a.

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This condition states, physically, that the number of neutrons striking a unit area of the surface of the semi-infinite medium per unit time is half the source strength (half the neutrons are lost to the void directly from the source). Note that this is *not* the net current, since some of the neutrons which enter will, after some diffusion, be lost by leakage back across the vacuum interface; these we have not yet accounted for. If (5.112) is used in (5.113), we obtain the constant C; the resulting expression for the flux is

$$\phi(x) = \frac{2q_0 e^{-x/L}}{1 + 2D/L} \tag{5.114}$$

The partial neutron current *out* of the medium at the boundary is

$$j_-(0) = \frac{1}{4}\phi(0) + \frac{D}{2}\phi'(0) = \frac{q_0(L - 2D)}{2(L + 2D)} \tag{5.115}$$

The current in j_+ is, of course, $q_0/2$; thus by (5.111) we have for the albedo of the semi-infinite medium

$$\text{Albedo} = \frac{L - 2D}{L + 2D} \tag{5.116}$$

This form for the albedo applies only to the semi-infinite medium; for other configurations, the albedo may be computed within the limits of diffusion theory from the general relation (where B denotes the position of the boundary):

$$\text{Albedo} = \frac{j_-}{j_+} = \frac{1 + 2D\phi'(B)/\phi(B)}{1 - 2D\phi'(B)/\phi(B)} \tag{5.117}\star$$

It should be emphasized that the diffusion-theory expression for the albedo, even when the adjoining media are both lightly absorbing, is in error since the diffusion theory does not give an accurate description of the angular distribution of the neutrons in the vicinity of the interface.

The essential parameters involved in computing the albedo [see (5.117)] are the diffusion coefficient in the medium and the ratio of the slope of the neutron flux to the value of the flux at the boundary. This ratio will depend upon the geometry, the diffusion coefficient, and the absorption of the material. To illustrate this dependence in the case of the semi-infinite medium we introduce the relation

$$\frac{2D}{L} = \frac{2\lambda_{tr}}{3}\left(\frac{\Sigma_a}{D}\right)^{\frac{1}{2}}$$

into (5.116) and obtain for the albedo α

$$\alpha = \frac{1 - 2\lambda_{tr}(\Sigma_a/D)^{\frac{1}{2}}/3}{1 + 2\lambda_{tr}(\Sigma_a/D)^{\frac{1}{2}}/3} \tag{5.118}$$

It is evident that for total reflection ($\Sigma_a \equiv 0$) the albedo is unity. It should be noted, however, that the result for heavy absorption (Σ_a large) cannot be obtained from the above relation because this result was

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derived from the diffusion-theory estimate of the partial currents which requires that Σ_a be small compared to Σ_t. Figure (5.17) shows a plot of the albedo for a semi-infinite medium as computed by (5.118). The diffusion-theory result is broken off at the higher values of Σ_a in accord with the preceding remarks. The upper broken-line curve indicates how one might expect the actual albedo of the material to vary.

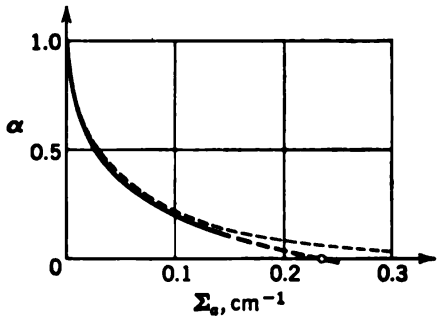

FIG. 5.17 Albedo for semi-infinite medium.

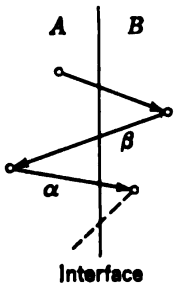

FIG. 5.18 Neutron crossings of interface.

An interesting application of the albedo concept is in the calculation of the average number of crossings of an imaginary plane by a neutron as the neutron diffuses through the medium. The principal ideas involved can be demonstrated by considering a one-dimensional system and starting with the assumption that the neutron has made at least one crossing (otherwise the spatial variation of the neutron flux must be taken into account). In order to retain some degree of generality, it will be assumed that the boundary in question is the common face between two media of different diffusion properties (see Fig. 5.18). Denote these media A and B and their respective albedos α and β. Thus β is the probability that a neutron which has crossed from A to B will be returned to A [see (5.111)]. The probability that it will not return to A is, of course, $1 - \beta$.

In the present case we wish to compute the probability $\mathfrak{p}_n$ that the neutron makes precisely n crossings and from this determine the average number of crossings. Now the probability that a capture will occur during any one crossing is independent of the number of previous crossings, and the probability for capture after exactly n crossings is the joint probability of surviving n crossings and of being captured before the $(n + 1)$st crossing. Since each of these events is independent, the joint probability is given by the product of the probabilities for the individual events. By this line of reasoning we find that, if the neutron starts in medium A,

$$\begin{aligned}
\mathfrak{p}_0 &= (1 - \alpha) \\
\mathfrak{p}_1 &= \alpha(1 - \beta) \\
\mathfrak{p}_2 &= \alpha\beta(1 - \alpha) \\
\mathfrak{p}_3 &= \beta\alpha^2(1 - \beta) \\
&\cdots\cdots\cdots
\end{aligned}$$

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In general,

$$\mathfrak{p}_\nu(\nu = 2n) = (1 - \alpha)(\alpha\beta)^n$$
$$\mathfrak{p}_\nu(\nu = 2n + 1) = \alpha(1 - \beta)(\alpha\beta)^n \qquad n = 0, 1, 2, \ldots \quad (5.119)$$

These relations may be checked by computing the sum of the joint probabilities $\mathfrak{p}_\nu$ for all ν. The sum should be unity; thus,

$$\sum_{n=0}^{\infty} \mathfrak{p}_\nu = (1 - \alpha) \sum_{n=0}^{\infty} (\alpha\beta)^n + \alpha(1 - \beta) \sum_{n=0}^{\infty} (\alpha\beta)^n = (1 - \alpha\beta) \sum_{n=0}^{\infty} (\alpha\beta)^n = 1 \quad (5.120)$$

We compute the average number of crossings $\bar{n}$ from the relation

$$\bar{n} = \sum_{n=0}^{\infty} \nu\mathfrak{p}_\nu = 2(1 - \alpha) \sum_{n=0}^{\infty} n(\alpha\beta)^n + \alpha(1 - \beta) \sum_{n=0}^{\infty} (2n + 1)(\alpha\beta)^n$$
$$= 2(1 - \alpha\beta) \sum_{n=0}^{\infty} n(\alpha\beta)^n + \alpha(1 - \beta) \sum_{n=0}^{\infty} (\alpha\beta)^n$$

The first series is easily identified as $\alpha\beta(1 - \alpha\beta)^{-2}$ and the second as $(1 - \alpha\beta)^{-1}$. The expression for $\bar{n}$ thereby reduces to

$$\bar{n} = \frac{\alpha(1 + \beta)}{1 - \alpha\beta} \quad (5.121)$$

In the event that $\alpha = \beta$,

$$\bar{n} = \frac{\alpha}{1 - \alpha} \quad (5.122)$$

5.4 Application of One-velocity Model to Multiplying Media

a. Fundamental Mode: Infinite-slab Reactor. With the remarks in the preceding section as an introduction, the more general problem of steady-state diffusion in a multiplying medium will now be examined. As indicated in Sec. 5.1a, the application of the one-velocity approximation to multiplying media is the second in a series of increasingly more detailed models which we will utilize for determining the nuclear characteristics of a reactor. This problem will be analyzed for three elementary geometric configurations: the slab, the sphere, and the cylinder. The calculation for the multiplying slab, however, will be treated in greater detail so as better to display the principal concepts involved. The general conclusions which can be drawn from the analysis of the slab problem will later be used in obtaining the solutions for the sphere and cylinder.

Consider, therefore, planar distributions of isotropic neutron sources in an infinite homogeneous slab. The medium is of infinite extent in both y and z directions and of width $2\tilde{a}$ in the x direction ($\tilde{a}$ is the extrapolated

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half-thickness of the slab); the origin of the x axis is placed at the center of the slab. These specifications reduce the spatial dependence of the flux to variations in x alone. It is assumed that all available neutrons (per unit volume per unit time) arise from fission reactions described by the term $\nu\Sigma_f\phi$, where ν denotes the average number of neutrons produced per reaction and Σ_f denotes the macroscopic fission cross section. In the present calculation we will retain the time dependence of the neutron flux in order to obtain a more general result (cf. Sec. 3.2b); thus we define $\phi \equiv \phi(x,t)$. The reduction of this result to the steady-state case will display the motivation for selecting the particular form of the time-independent solution in the cylinder and sphere problems.

This model in which the neutron source is taken to be $\nu\Sigma_f\phi$ (which implies that neutrons from fission appear at the one velocity of diffusion) is not expected to apply directly to operating reactors; however, the techniques to be used later are well illustrated by this formulation of the problem, and the results are useful, if properly adapted, to more general situations. We take then as our neutron-balance relation the time-dependent diffusion equation (5.21) along with the assumed source term

$$S(\mathbf{r},t) \equiv \nu\Sigma_f\phi(\mathbf{r},t) \tag{5.123}$$

The general one-velocity diffusion equation for the multiplying medium is then

$$-D\nabla^2\phi(\mathbf{r},t) + \Sigma_a\phi(\mathbf{r},t) = \nu\Sigma_f\phi(\mathbf{r},t) - \frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},t) \tag{5.124}$$

where v is the "one velocity" of diffusion (see Sec. 5.1a). The appropriate form for the infinite-slab problem is

$$\frac{\partial^2}{\partial x^2}\phi(x,t) + B^2\phi(x,t) = \frac{1}{Dv}\frac{\partial}{\partial t}\phi(x,t) \tag{5.125}$$

where we have introduced the definition

$$B^2 \equiv \frac{\nu\Sigma_f - \Sigma_a}{D} \tag{5.126}\star$$

The solution to (5.125) must satisfy the conditions [see (5.72)]

$$(1)\ \phi(\pm\tilde{a},t) = 0 \qquad (2)\ \phi(x,0) = \phi_0(x) \tag{5.127}$$

in which $\phi_0(x)$ is some specified initial distribution.

Equation (5.125) may be solved by the separation of variables;[1] therefore, define functions $F(x)$ and $G(t)$ such that

$$\phi(x,t) \equiv F(x)\,G(t) \tag{5.128}$$

[1] H. Margenau and G. M. Murphy, "The Mathematics of Physics and Chemistry," chap. 7, D. Van Nostrand Company, Inc., Princeton, N.J., 1943.

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The substitution of this expression into (5.125) yields

$$\frac{F''(x)}{F(x)} + B^2 = \frac{1}{Dv}\frac{G'(t)}{G(t)} = -c^2 \tag{5.129}$$

The arbitrary constant c^2 (here assumed positive) is determined by conditions on the time behavior of the flux. Note that a relation of this type must be satisfied by F and G inasmuch as x and t are independent variables, and (5.129) can hold only if c is a constant. Thus (5.129) may be separated into two ordinary differential equations

$$F''(x) + (B^2 + c^2) F(x) = 0 \tag{5.130}$$

$$G'(t) = -Dvc^2 G(t) \tag{5.131}$$

The general solution to (5.131) is

$$G(t) = G_0 e^{-Dvc^2 t} \qquad G_0 \equiv G(0) \tag{5.132}$$

and if we call

$$\beta^2 \equiv B^2 + c^2 \tag{5.133}$$

we can write the general solution to (5.130) in the form

$$F(x) = A \cos \beta x + C \sin \beta x \tag{5.134}$$

We evaluate the constants A and C by the application of boundary condition (1) from (5.127). The result is

$$\begin{aligned} F(\tilde{a}) &= 0 = A \cos \beta\tilde{a} + C \sin \beta\tilde{a} \\ F(-\tilde{a}) &= 0 = A \cos \beta\tilde{a} - C \sin \beta\tilde{a} \end{aligned} \tag{5.135}$$

The successive addition and subtraction of these equations yield the requirements that

$$2A \cos \beta\tilde{a} = 0 \qquad 2C \sin \beta\tilde{a} = 0 \tag{5.136}$$

The only nontrivial solutions to these relations are, respectively,

$$\begin{aligned} \beta\tilde{a} &= \frac{\pi}{2}(2n + 1) \equiv \alpha_n\tilde{a} \qquad \text{where } \alpha_n \equiv \frac{\pi}{2\tilde{a}}(2n + 1) \\ \beta\tilde{a} &= \pi(n + 1) \equiv \gamma_n\tilde{a} \qquad \text{where } \gamma_n \equiv \frac{\pi}{\tilde{a}}(n + 1) \end{aligned} \tag{5.137}$$

and $n = 0, 1, 2, \ldots$. Thus solutions of the form (5.134) exist for $F(x)$ provided that β has any of the values α_n in the cosine term and the values γ_n in the sine term. The α_n and γ_n are the eigenvalues[1] of (5.134), and for each value of n there exists a solution $F_n(x)$ given by

$$F_n(x) = A_n \cos \alpha_n x + C_n \sin \gamma_n x \tag{5.138}$$

which are the eigenfunctions of the differential equation. Moreover, we

[1] A. Sommerfeld, "Partial Differential Equations in Physics," chap. V, Academic Press, Inc., New York, 1949.

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note from the definition of β [see (5.133)] that for each value of n there must be a corresponding value for c. Thus if we write $c^2 = \beta^2 - B^2$, then it follows from (5.137) that

$$\begin{aligned} c^2 &= \alpha_n^2 - B^2 \equiv \lambda_n^2 \qquad \text{when } \beta \text{ takes on values } \alpha_n \\ c^2 &= \gamma_n^2 - B^2 \equiv \omega_n^2 \qquad \text{when } \beta \text{ takes on values } \gamma_n \end{aligned} \tag{5.139}$$

These are the values of c for various n which must be used in the solution for $G(t)$ [see (5.132)]. Hence, we construct a solution for the diffusion equation (5.125) in the form (5.128) for each value of $n = 0, 1, \ldots$.

$$F_n(x)\, G_n(t) \equiv G_0(A_n e^{-Dv\lambda_n^2 t} \cos \alpha_n x + C_n e^{-Dv\omega_n^2 t} \sin \gamma_n x) \tag{5.140}$$

The solution to Eq. (5.125) is given by the sum[1]

$$\phi(x,t) \equiv \sum_{n=0}^{\infty} F_n(x)\, G_n(t) \tag{5.141}$$

The analysis leading up to the result (5.140) has been concerned only with boundary conditions (1) of Eq. (5.127). The requirement is that, if this condition is to be satisfied by the spatially dependent portion of $\phi(x,t)$, then β may take on only the values indicated in (5.137). This in turn places a constraint on the constant c because of the relationship (5.133); consequently, c is limited to the values (5.139). Given, then, a value of n, α_n, γ_n and in turn λ_n and ω_n may be computed. Note, however, that the evaluation of λ_n and ω_n involves the parameter B^2 which is determined by the nuclear properties of the system. (We will consider this matter in detail shortly.) The remaining undefined quantities in (5.140) are the coefficients A_n and C_n. These are determined from the application of the initial condition (2) of (5.127). Thus if we compute $\phi(x,0)$ from (5.141) using (5.140), we obtain

$$\phi(x,0) \equiv \phi_0(x) = G_0 \sum_{n=0}^{\infty} (A_n \cos \alpha_n x + C_n \sin \gamma_n x) \tag{5.142}$$

The application of the orthogonality properties of the trigonometric functions yields the coefficients

$$\begin{aligned} A_n &= \frac{1}{\tilde{a} G_0} \int_{-\tilde{a}}^{\tilde{a}} \phi_0(x) \cos \alpha_n x \, dx \\ C_n &= \frac{1}{\tilde{a} G_0} \int_{-\tilde{a}}^{\tilde{a}} \phi_0(x) \sin \gamma_n x \, dx \end{aligned} \tag{5.143}$$

The solution of the time-dependent problem is now complete. Given a system of certain nuclear properties (implied in B^2) and any initial distribution of neutrons (according to ϕ_0), one can compute the flux dis-

[1] Margenau and Murphy, *op. cit.*, pp. 240–248.

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tribution for all subsequent time from the general solution (5.141). The coefficients involved in this expression are to be computed by (5.143), and the various eigenvalues, from the relations (5.137) and (5.139).

Let us examine now the distribution of the neutron population in the present system for large values of t. We inquire, then, into the behavior of the flux $\phi(x,t)$, given the nuclear properties of the system and a knowledge of the flux at some time instant long past. Therefore, assume that the flux is known at some arbitrary time $t = 0$; then, for all subsequent time (during which no changes are introduced into the system), the flux is given by

$$\phi(x,t) = G_0 \sum_{n=0}^{\infty} A_n e^{-D v \lambda_n^2 t} \cos \alpha_n x \left[1 + \frac{C_n \sin \gamma_n x}{A_n \cos \alpha_n x} e^{-D v t (\omega_n^2 - \lambda_n^2)} \right] \quad (5.144)$$

according to (5.140) and (5.141). Now by (5.139) and (5.137),

$$\omega_n^2 - \lambda_n^2 = \gamma_n^2 - \alpha_n^2 = \left[\frac{\pi(n+1)}{\tilde{a}} \right]^2 - \left[\frac{\pi(2n+1)}{2\tilde{a}} \right]^2 = \left(\frac{\pi}{2\tilde{a}} \right)^2 (4n+3) > 0 \quad (5.145)$$

It follows, then, that as $t \to \infty$, $\phi(x,t)$ takes the asymptotic form

$$\phi(x,t) \sim G_0 \sum_{n=0}^{\infty} A_n e^{-D v \lambda_n^2 t} \cos \alpha_n x \quad (5.146)$$

Thus with increasing time, the antisymmetric components of the initial distribution $\phi_0(x)$ (given by the sine terms) decay more rapidly than the symmetric components (the cosine terms). We may conclude from this result that, regardless of the initial shape of the flux distribution, the later distributions will become progressively more symmetric with increasing time, provided of course that no further changes are made in the nuclear characteristics of the system. This behavior is a consequence of the assumption that the fissionable material is uniformly distributed throughout the multiplying medium, i.e., Σ_f is independent of x.

There is one other important conclusion which may be drawn from (5.146), namely, there is always one term in the series (5.146) which will predominate, and with increasing values of t, this single term will yield increasingly better approximations to the series. We demonstrate this trend by means of the following argument: Consider the quantity λ_n, which may be written in the form

$$\lambda_n^2 \equiv \alpha_n^2 - B^2 \equiv \left[\frac{\pi(2n+1)}{2\tilde{a}} \right]^2 - \frac{1}{D} (\nu\Sigma_f - \Sigma_a) \quad (5.147)$$

Suppose that we chose the width of the slab $\tilde{a}$ (for fixed composition)

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such that a particular member of the set λ_n vanishes; i.e., let us take $\tilde{a}$ such that $\lambda_m^2 \equiv 0$, where $m \neq 0$. Then by (5.147)

$$\left[\frac{\pi(2m+1)}{2\tilde{a}}\right]^2 = \frac{1}{D}(\nu\Sigma_f - \Sigma_a) \tag{5.148}$$

All other λ_n are given by [using (5.148) in (5.147)]

$$\lambda_n^2 = \left[\frac{\pi(2n+1)}{2\tilde{a}}\right]^2 - \left[\frac{\pi(2m+1)}{2\tilde{a}}\right]^2 = \left(\frac{\pi}{2\tilde{a}}\right)^2 [(2n+1)^2 - (2m+1)^2] \tag{5.149}$$

It follows that if

$$\begin{array}{ll} m > n & \lambda_n^2 < 0 \\ m = n & \lambda_n^2 = 0 \\ m < n & \lambda_n^2 > 0 \end{array} \tag{5.150}$$

Thus for $m \neq 0$, we conclude that (1) all terms in (5.146) with $n > m$ have negative exponents and therefore decrease with time; and (2) all terms with $n < m$ have positive exponents and increase with time. Thus, the flux is an increasing function of time, and no steady-state condition is possible for this system[1] (unless, by some chance, all A_n for $n = 0, 1, \ldots m - 1$ are identically zero).

Consider now the special case when $m = 0$; then,

$$\lambda_0^2 \equiv 0 \equiv \left(\frac{\pi}{2\tilde{a}}\right)^2 - \frac{1}{D}(\nu\Sigma_f - \Sigma_a) \tag{5.151}$$

and for all other $n > 0$

$$\lambda_n^2 = \left[\frac{\pi(2n+1)}{2\tilde{a}}\right]^2 - \left(\frac{\pi}{2\tilde{a}}\right)^2 = 4n(n+1)\left(\frac{\pi}{2\tilde{a}}\right)^2 \tag{5.152}$$

Evidently all the λ_n are positive; therefore, after sufficiently long time all terms beyond the first ($n > 0$) will become negligibly small in comparison to the first term. For this case (5.146) reduces to the form

$$\phi(x,t) \sim G_0 A_0 \cos \alpha_0 x = G_0 A_0 \cos\left(\frac{\pi x}{2\tilde{a}}\right) \tag{5.153}$$

which is independent of time. The coefficient A_0 is given by (5.143)

$$A_0 = \frac{1}{\tilde{a}G_0}\int_{-\tilde{a}}^{\tilde{a}} \phi_0(x) \cos\left(\frac{\pi x}{2\tilde{a}}\right) dx \tag{5.154}$$

We conclude, therefore, that if at time $t = 0$ we start a multiplying slab with a neutron flux of the form $\phi_0(x)$ and choose the composition and the

[1] A rigorous proof of this statement is given, for example, by R. V. Churchill, "Fourier Series and Boundary Value Problems," p. 105, McGraw-Hill Book Company, Inc., New York, 1941.

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dimensions of the slab so that

$$\left(\frac{\pi}{2\tilde{a}}\right)^2 = \frac{\nu\Sigma_f - \Sigma_a}{D} \tag{5.155}$$

then after some time, the flux distribution in the slab will approach a steady-state condition given by

$$\phi(x,t) \to \phi(x) = A \cos\left(\frac{\pi x}{2\tilde{a}}\right) \qquad \text{as } t \to \infty \tag{5.156}$$

where $A \equiv A_0 G_0$ and is given by the initial conditions. Furthermore, no other choice than (5.155) will lead to steady-state operation [see the discussion following (5.150)]. It is true that if it were possible to start the slab with a flux of the form

$$\phi_0(x) = C \cos\frac{(2m+1)\pi x}{2\tilde{a}} \tag{5.157}$$

where $m > 0$, one could obtain a steady-state flux of this shape, since all the coefficients A_n in (5.146) would be zero except the mth [see (5.143)], and the slab width a could be chosen so that the exponent λ_m in the mth term would be zero. However, such a function as (5.157) with $m > 0$ is negative in some regions of the slab; thus initial distributions of the form (5.157) are unacceptable on the basis of a physical argument.

In the case that $\phi_0(x) \equiv 0$, there will never be a nonzero value of the neutron flux in the system, no matter what choice is made for the half-width, for from (5.143) all A_n and $C_n \equiv 0$. However, if there is even an infinitesimally small initial flux, it may be built up to any desired magnitude by altering the composition or size of the slab. Suppose, for example, that the slab is supporting some steady-state flux level $\phi_0(x)$. By the arguments given above $\phi_0(x)$ must have the form[1] (5.156)

$$\phi_0(x) = \alpha_0 \cos\left(\frac{\pi x}{2\tilde{a}}\right) \tag{5.158}$$

and the condition (5.155) must be satisfied. At $t = t_0$, the composition of the slab is changed uniformly so that

$$\frac{\nu\Sigma_f - \Sigma_a}{D} > \left(\frac{\pi}{2\tilde{a}}\right)^2 \tag{5.159}$$

Then by (5.151) $\lambda_0^2 < 0$, and the flux will no longer remain at steady state, but will rise according to (5.146). Because of the assumption that the initial distribution $\phi_0(x)$ is a simple cosine function, all coefficients $A_n = 0$ for $n > 0$ and all $C_n \equiv 0$ for $n \geq 0$. Thus the time dependence will be

[1] Note that this is true only if the slab is homogeneous and contains no irregularities in composition or form.

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given by

$$\phi(x,t) = \mathfrak{A}_0 e^{-Dv\lambda_0^2 t} \cos\left(\frac{\pi x}{2\tilde{a}}\right) \tag{5.160}$$

This function is shown in Fig. 5.19 for various values of t. The initial magnitude of the flux was characterized by the amplitude $\mathfrak{A}_0$; thus to achieve any other higher flux level we need wait only until the desired increase is achieved by the factor exp $(-Dv\lambda_0^2 t)$. If this level is reached at $t = t'$, a suitable readjustment in the composition of the slab can then be made so as to satisfy the condition (5.155). The flux will now cease to rise and can be caused to stabilize at the desired level. Conditions of the types (5.155) and (5.159) can be derived for other reactor geometries. In practice, one can impose a condition such as (5.159) upon the multiplying system by external means and thereby exercise control over the neutron population. This is the basis of all reactor control methods (see Sec. 1.3b).

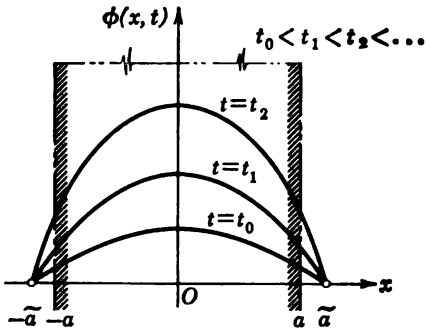

FIG. 5.19 Flux distribution in infinite-slab reactor.

Equation (5.156) gives the flux distribution at steady state in an infinite-slab reactor (with fission sources only) based on the one-velocity model. This result may also be obtained by starting with the steady-state diffusion equation directly; however, it is not possible, then, to prove rigorously that only the member of the set,

$$\cos\frac{(2n+1)\pi x}{2\tilde{a}} \qquad n = 0, 1, 2, \ldots$$

which persists in the steady state, is the one corresponding to $n = 0$. This term is commonly called the *lowest mode* or *fundamental mode*. Thus the arguments which have been given are really necessary for the completion of the discussions given below in which relations analogous to (5.156) are obtained for the sphere and the infinite cylinder by starting directly with the steady-state diffusion equation.

b. The Sphere and the Infinite-cylinder Reactors. The solutions for the steady-state flux distribution in a multiplying system with spherical symmetry or in a system with axial (cylindrical) symmetry can be carried out by undertaking a series of arguments similar to those used in the case of the slab reactor. The method is straightforward and will yield in each case a general result of the form (5.141). In these applications the time-dependent portion of the solution will have the form (5.132). The space-dependent part of the solution, however, will involve some

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new functions. In the case of the sphere (as we will demonstrate shortly) these functions are the trigonometric functions divided by the radial coordinate r, and in the case of the cylinder, the ordinary Bessel functions. It can be shown, as in the slab problem, that long time periods after "starting" the system (i.e., as $t \to \infty$) one term will predominate in the series solutions for both the sphere problem and the cylinder problem. Moreover, if it is required that either of these systems operate at steady state, there exists only one term from each of these series (corresponding to the fundamental mode) which is acceptable as a solution. These terms will correspond to the result (5.156).

We will not use this general approach in the present analysis. Instead, we will start with the general steady-state diffusion equation and establish the appropriate solution in each case by a suitable identification of the fundamental mode, guided in our choice by our experience with the slab problem. The steady-state diffusion equation is [cf. Eq. (5.124)]

$$\nabla^2\phi(\mathbf{r}) + B^2\phi(\mathbf{r}) = 0 \tag{5.161}\star$$

with B^2 defined by (5.126). For the spherically symmetric problem, this differential equation reduces to [see (5.80)]

$$\frac{1}{r}[r\phi(r)]'' + B^2\phi(r) = 0 \tag{5.162}$$

which has the general solution

$$\phi(r) = \frac{1}{r}(A \sin Br + C \cos Br) \tag{5.163}$$

This solution must satisfy the boundary conditions (5.72a) and (5.72c). If the outer boundary of the sphere is given by $r = R$, these boundary conditions take the form

$$\begin{array}{l} (1)\ \phi(r) \text{ must be finite for all } r \leq R \\ (2)\ \phi(R) = 0 \end{array} \tag{5.164}$$

The condition that the flux remain finite for all r, appropriate here since there are no *localized* sources, yields the requirement that $C \equiv 0$. The values of B needed to meet condition (2) are

$$B = \frac{n\pi}{R} \qquad n = 0, 1, \ldots \tag{5.165}$$

Note that we have retained the symbol B to denote the various eigenvalues, instead of the notation α_n used in the slab problem. As indicated in Eqs. (5.137) and (5.165), the eigenvalues form an infinite set; however, only the first of these, corresponding to the fundamental mode, is required since this mode gives the steady-state solution. It is understood, then, that the B which appears in the argument of ϕ refers to the lowest eigenvalue. Since the quantity B was originally defined by (5.126), this

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equivalence implies that there must exist a certain relationship between the nuclear properties of the system and its geometry if the system is to be at steady state [cf. Eqs. (5.151), (5,155), and (5.156)]. A detailed study of this relationship is given in a subsequent analysis.

On the basis of the preceding remarks we select the eigenvalue corresponding to $n = 1$. This choice is entirely consistent with our experience with the time-dependent behavior of the flux in the slab reactor. It can also be argued that larger values of n are physically unacceptable because they predict negative values of the neutron flux within the medium. However, this argument is not complete, since it does not rule out the possibility of a superposition of solutions for the various values of n; the arguments presented in connection with Eq. (5.149) must be used. It follows from these remarks that the solution to (5.162) is

$$\phi(r) = \frac{A}{r} \sin \frac{\pi r}{\tilde{R}} \qquad (5.166)\star$$

Note that as in the case of the slab problem there still remains an arbitrary constant A which can be determined only from the initial conditions of the system.

The solution for the cylindrically symmetric system is obtained from (5.161); the appropriate form for the present application is [see Eq. (5.86)]

$$\frac{1}{\rho} [\rho\phi'(\rho)]' + B^2\phi(\rho) = 0 \qquad (5.167)$$

In this case we consider a cylinder of radius $\mathfrak{R}$ and of infinite height (with clearly no lengthwise dependence of the flux). As in the sphere problem, we require that the flux be finite for all $\rho \leq \mathfrak{R}$ and that $\phi(\tilde{\mathfrak{R}}) = 0$. Equation (5.167) is Bessel's equation of order zero; its general solution is

$$\phi(\rho) = AJ_0(\rho B) + CY_0(\rho B) \qquad (5.168)$$

The Bessel function of the second kind $Y_0(\rho B)$ has a singularity at the origin; therefore, the constant C must be taken to be zero so as to satisfy the requirement that ϕ remain finite. The boundary condition at $\tilde{\mathfrak{R}}$ requires, in addition, that

$$J_0(B\tilde{\mathfrak{R}}) = 0 \qquad (5.169)$$

The Bessel function J_0 has an infinite set of zeros[1] $\nu_1, \nu_2, \ldots$. As in the two previous cases we select the first root of this equation. Let its value[2] be ν_1; then the corresponding eigenvalue is

$$B = \frac{\nu_1}{\tilde{\mathfrak{R}}} \qquad \text{where } \nu_1 = 2.405, \text{ etc.} \qquad (5.170)$$

[1] See Jahnke and Emde, *op. cit.*, p. 126, for a sketch of this function and p. 166 for zeroes.

[2] Hildebrand, *op. cit.*, p. 575.

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and the solution to (5.167) is

$$\phi(\rho) = AJ_0\left(\frac{\nu_1\rho}{\tilde{\mathfrak{R}}}\right) \tag{5.171}\star$$

It is of interest to compare the shape of the neutron flux for the three elementary geometries considered. The normalized solutions (5.156), (5.166), and (5.171) are shown in Fig. 5.20.

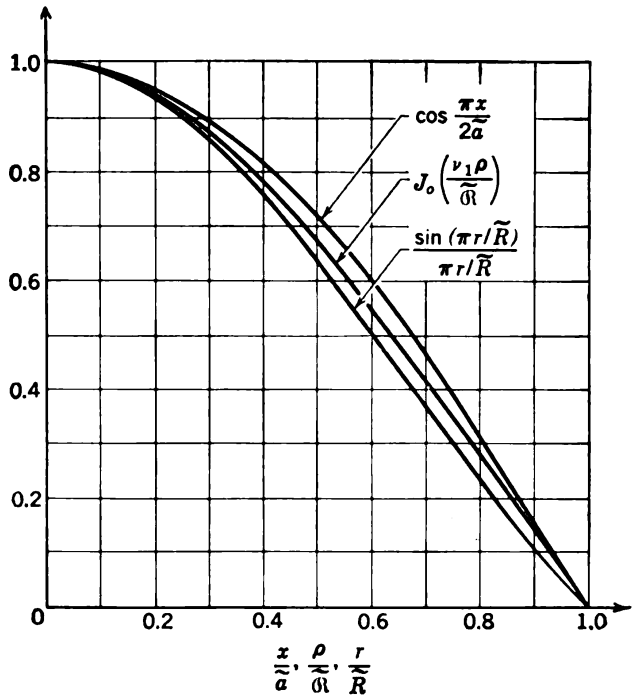

FIG. 5.20 Normalized solutions for flux.

c. Criticality Condition and Multiplication Constant. In each of the three geometries for single multiplying media discussed above, it was found that a certain relationship was required between the composition of the medium and its dimensions if the system was to operate as a stable chain reactor. This relation was expressed in the form

$$\frac{1}{D}(\nu\Sigma_f - \Sigma_a) = B^2 \qquad \text{for all three geometries} \tag{5.126}$$

and

$$\begin{aligned} B^2 &= \left(\frac{\pi}{2\tilde{a}}\right)^2 && \text{for the slab} \\ B^2 &= \left(\frac{\nu_1}{\tilde{\mathfrak{R}}}\right)^2 && \text{for the cylinder} \\ B^2 &= \left(\frac{\pi}{\tilde{R}}\right)^2 && \text{for the sphere} \end{aligned} \tag{5.172}\star$$

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For each geometry, Eq. (5.126) and one of Eqs. (5.172) must hold simultaneously if steady-state operation is to be achieved. Note that by imposing these requirements on each system the composition or size is completely specified; thus, if the size of the system is selected, then (5.172) yields the corresponding value of B^2, and the substitution of this number into (5.126) yields an expression for the composition (note that the cross sections are functions of the fuel concentration). Conversely, if the fuel concentration is given, (5.126) yields the B^2 value, which when used in one of the Eqs. (5.172) specifies the size of the system which will support a stable chain reaction.

A set of equations of the type (5.126) and (5.172) is called the *critical condition* or *criticality condition.* It will be found that similar criticality conditions must hold for the steady-state operation of all systems in which the only source of neutrons is the fission process. An examination of Eqs. (5.146) and (5.147) shows (in the case of the slab, for example) that there are three important cases for the quantity

$$\lambda_0^2 = \left(\frac{\pi}{2\tilde{a}}\right)^2 - \frac{1}{D}(\nu\Sigma_f - \Sigma_a) \tag{5.151}$$

namely,

(1) $\lambda_0^2 = 0$ gives, as argued above, steady-state operation

(2) $\lambda_0^2 > 0$ all terms in the time-dependent series have decreasing exponential factors so that the flux steadily diminishes toward zero

(3) $\lambda_0^2 < 0$ at least the first term of the series has an increasing exponential factor; thus the flux is a monotonically increasing function of time

The situations corresponding to (1), (2), and (3) have been designated *critical*, *subcritical*, and *supercritical*, respectively.

Consider now the "multiplication constant" of such a system; as in Eq. (3.16), the multiplication constant is defined as the ratio of the number of neutrons produced in a given generation in a reactor to the number produced in the preceding generation. If the reactor is to be in steady-state operation, the neutron population must be stationary in time (on the average), so that neutron gains always just balance neutron losses, and the multiplication constant is unity. This may be seen directly from the steady-state differential equation (using the slab case as an example):

$$-D\phi''(x) + \Sigma_a\phi(x) = \nu\Sigma_f\phi(x) \tag{5.173}$$

The steady-state solution is given by

$$\phi(x) = A\cos Bx \qquad B = \frac{\pi}{2\tilde{a}} \tag{5.174}$$

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The substitution of this expression into (5.173) yields

$$DB^2\phi(x) + \Sigma_a\phi(x) = \nu\Sigma_f\phi(x)$$

Integration of this equation over the entire volume of the reactor gives (in the case of the infinite slab or cylinder the integration is carried out for a unit height of the system)

$$DB^2\Phi + \Sigma_a\Phi = \nu\Sigma_f\Phi \tag{5.175}$$

where we have defined

$$\Phi \equiv \int_V \phi(x)\, dx \tag{5.176}$$

Equation (5.175) may be stated verbally as

(Total leakage losses) + (total absorption losses) = (total fission gains)

The left side of (5.175) is the total number of neutrons lost from the reactor (per unit time), and these give rise to $\nu\Sigma_f\Phi$ neutrons on the right side; thus the multiplication constant for this case is

$$\frac{\nu\Sigma_f\Phi}{DB^2\Phi + \Sigma_a\Phi} = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a} = 1 \tag{5.177}$$

where we have used (5.175).

If the reactor is not operating at steady state, the calculation of the multiplication constant is more difficult, since, in the differential equation, there is now a time-derivative term to be considered. A convenient method for treating this case involves the use of the quantity ν_c (introduced in Sec. 4.6d), a fictitious number of neutrons per fission, which may be adjusted so that for any composition and configuration the fission sources can be made just to balance the losses. Thus we choose as the applicable differential equation for the nonstationary system

$$-D\nabla^2\phi(\mathbf{r}) + \Sigma_a\phi(\mathbf{r}) = \nu_c\Sigma_f\phi(\mathbf{r}) \tag{5.178}$$

where for the steady-state case $\nu_c \equiv \nu$, the experimental value. This equation, then, describes a fictitious steady-state system which could be achieved by using the fuel concentration of the nonstationary system (implied in the macroscopic cross sections) if the neutron yield per fission were ν_c (in general, different from ν). Clearly, if the flux in this system is to be time independent, then the solution to (5.178) must satisfy a relation of the type (5.177); but, the solution to (5.178) is of the form (5.174); we have, therefore, the immediate result that

$$\nu_c = \frac{DB^2 + \Sigma_a}{\Sigma_f} \tag{5.179}$$

for all three elementary geometries. The value of B^2 is given in each case by Eqs. (5.172). Equation (5.179) states that, if $(DB^2 + \Sigma_a)\Phi$ neutrons are lost from the reactor, it requires $\nu_c\Sigma_f\Phi$ neutrons to replace them;

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but, only $\nu\Sigma_f\Phi$ neutrons are actually produced (where ν is the true average number of neutrons per fission for the given fuel material). Thus the multiplication constant is, by definition [cf. Eq. (3.16)],

$$\text{Multiplication constant} \equiv k = \frac{\nu\Sigma_f}{DB^2 + \Sigma_a} = \frac{\nu}{\nu_c} \tag{5.180}$$

where the last equality follows from (5.179). The usual problem that one encounters in practice is to determine the multiplication constant of a specified system. In this situation all the nuclear characteristics of the reactor are known (i.e., all the cross sections, including the fuel, have been specified), and the geometric configuration and size of the system have been selected. The calculation of k requires first the evaluation of B^2 from the appropriate relation in the set (5.172). The substitution of this result, along with the cross sections, into (5.179) yields ν_c. The multiplication constant is then computed from (5.180).

It is convenient in describing the present systems, which take into account the neutron leakages, to separate (5.180) into the factors

$$k = \left(\frac{\nu\Sigma_f}{\Sigma_a}\right)\left(1 + \frac{DB^2}{\Sigma_a}\right)^{-1} \tag{5.181}$$

But, by our previous definitions [see (3.18)], $\nu\Sigma_f/\Sigma_a$ is the multiplication constant for the infinite medium in the one-velocity approximation. For clarity in notation we introduce the symbol k_∞ for the infinite medium.

$$k_\infty \equiv \frac{\nu\Sigma_f}{\Sigma_a} \tag{5.182}\star$$

The substitution of this expression, along with the definition of L^2 from (5.75), into (5.181) yields

$$k = k_\infty(1 + L^2B^2)^{-1} \tag{5.183}\star$$

Again note that B^2 is obtained from (5.172). The multiplication constant k for the finite medium is customarily called the "effective multiplication constant," since it is the infinite-medium multiplication constant k_∞ multiplied by a reduction factor $(1 + L^2B^2)^{-1}$; this factor is the one-velocity nonleakage probability and will be discussed in detail later. We summarize the above results by combining the information from (5.151) and (5.183):

(1) $k = 1$ corresponds to $\lambda_0^2 = 0$, and the system is critical
(2) $k < 1$ corresponds to $\lambda_0^2 > 0$, and the system is subcritical
(3) $k > 1$ corresponds to $\lambda_0^2 < 0$, and the system is supercritical

d. Sample Calculation. We illustrate the procedure described in the preceding section by means of a simplified calculation for a bare homogeneous spherical reactor of radius R. The problem is to compute

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the critical fuel concentration and total fuel mass for this reactor as a function of its size. For convenience in computation we will assume that the moderator density N_M in the system is unaffected by changes in the fuel concentration N_F; as we noted previously (see Sec. 4.8c) this is a good approximation for systems with low fuel concentration. We further assume that the diffusion coefficient D is also unaffected by such changes, being principally determined by the moderator concentration.

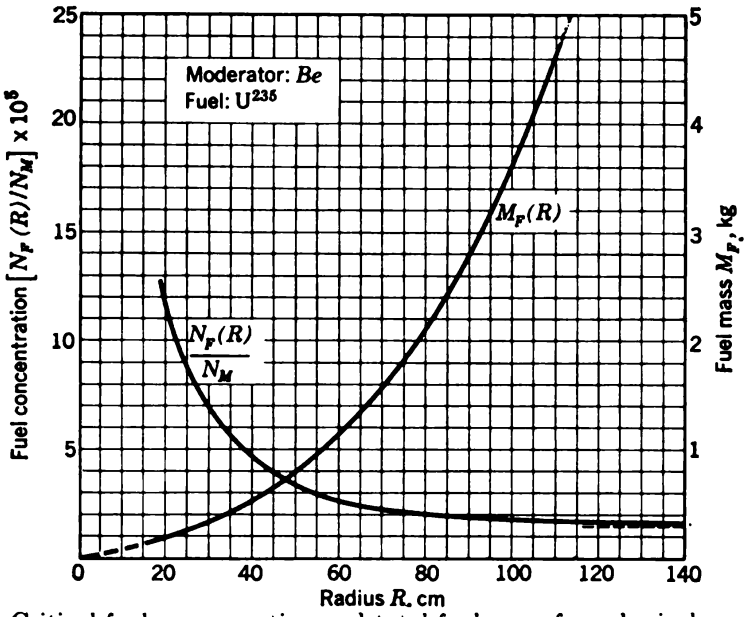

FIG. 5.21 Critical fuel concentration and total fuel mass for spherical reactor in the one-velocity model.

Then, in the simple one-velocity approximation, we must have, from Eqs. (5.126) and (5.172),

$$\frac{N_F[\nu\sigma_f - \sigma_a^{(F)}] - N_M\sigma_a^{(M)}}{D} = \left(\frac{\pi}{R + 0.7104\lambda_{tr}}\right)^2 \equiv B^2 \qquad (5.184)$$

This relationship is the requirement for criticality. The solution of this equation for N_F gives the critical fuel concentration [with $\eta \equiv \nu\sigma_f/\sigma_a^{(F)}$]

$$N_F = \frac{N_M\sigma_a^{(M)} + DB^2}{\sigma_a^{(F)}(\eta - 1)} \qquad (5.185)$$

Figure 5.21 shows a plot of N_F/N_M as a function of the radius of the sphere for a system of pure U^{235} and beryllium. The nuclear density of the beryllium was taken as 0.124×10^{24} nuclei/cm^3, and the microscopic cross sections are from Table 2.2. It is of interest to note the asymptotic behavior of the function $N_F(R)/N_M$. The asymptote corresponds to the minimum fuel concentration with which a system of U^{235} and beryllium can go critical (on the basis of the one-velocity model); this is the con-

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centration required for an infinite medium of these materials and can be obtained from (5.185) by taking the limit as $R \to \infty$. The result is

$$\frac{N_F(\infty)}{N_M} = \frac{\sigma_a^{(M)}}{\sigma_a^{(F)}(\eta - 1)}$$

The critical mass M_F of U^{235} may be obtained as a function of R by the use of the relation

$$M_F(R) = \frac{4\pi R^3 N_F A_F}{3\mathcal{A}} = \left[\frac{4\pi A_F}{3\mathcal{A}\sigma_a^{(F)}(\eta - 1)}\right] R^3[\Sigma_a^{(M)} + DB^2] \quad (5.186)$$

This function is shown in the figure. It is evident from (5.186) that, for large R, $M_F(R) \sim R^3$, and for small R, $M_F(R) \sim R$. This is the behavior predicted by the one-velocity model. Although the indicated trend for large systems ($R \to \infty$) is in good agreement with more accurate models, the dependence of the fuel mass on R for small systems is given incorrectly by (5.186). We will show later, in Chap. 6, that the critical fuel mass does not decrease to zero with decreasing size but, in fact, first passes through a minimum as R decreases and then rises again to steadily increasing magnitudes with further decreases in R. This trend is not predicted by (5.186) because the one-velocity model does not provide an adequate estimate of all the neutron losses resulting from escapes through the boundaries of the system.

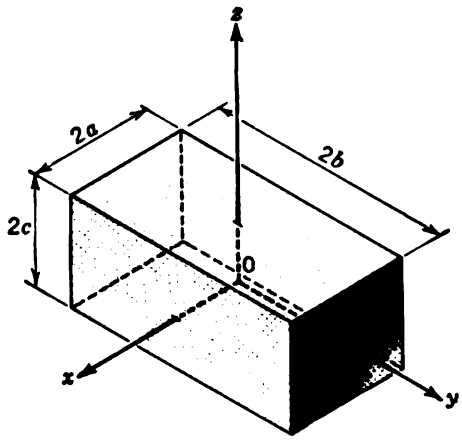

FIG. 5.22 Rectangular block reactor.

e. Rectangular Block and Finite Cylinder. The methods used in the analysis of the infinite slab and cylinder and of the sphere are easily extended to the analysis of the rectangular-block and finite-cylinder reactors. The procedure is straightforward and involves the use of the technique of separation of variables.

Consider first the homogeneous multiplying medium in the form of a rectangular parallelepiped, as shown in Fig. 5.22. The appropriate form of the diffusion equation which describes the steady-state flux distribution in this medium of absorption cross section Σ_a and fission cross section Σ_f is

$$\nabla^2\phi(x,y,z) + B^2\phi(x,y,z) = 0 \quad (5.187)$$

where $B^2 = (\nu\Sigma_f - \Sigma_a)/D$. We look for a solution of this equation which may be written in the form

$$\phi(x,y,z) = F(x)\,G(y)\,H(z) \quad (5.188)$$

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The substitution of this expression into (5.187) yields

$$\frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} + \frac{H''(z)}{H(z)} = -B^2 \tag{5.189}$$

In order for this equation to be satisfied, each term must be a constant, independent of the variables x, y, and z. Let these constants be k_1^2, k_2^2, and k_3^2; then

$$\frac{F''(x)}{F(x)} = -k_1^2 \qquad \frac{G''(y)}{G(y)} = -k_2^2 \qquad \frac{H''(z)}{H(z)} = -k_3^2 \tag{5.190}$$

It follows from (5.189) that

$$B^2 = k_1^2 + k_2^2 + k_3^2 \tag{5.191}$$

An appropriate set of boundary conditions may be stated in terms of the extrapolated dimensions of the parallelepiped. In the present calculation we will ignore the difficulties in the concept of the extrapolated boundary at the corners which have infinite curvature by arguing that the necessary correction will be small and, furthermore, that the region of primary importance in the reactor is near the center of the block. With this approximation in mind we can confine the requirements on the neutron flux $\phi(x,y,z)$ to the following pair of boundary conditions: (1) The neutron flux is symmetric about the three midplanes; thus, for example, $\phi(x,y,z) = \phi(-x,y,z)$. (2) The flux vanishes at the outer extrapolated surfaces of the rectangular block; for example, $\phi(\pm\tilde{a},y,z) = 0$, where $\tilde{a}$ is the extrapolated half-width of the block in the x direction (see Fig. 5.22).

It should be noted that boundary condition 1 may be stated in an alternate manner, namely, that the component of the net current in each of the x, y, and z directions be zero at the midplanes. Thus,

$$J_x(0,y,z) = J_y(x,0,z) = J_z(x,y,0) = 0$$

These two statements are equivalent, and either may be used in the case of a symmetric function. Boundary condition 2 is the appropriate form of the statement (5.72*c*).

The solutions of the three ordinary differential equations (5.190) are of the same form. In the case of $F(x)$ we obtain the familiar result

$$F(x) = A \sin k_1 x + C \cos k_1 x$$

The application of the boundary conditions 1 and 2 yields the solution

$$F(x) = C \cos \frac{\pi x}{2\tilde{a}} \tag{5.192}$$

where we have selected the fundamental mode by the usual arguments. The solutions to $G(y)$ and $H(z)$ may be obtained in the same way, and the

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complete solution to (5.187) is easily shown to be

$$\phi(x,y,z) = \mathfrak{C} \cos \frac{\pi x}{2\tilde{a}} \cos \frac{\pi y}{2\tilde{b}} \cos \frac{\pi z}{2\tilde{c}} \tag{5.193}$$

We have selected for the eigenvalues k_1, k_2, and k_3 the values corresponding to the fundamental modes. It follows that (5.191) may be written

$$B^2 = \left(\frac{\pi}{2\tilde{a}}\right)^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2 \tag{5.194}$$

The result (5.193) gives the steady-state flux distribution in the rectangular block reactor with half dimensions a, b, and c. Note that, as in the case of the more elementary geometries previously considered, the solution involves an arbitrary constant $\mathfrak{C}$ which can be determined only from the initial conditions on the system. In order that the system support the steady-state flux distribution (5.193), we require that

$$\left(\frac{\pi}{2\tilde{a}}\right)^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2 = \frac{1}{D}(\nu\Sigma_f - \Sigma_a) \tag{5.195}$$

This is the condition of criticality. Given the dimensions of this reactor, (5.195) yields an equation for the critical fuel concentration. Given the fuel concentration, and say the proportions of the block, (5.195) yields the size of the system. The multiplication constant for this reactor may be derived by means of the usual method (see, for example, Sec. 5.4c). It is easily shown to be given by

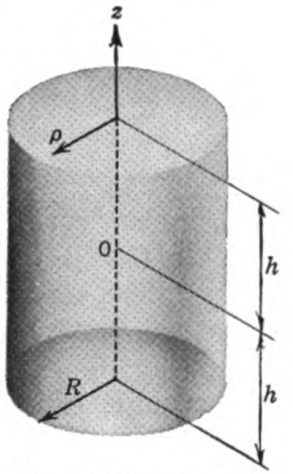

FIG. 5.23 Cylindrical reactor.

$$k = \left(\frac{\nu\Sigma_f}{\Sigma_a}\right)(1 + L^2B^2)^{-1} \tag{5.196}$$

where for criticality, $k = 1$. The value of B^2 to be used in this relation is given by (5.194).

Another geometric configuration which is of practical importance in reactor design is the cylinder of finite length. Consider therefore a homogeneous multiplying medium in the form of a right circular cylinder of height $2h$ and radius $\mathfrak{R}$ (see Fig. 5.23). This system has cylindrical symmetry, and the appropriate statement of the steady-state diffusion equation is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\phi(\rho,z)\right] + \frac{\partial^2}{\partial z^2}\phi(\rho,z) + B^2\phi(\rho,z) = 0 \tag{5.197}$$

It has been assumed here that the neutron-fission source is given by

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$\nu\Sigma_f\phi(\rho,z)$, and B^2 is defined in the usual way (5.126). We solve this differential equation by separation of variables; therefore, take

$$\phi(\rho,z) \equiv R(\rho)\, Z(z) \tag{5.198}$$

Equation (5.197) may now be written

$$\frac{1}{R(\rho)}\,[\rho R'(\rho)]' + \frac{Z''(z)}{Z(z)} = -B^2 \tag{5.199}$$

As before, let

$$\frac{1}{R(\rho)}\,[\rho R'(\rho)]' \equiv -k_\rho^2 \qquad \frac{Z''(z)}{Z(z)} \equiv -k_z^2 \qquad B^2 \equiv k_\rho^2 + k_z^2 \tag{5.200}$$

The general solutions to these equations are

$$\begin{aligned} Z(z) &= A_z \sin k_z z + C_z \cos k_z z \\ R(\rho) &= A_\rho J_0(k_\rho \rho) + C_\rho Y_0(k_\rho \rho) \end{aligned} \tag{5.201}$$

The boundary conditions which must be satisfied by these solutions are given in (5.72). Note that the symmetry condition requires that $A_z \equiv 0$. The condition that the flux remain finite at all points within the medium requires that we take $C_\rho \equiv 0$, inasmuch as $Y_0(\rho)$ has a singularity at $\rho = 0$. The extrapolated boundary condition yields the additional requirement that [see (5.170)]

$$k_\rho \tilde{\mathcal{R}} = \nu_1 \qquad k_z \tilde{h} = \frac{\pi}{2} \tag{5.202}$$

where ν_1 is the first zero of J_0. We combine all these results to obtain the solution

$$\phi(\rho,z) = \mathcal{C} J_0\left(\frac{\nu_1 \rho}{\tilde{\mathcal{R}}}\right) \cos\frac{\pi z}{2\tilde{h}} \tag{5.203}$$

As before ($\sim$) denotes the extrapolated boundaries. The criticality requirement for the cylindrical reactor is

$$B^2 = \frac{1}{D}\,(\nu\Sigma_f - \Sigma_a) = \left(\frac{\nu_1}{\tilde{\mathcal{R}}}\right)^2 + \left(\frac{\pi}{2\tilde{h}}\right)^2 \tag{5.204}$$

and the multiplication constant is given by (5.196). In this case, however, we use (5.204) for B^2. The general results for the rectangular-block, finite-cylinder, and sphere reactors have been summarized in Table 5.3.

f. Optimum Proportions for the Bare Reactor. Suppose that a given homogeneous mixture of materials with a certain k_∞ and L^2 is to be made into a critical reactor. For the three configurations listed in Table 5.3, three different total volumes of mixture will be required. Since for all

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three systems $k = 1$, we must have

$$\left(\frac{\pi}{\tilde{R}}\right)^2 = \left(\frac{\nu_1}{\tilde{\mathfrak{R}}}\right)^2 + \left(\frac{\pi}{2\tilde{h}}\right)^2 = \left(\frac{\pi}{2\tilde{a}}\right)^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2 \tag{5.205}$$

The actual critical volume of the block and cylinder will depend on which of the possible proportions is chosen, that is, which ratio $a:b:c$ and $R:h$. In each of these two cases, however, the ratio may be specified so as to

TABLE 5.3 SUMMARY OF RESULTS FOR THE ONE-VELOCITY REACTOR

	Rectangular block	Right cylinder	Sphere
Dimensions	$2a \times 2b \times 2c$	Radius $\mathfrak{R}$, height $2h$	Radius R
Spatial flux form[a]	$\cos\frac{\pi x}{2\tilde{a}} \cos\frac{\pi y}{2\tilde{b}} \cos\frac{\pi z}{2\tilde{c}}$	$J_0\left(\frac{2.405\rho}{\tilde{\mathfrak{R}}}\right)\cos\frac{\pi z}{2\tilde{h}}$	$\frac{1}{r}\sin\frac{\pi r}{\tilde{R}}$
Buckling, B^2	$\left(\frac{\pi}{2\tilde{a}}\right)^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2$	$\left(\frac{2.405}{\tilde{\mathfrak{R}}}\right)^2 + \left(\frac{\pi}{2\tilde{h}}\right)^2$	$\left(\frac{\pi}{\tilde{R}}\right)^2$

Criticality condition: $k = 1$, where $k \equiv k_\infty p_{\mathrm{NL}}$, and $k_\infty \equiv \eta f p_{\mathrm{th}} \epsilon$, $p_{\mathrm{NL}} \equiv (1 + L^2B^2)^{-1}$, and B^2 is given for each geometry.

[a] x, y, z are measured from the center of the block; z is measured from the geometrical center of the cylinder, and ρ is measured normal to the axis; r is measured from the center of the sphere; $\tilde{A} \equiv A + 0.7104\lambda_{\mathrm{tr}}$.

give the *minimum* critical volume and, therefore, minimum fuel mass. To determine this we require that the volume be a minimum as the ratio of the dimensions is varied and the buckling is held fixed (i.e., the composition of the mixture is held fixed). The results of this minimization are (if it is assumed that the extrapolation distance is negligible in comparison with the dimensions of the reactor; that is, $\tilde{a} \simeq a$, etc.):

$$\begin{aligned} a = b = c \qquad & \text{for rectangular block reactor of minimum critical volume} \\ \mathfrak{R} = \frac{\sqrt{2}\,\nu_1 h}{\pi} \cong 1.08h \qquad & \text{for cylinder} \end{aligned} \tag{5.206}$$

It is of interest to compare the total volume of material needed to produce a bare critical assembly in each of the three cases considered above. If we use for the block and cylinder geometries the optimum proportions (5.206), and neglect the extrapolation distance (a reasonable approximation for a large reactor), then it may be shown that these volumes are:

$$\begin{aligned} &\textit{Cube:} && V = \frac{\pi^3 3^{\frac{3}{2}}}{B^3} \simeq \frac{161}{B^3} \\ &\textit{Cylinder:} && V = \frac{\pi^2 3^{\frac{3}{2}} \nu_1^2}{2B^3} \simeq \frac{150}{B^3} \\ &\textit{Sphere:} && V = \frac{4\pi^4}{3B^3} \simeq \frac{130}{B^3} \end{aligned} \tag{5.207}$$

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It is seen that the least total volume of material is used in the spherical configuration. Thus the sphere yields the minimum critical fuel mass for a given concentration of fuel. This result may ultimately be traced to the greater leakage of neutrons from the cubical and cylindrical forms, which have greater surface-to-volume ratios. The question of the most efficient shape for a bare assembly is one of making the best use of all the material involved in the reactor. For example, the material at the corners of the cube is very inefficiently used, since neutrons readily escape from such regions, and in a homogeneous reactor there is as much fuel invested (per unit volume) in the corners as at the center.

Another measure of the efficiency of use of the material in the reactor is often taken to be the "peak-to-average" flux ratio. If the flux profile is relatively flat over the reactor volume, all regions are contributing fairly equally to the total reaction rate; however, if the flux is strongly peaked at the center of the reactor, the fuel in the outer regions is being poorly used. Some measure of this is the ratio mentioned, that of the highest value of the flux to its average value over the reactor. In the case of the bare reactor configurations considered here,

$$\text{Peak-to-average ratio} = \frac{\phi(0)}{\bar{\phi}} = \frac{\text{value of flux at center of reactor}}{\text{average value of flux over reactor}}$$

Thus

$$\text{Peak-to-average ratio} = \frac{V_R \phi(0)}{\int_{V_R} \phi(\mathbf{r})\, d\mathbf{r}} \tag{5.208}$$

where V_R denotes the volume of the reactor. It may be easily shown that for the bare assemblies discussed above the peak-to-average ratio has the values:

$$\begin{aligned} &\textit{Block:} && \left(\frac{\pi}{2}\right)^3 \simeq 3.88 \\ &\textit{Cylinder:} && \frac{\nu_1 \pi}{4 J_1(\nu_1)} \simeq 3.65 \\ &\textit{Sphere:} && \frac{\pi^2}{3} \simeq 3.29 \end{aligned} \tag{5.209}$$

Note that $J_1(\nu_1) = 0.519$ and $\nu_1 = 2.405$. It is evident from these results that the sphere has the more favorable flux distribution.

g. Fission Sources and the Nonleakage Probability. We conclude the present section on the application of the one-velocity model to multiplying media with some remarks about the approximation of neutron-source strength proportional to flux (see Sec. 5.4a) and the concept of the nonleakage probability [see Eq. (5.183)].

In the problems that have so far been considered, the neutron-source term $S(\mathbf{r})$ for fission has been assumed to be $\nu\Sigma_f\phi(\mathbf{r})$. We should like now to examine the implications of this assumption. Neutrons produced

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by a fission reaction occupy the complete energy spectrum; however, the majority have energies considerably greater than thermal. These high-energy neutrons are eventually slowed to thermal speeds by experiencing numerous scattering collisions with the nuclei of the medium. The "thermalized" neutrons then enter a one-velocity diffusion process and wander through the system until they are eventually absorbed or escape through the boundaries of the reactor. The diffusion equation (5.22) is useful for describing the spatial distribution of these neutrons, provided that a suitable expression is developed for the source term.

This problem was previously encountered in the case of the infinite-medium slowing-down process for a multiplying material. In that instance we identified the source term for the thermal neutrons as the slowing-down density at thermal energy $q(E_{th})$ [cf. Eq. (4.252)]. In the present case we require a somewhat more generalized function which takes into account the spatial distribution of the slowing-down density. Thus a suitable description for the source term in the diffusion equation for thermal neutrons would be

$$S(\mathbf{r}) \equiv q(\mathbf{r},E_{th}) \tag{5.210}$$

where $q(\mathbf{r},E_{th})$ denotes the number of neutrons slowing past E_{th} per unit volume per unit time around space point $\mathbf{r}$ in the reactor. Unfortunately, the analytical machinery so far developed for treating multiplying media does not provide us with a detailed statement of the function $q(\mathbf{r},E_{th})$; this approach to the estimation of $S(\mathbf{r})$ will be deferred until later. For the present we will apply an intuitive formulation which will draw upon the concepts already developed. Let us define, then, the following source function:

$$\begin{aligned} S(\mathbf{r}) &\equiv \nu\Sigma_f\phi(\mathbf{r})p_{th}\epsilon \\ &= \text{number of thermal neutrons which appear per unit volume per unit time around space point } \mathbf{r} \end{aligned} \tag{5.211}$$

We have used in this definition the quantities:

$\nu \equiv$ average number of neutrons produced per fission

$p_{th} \equiv$ probability that a neutron born from fission reaches thermal energy before it is absorbed [see Eq. (4.283)]

$\epsilon \equiv$ fast-fission effect; the ratio of all fissions to thermal fissions [see Eq. (4.269)]

Equation (5.211) states that the rate at which neutrons become thermal at $\mathbf{r}$ is given by the fission rate[1] at $\mathbf{r}$, $\Sigma_f\phi\epsilon$, times the number of neutrons per fission ν corrected by p_{th}, for the losses which occur because of absorption during the slowing-down process. It is important to note that

[1] Note that, if $\Sigma_f\phi$ is the rate at which fissions occur because of thermal neutrons, $\epsilon\Sigma_f\phi$ is the rate at which fissions occur due to neutrons of all energies.

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(5.211) fails to account for any losses in the neutron population because of leakage during slowing down. Thus in (5.211) we have made the tacit assumption that there is no leakage of high-energy neutrons from the system. Moreover (5.211) implies that the migration of neutrons during the slowing-down process does not alter their spatial distribution (this assumption is valid in a single-medium system). Taking the thermal source function as having the space form of the thermal flux does *not* imply, physically, that the fission neutrons slow down to thermal and remain at the space point at which they originally appeared. It will be shown later that in a bare system the neutrons wander throughout the reactor during the slowing-down process and always move about in such a way that the space form of the neutron density (flux) at all energies is the same as at thermal energy.[1]

In all the preceding calculations in this section the one-velocity approximation was applied, and the source term was appropriately selected as $\nu\Sigma_f\phi$. This choice is consistent with the form (5.211) in that it is the special case of that function when $\epsilon p_{\text{th}} \equiv 1$; i.e., all neutrons born at $\mathbf{r}$ are immediately reduced to thermal (the one-velocity postulate). It should be recognized that, if the diffusion equation is used to obtain the thermal-flux distribution in a reactor, it should be written in terms of the corrected source function (5.211); although this formulation is incomplete, it is better than the simple form $\nu\Sigma_f\phi$.

The factors which appear in the source relation (5.211) may be expressed in an alternate set of reactor terminology which displays more clearly its relation to the multiplication constant of the system. Consider then the quantity $\nu\Sigma_f$, which may be written

$$\nu\Sigma_f = \left[\frac{\nu\Sigma_f}{\Sigma_a^{(F)}}\right]\left[\frac{\Sigma_a^{(F)}}{\Sigma_a}\right]\Sigma_a = \eta f \Sigma_a \tag{5.212}$$

where $\Sigma_a^{(F)}$ is the macroscopic absorption cross section of the fuel in the reactor and Σ_a is the total macroscopic absorption cross section for the mixture; η and f are defined in the usual way. The substitution of (5.212) into (5.211) gives

$$S(\mathbf{r}) = \eta f p_{\text{th}} \epsilon \phi(\mathbf{r}) \Sigma_a \equiv k_\infty \Sigma_a \phi(\mathbf{r}) \tag{5.213}$$

where the multiplication constant for the infinite medium, k_∞, is given by (4.275). Note that the expression used here for k_∞ differs from (5.182) through the additional factors ϵp_{th}. Clearly, (5.182) is the special one-velocity case of (4.275) used in (5.213). If (5.213) is used as the source term in the steady-state diffusion equation (5.22), we obtain

$$-D\nabla^2\phi(\mathbf{r}) + \Sigma_a\phi(\mathbf{r}) = k_\infty \Sigma_a \phi(\mathbf{r}) \tag{5.214}$$

[1] See discussion accompanying Eq. (6.87).

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This equation may be written in the form (5.161) if we identify B^2 as

$$B^2 = \frac{1}{D}(k_\infty \Sigma_a - \Sigma_a) = \frac{k_\infty - 1}{L^2} \tag{5.215}$$

with L^2 given by (5.75). This statement of B^2 is entirely consistent with (5.126). The definition of k_∞ in this relation is determined by the factors selected for the thermal-source term. Either of the definitions (4.275) or (5.182) may be used, depending upon the accuracy desired in a particular application.

We showed previously that the multiplication constant k for the finite medium could be written in the form (5.183) and identified the factor $(1 + L^2B^2)^{-1}$ as the nonleakage probability for the neutrons based on the one-velocity approximation. In the case that the diffusion equation is used to describe the distribution of thermal neutrons in a reactor, this factor gives the nonleakage probability for the thermals. We demonstrate this relationship by applying two separate lines of reasoning: According to our usual definition of the multiplication constant of a system, we can write

$$k = \frac{n_1}{n_0} \tag{3.16}$$

In our present situation, n_1 and n_0 represent the total neutron populations in the first and second generations for a finite system. Thus if n_0 neutrons are absorbed by a finite medium, these neutrons produce a certain number of fissions which yield further neutrons. Not all these second-generation neutrons, however, are available to the chain reaction; a certain fraction leak out through the boundaries. Therefore, if we call n_1^∞ the number of neutrons produced by the reactions because of the n_0 neutrons if the system had been infinite and let p_{NL} be the nonleakage probability for a neutron in the finite system, then it follows that $n_1^\infty p_{NL}$ is the number of second-generation neutrons available to the finite system for maintaining the chain. Thus, $n_1 = n_1^\infty p_{NL}$, and

$$k = \frac{n_1^\infty p_{NL}}{n_0} = k_\infty p_{NL} \tag{5.216}$$

By comparison of this equation with (5.183), we conclude that

$$p_{NL} \equiv \frac{1}{1 + L^2B^2} \tag{5.217}\star$$

We can derive this result by an alternate approach. By definition, the nonleakage probability must be given by the fraction of neutrons absorbed in the system. Let us compute, therefore, the ratio of the total number of neutrons absorbed per unit time in the reactor to the total

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number produced per unit time. We write this quantity in the form

$$p_{\text{NL}} = \frac{\int_V \Sigma_a \phi(\mathbf{r})\, d\mathbf{r}}{\int_V S(\mathbf{r})\, d\mathbf{r}} \tag{5.218}$$

where the indicated integrations extend over the volume of the reactor. If we use for $S(\mathbf{r})$ the equation (5.22) and recognize that the cross sections are independent of the spatial coordinates for a homogeneous system, then (5.218) reduces to [where we have used (5.161)]

$$p_{\text{NL}} = \frac{\Sigma_a \int_V \phi(\mathbf{r})\, d\mathbf{r}}{\Sigma_a \int_V \phi(\mathbf{r})\, d\mathbf{r} + DB^2 \int_V \phi(\mathbf{r})\, d\mathbf{r}} = \frac{1}{1 + L^2B^2} \tag{5.219}$$

We may further conclude from these results that the probability that a neutron does escape from the reactor, i.e., the *leakage probability* p_L, is given by

$$p_L = 1 - p_{\text{NL}} = \frac{L^2B^2}{1 + L^2B^2} \tag{5.220}$$

These results will be of frequent use to us in later analyses. A simple application which may be demonstrated here is the calculation of the *lifetime* of a thermal neutron in a finite reactor. Before computing this quantity, consider first the average time spent by a thermal neutron in an infinite medium before it is absorbed. If we call this number l_0, then we may define

$$l_0 = \frac{\text{mean distance traveled by neutron to absorption}}{\text{mean speed}} = \frac{\lambda_a^{\text{th}}}{v_{\text{th}}} = \frac{1}{\Sigma_a^{\text{th}} v_{\text{th}}} \tag{5.221}$$

where $\lambda_a^{\text{th}} = 1/\Sigma_a^{\text{th}}$ is the absorption mean free path for a thermal neutron and v_{th} is the speed of a thermal neutron [see Eq. (4.253) and discussion]. In a finite medium the mean lifetime l of a thermal neutron is given by

$$l = \frac{\text{mean distance traveled by neutron before it is absorbed } \textit{or} \text{ escapes from the system}}{\text{mean speed}} = \frac{\lambda}{v_{\text{th}}} = \frac{1}{(\Sigma_a^{\text{th}} + \Sigma_L^{\text{th}}) v_{\text{th}}} \tag{5.222}$$

We identify the cross section for escape Σ_L^{th} (i.e., the number of escapes per unit distance traveled) from the relation (5.175). Evidently, $\Sigma_L^{\text{th}} \equiv DB^2$. This concept of DB^2 as the expected number of escapes per unit path is applicable only to finite single-medium systems. The ratio $\Sigma_L^{\text{th}}/\Sigma_a^{\text{th}}$ is obtained by computing the total number of escapes and absorptions in the system by means of the method used for determining

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p_{NL} [see also (5.220)]. With this expression for Σ_L^{th}, (5.222) reduces to

$$l = \frac{1}{(\Sigma_a^{th} + DB^2)v_{th}} = \frac{1}{\Sigma_a^{th} v_{th}(1 + L^2B^2)} = l_0 p_{NL} \qquad (5.223)$$

with the aid of (5.221). Note that $l < l_0$; that is, the mean lifetime of a thermal neutron in a finite system is less than the lifetime in an infinite system, and this is due to the fact that neutrons are lost to the system by escapes, as well as by absorptions.

5.5 Diffusion Length

Two physical parameters required in every application of the one-velocity model are the diffusion coefficient D and the diffusion length L. So far, our interest in the one-velocity model has been confined to descriptions of the spatial distribution of thermal neutrons, and these parameters have therefore been defined in terms of the thermal cross sections. We will show later that similar quantities can be defined in the treatment of multivelocity systems by the energy-group method, or for that matter, in any system wherein it is convenient to describe the "diffusion properties" of a particular group of neutrons in terms of a single speed.

The diffusion coefficient and the diffusion length are fundamental macroscopic properties of a material which are useful in the one-velocity formulation. Both quantities can be measured directly in the laboratory by suitable experiments. The direct measurement of the diffusion coefficient, however, entails the use of a pulsed beam of neutrons. Inasmuch as this experiment involves a time-dependent phenomenon, a discussion of the experiment will be deferred until after a suitable model has been developed for the analysis of nonstationary problems in neutron diffusion. An experiment for the direct determination of the diffusion length, however, is based on a steady-state phenomenon, and the important features of this experiment can be displayed by means of the models and concepts already developed. Because of the close relationship between the parameters L and D, it would be desirable to examine the techniques for their measurement simultaneously. This is not possible because of the complications mentioned above; thus for the present we confine our attention to the study of the diffusion length and an experiment for its measurement.

It should be noted that a separate measurement of both D and L is not always required. Because of the relationship $L^2 = D/\Sigma_a$, it is clear that a knowledge of one parameter yields the other, provided of course that Σ_a is known. This is precisely the situation encountered in the L experiment; thus, in order to determine D we require a separate measurement of Σ_a. The diffusion-coefficient experiment based on the pulsed-neutron-

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beam technique, on the other hand, provides a direct measurement not only of D but also of Σ_a (refer to Sec. 9.2).

a. The Root-mean-square Radial Distance. The elementary problem of neutron diffusion from a plane source in an infinite medium gives some indication of the physical significance of the diffusion-length concept. It was shown in the treatment of Sec. 5.2a that the flux distribution in a nonmultiplying medium had an exponential behavior given by the factor $\exp(-x/L)$. In this form one can identify the diffusion length L as an attenuation, or relaxation, length (see Fig. 5.24*a*). Thus L is the distance from the source plane at which the neutron density is reduced to

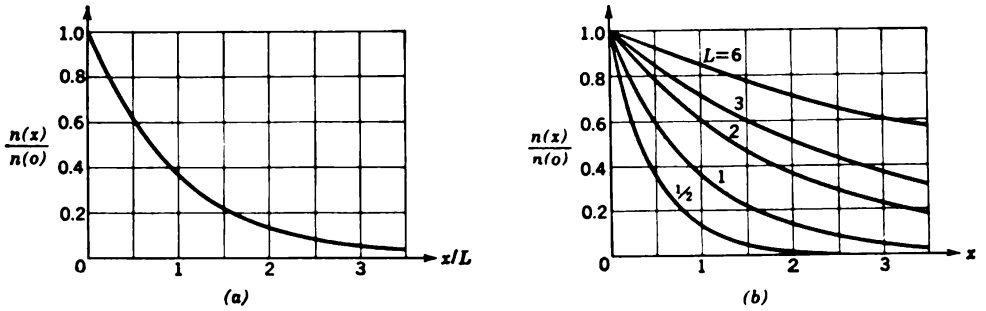

FIG. 5.24 Neutron attenuation in diffusion lengths.

$1/e$ of its value at the source. Each additional distance of multiple L from the source causes an additional reduction by the factor $1/e$ from the previous station. Figure 5.24*b* shows moreover that, at any given distance from the source plane, the neutron density is smaller in those materials which have the smaller value of L. Thus materials of low L (large Σ_a or small D) "attenuate" neutrons more rapidly than materials of large L. One can think of the diffusion length, then, as a measure of the average depth of penetration into a medium by a source neutron. In highly absorbing materials, or in materials of small diffusion coefficient, this penetration tends to be small. It should be noted that a proper measure of the penetration is the "crow-flight" distance from source to point of capture. The connection between the diffusion length and the crow flight, or root-mean-square distance, is now demonstrated.

For this calculation we use the problem of neutron diffusion from a point source in an infinite medium. Similar calculations can be made for the plane and line source, but the basic relationship between L and the crow-flight distance will prove to be the same in all cases. Consider then the diffusion of neutrons in an infinite medium from a point source of strength q_0 neutrons per unit time. For convenience, we place the source at the origin of a suitable coordinate system. As previously noted, this situation has spherical symmetry, and the neutron distribution may be described completely by the radial coordinate r alone. Let us compute now the quantity $\overline{r^2}$ for this system. We define $\overline{r^2}$ as the average square

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radial distance from the origin at which a neutron is absorbed. If $f(r)$ is the appropriate frequency function, that is, if

$$f(r)\,d\mathbf{r} \equiv \text{probability that a neutron is absorbed in volume } d\mathbf{r} \text{ at distance } r \tag{5.224}$$

then we define

$$\overline{r^2} \equiv \int_V r^2 f(r)\,d\mathbf{r} \tag{5.225}$$

The frequency function can be computed from our knowledge of the distribution of absorption reactions throughout the entire medium. Now q_0 is the total number of absorptions per unit time in this system and $\Sigma_a\phi(r)\,d\mathbf{r}$ the absorptions per unit time in $d\mathbf{r}$ at r. Thus the probability of a neutron being absorbed in $d\mathbf{r}$ at r, namely, the function $f(r)\,d\mathbf{r}$, is simply

$$f(r)\,d\mathbf{r} = \frac{\Sigma_a\phi(r)\,d\mathbf{r}}{q_0} \tag{5.226}$$

The neutron flux due to a point source in an infinite nonmultiplying medium was obtained in Sec. 5.2b, and is given by Eq. (5.84). If this result is used in (5.226), it is easily shown that (5.225) yields

$$\overline{r^2} = \frac{\Sigma_a}{q_0}\int_V \frac{q_0 e^{-r/L}\,r\,d\mathbf{r}}{4\pi D} = \frac{\Sigma_a}{D}\int_0^\infty r^3 e^{-r/L}\,dr = 6L^2 \tag{5.227}$$

The diffusion length L is related, therefore, to the mean-square radial distance by the equation

$$L = \left(\frac{\overline{r^2}}{6}\right)^{\frac{1}{2}} \tag{5.228}$$

Although the present treatment is based on the quantity $\overline{r^2}$, a similar calculation can be made for $\bar{r}$, and it can be shown that for a point source

$$L = \tfrac{1}{2}\bar{r} \tag{5.229}$$

The important fact here is that L is a measure of the average value of the radial distance squared at which a neutron is absorbed; thus, as previously indicated, one can think of L as the average penetration into a medium by the source neutrons.

The detailed form of the relations (5.228) and (5.229) (or the corresponding equations for the other source geometries) is of interest only when the value of L for a particular medium is to be obtained by a purely analytical method. For example, the diffusion length L can be computed for a given medium by means of the Monte Carlo technique. In a calculation of this type one selects a suitable neutron source for an infinite medium of the material in question and traces the detailed trajectories of a large number of source neutrons from birth to capture. From these

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histories one can compute either $\overline{r^2}$ or $\bar{r}$ by an appropriate relation of the type (5.228) or (5.229), and thence the value of L.

b. The Diffusion-length Experiment. The diffusion length of thermal neutrons in a particular material can be determined experimentally by measuring the axial flux distribution in a long rectangular parallelepiped at one end of which is placed a source of thermal neutrons. The connection between this flux distribution and the diffusion length may be demonstrated by a direct application of the one-velocity model. Consider therefore the configuration shown in Fig. 5.25. The medium in question

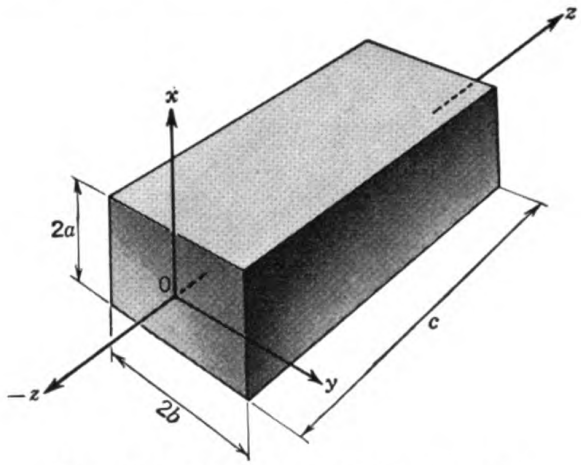

FIG. 5.25 Diffusion-length experiment.

is a parallelepiped of length c and sides $2a$ and $2b$. Thermal neutrons are introduced at $z = 0$ according to some distribution $S(x,y)$. The only requirement on this source is that it emit neutrons isotropically and that it be symmetric about the x and y axes. These requirements are imposed only to simplify the analysis; a similar development to that which follows can be carried out for a more general source distribution.

The neutron sources in the specified system then are all confined to one end face of the block. The appropriate differential equation for describing the spatial distribution of the thermal neutrons inside the medium is therefore

$$-D\nabla^2\phi(x,y,z) + \Sigma_a\phi(x,y,z) = 0 \qquad z > 0 \tag{5.230}$$

The solution to this equation must satisfy the boundary conditions[1]

$$\begin{array}{ll} (1)\ j_+(x,y,0) = \tfrac{1}{2}S(x,y) & (2)\ \phi(\pm\tilde{a},y,z) = 0 \\ (3)\ \phi(x,\pm\tilde{b},z) = 0 & (4)\ \phi(x,y,\tilde{c}) = 0 \\ (5)\ \phi(+x,y,z) = \phi(-x,y,z) & (6)\ \phi(x,+y,z) = \phi(x,-y,z) \end{array} \tag{5.231}$$

We look for a solution of the form $\phi(x,y,z) = F(x)\,G(y)\,H(z)$. The

[1] Conditions (5) and (6) follow from the specification that $S(x,y)$ is symmetric in x and y.

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substitution of this relation into (5.230) yields the set

$$\frac{F''(x)}{F(x)} \equiv -k_1^2 \qquad \frac{G''(y)}{G(y)} \equiv -k_2^2 \qquad \frac{H''(z)}{H(z)} \equiv +k_3^2 \tag{5.232}$$

where

$$-k_1^2 - k_2^2 + k_3^2 = \frac{1}{L^2} \tag{5.233}$$

The algebraic signs of these constants have been selected in anticipation of the proper solution to (5.230). The general solutions to equations (5.232) are

$$F(x) = A_1 \sin k_1 x + C_1 \cos k_1 x \tag{5.234}$$
$$G(y) = A_2 \sin k_2 y + C_2 \cos k_2 y \tag{5.235}$$
$$H(z) = A_3 e^{-k_3 z} + C_3 e^{k_3 z} \tag{5.236}$$

The application of the symmetry condition yields immediately

$$A_1 = A_2 = 0$$

The end condition (4) requires that

$$H(\tilde{c}) = 0 = A_3 e^{-k_3 \tilde{c}} + C_3 e^{k_3 \tilde{c}} \tag{5.237}$$

which yields $C_3 = -A_3 e^{-2k_3 \tilde{c}}$. It follows that $H(z)$ may be written

$$H(z) = A_3 e^{-k_3 z}[1 - e^{-2k_3(\tilde{c}-z)}] \tag{5.238}$$

The extrapolation boundary conditions at the sides require that

$$F(\tilde{a}) = C_1 \cos k_1 \tilde{a} = 0 \qquad \text{with } k_{1n} = \frac{\pi}{2\tilde{a}}(2n+1) \tag{5.239}$$

$$G(\tilde{b}) = C_2 \cos k_2 \tilde{b} = 0 \qquad \text{with } k_{2m} = \frac{\pi}{2\tilde{b}}(2m+1) \tag{5.240}$$

and $m, n = 0, 1, \ldots$. The corresponding eigenfunctions are, therefore,

$$F_n(x) = C_{1n} \cos k_{1n} x \qquad G_m(y) = C_{2m} \cos k_{2m} y \tag{5.241}$$

Now from (5.233), we have that

$$k_3^2 = k_1^2 + k_2^2 + \frac{1}{L^2} \tag{5.242}$$

It follows that, for each set of values of n and m, there is a corresponding value of k_3. It is convenient to define then a number k_{3mn} such that

$$k_3^2 \to k_{3mn}^2 \equiv k_{1n}^2 + k_{2m}^2 + \frac{1}{L^2} \tag{5.243}$$

These results may be used to obtain a solution for $\phi(x,y,z)$. Thus we write

$$\phi(x,y,z) = \sum_{n,m=0}^{\infty} A_{mn} \cos k_{1n} x \cos k_{2m} y \, e^{-k_{3mn} z}[1 - e^{-2k_{3mn}(\tilde{c}-z)}] \tag{5.244}$$

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where we have combined the constants A_3, C_{1n}, and C_{2m} into a single coefficient A_{nm}. The coefficients A_{nm} may be evaluated by the application of the boundary condition (1) and the orthogonality properties of the eigenfunctions $F_n(x)$ and $G_m(y)$. We do not require so detailed a description of the flux for this application, however; for our present purpose it

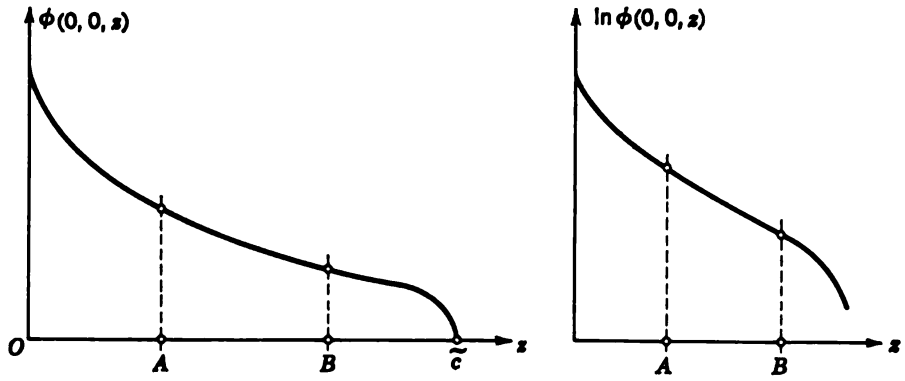

FIG. 5.26 Axial flux distribution: (*a*) linear plot; (*b*) semilog plot.

will suffice to examine the general form of the flux distribution along the central (z) axis. This functional dependence is given by

$$\phi(0,0,z) = \sum_{n,m=0}^{\infty} A_{nm} e^{-k_{3mn}z}[1 - e^{-2k_{3mn}(\tilde{c}-z)}] \tag{5.245}$$

We note however that the quantities k_{3mn} increase rapidly with increasing values of m and n; thus the higher terms of the infinite series diminish more rapidly than the first. Therefore, in first approximation we can take

$$\phi(0,0,z) \simeq A_{00} e^{-k_{300}z}[1 - e^{-2k_{300}(\tilde{c}-z)}] \tag{5.246}$$

The factor in the brackets represents the "end effects" of the parallelepiped. Note that for very long media (large c) this factor approaches unity. Thus the function exp $(-k_{300}z)$, which gives the usual exponential attenuation in a nonmultiplying medium, is combined with a correction factor which accounts for the end effects in the finite medium. However, even in the case of a short medium (c small), the end effect is not significant until the values of z in question approach the magnitude of c, i.e., when $(\tilde{c} - z) \to 0$. Only then is the end effect important. Figure 5.26*a* shows a typical graph of the approximate function given in (5.246). It is evident from (5.246) that there exists some intermediate range of z sufficiently far removed from both ends of the test medium that the axial flux distribution in this region varies essentially as the exponential exp $(-k_{300}z)$. A region of this sort is indicated by the interval AB in

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Fig. 5.26. Note that we qualify the preceding statement by including both ends of the medium. The immediate vicinity of the face $z = 0$ is also a special region of this medium and, from a practical standpoint, can also introduce certain complications. Some of these will be discussed in the next chapter. Suffice it to say for the present that the flux distribution near the face $z = 0$ involves the higher harmonics. Thus both ends of the medium are poorly described by the simple relation (5.246) and are to be avoided in the experiment when possible. The particular importance of a region such as AB to the diffusion-length experiment is due to the especially simple form of the axial flux distribution in this range. We have noted that in this range the flux distribution is exponential, and the exponent includes the constant k_{300} which is a function of the diffusion length L. Thus a knowledge of k_{300} yields the value of L.

A measurement of the quantity k_{300} is obtained from a knowledge of the actual flux distribution in the test specimen. The usual experimental procedure is to obtain a measure of the *relative* flux at various stations along the axis of the parallelepiped by means of metallic foils. The activation of these foils due to neutron captures is directly proportional to the flux level at the foil. Thus a plot of the foil activities as a function of position along the axis will yield a curve having the spatial form of the axial thermal flux. Moreover, if this curve is drawn on semilog paper, it will display a linear behavior in the intermediate range AB, as shown in Fig. 5.26*b*. The slope of this curve yields the quantity k_{300}. But, from (5.243) we have that

$$k_{300}^2 = k_{10}^2 + k_{20}^2 + \frac{1}{L^2} \tag{5.247}$$

and from (5.239) and (5.240)

$$k_{300}^2 = \left(\frac{\pi}{2\tilde{a}}\right)^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \frac{1}{L^2} \tag{5.248}$$

The quantities $\tilde{a}$ and $\tilde{b}$ are known from the size of the specimen and k_{300} from the slope; thus, (5.248) yields the diffusion length L.

In a practical situation it may not be possible to obtain a purely exponential region such as AB of Fig. 5.26. The specimen may be too short for such a region to exist. In this case the end effects on the flux distribution can be removed by recognizing that these are confined to the quantity in the square bracket. Thus flux measurements toward the outer end of the specimen can be analyzed by means of (5.246). If because of some practical limitation the outer end of the medium is not accessible and the flux measurements must be made near the source plane, then the end effect might well be negligible; however, the one-term expansion for $\phi(0,0,z)$ will no longer be acceptable and some higher terms in the series must be retained. In this case it will be necessary to find the best fit to

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the experimental data with a series of exponentials. Thus, for example, one might use

$$\phi(0,0,z) \simeq A_{00}e^{-k_{00}z} + A_{10}e^{-k_{10}z} + A_{01}e^{-k_{01}z} \tag{5.249}$$

The fit to the data is made by adjusting the coefficients A_{nm}.

5.6 Sample Reactor Calculation with One-velocity Model

A sample reactor calculation is presented in this section to demonstrate some of the ideas developed in connection with the one-velocity model. It must be emphasized that this computation, which is based on the one-velocity approximation, cannot be expected to yield accurate results. This model, in its present form, has severe limitations, and some of these were discussed in Sec. 5.4g. The most serious deficiency of the model is that it fails to account for neutron losses during the slowing-down process. Even if the reactor in question is a highly thermal system, the one-velocity model in its present form is inadequate for describing the neutron physics. The difficulty is due to the form of the source term used in the diffusion equation. It was pointed out in Sec. 5.4g that this term should properly include certain factors (ϵ and p) to account for some of the processes which can affect the neutron population at thermal energy through their influence on the slowing-down process. But these factors alone do not suffice. There is still a third factor which must be introduced to describe the losses in the neutron population due to leakages during slowing down. This factor is especially important in the case of small reactors wherein the surface-to-volume ratio of the system is relatively large.

At this stage we are not yet prepared to take all these factors into account. Thus the example which follows is not suggested as a proper detailed analysis of a reactor. Its purpose is to demonstrate the magnitude of some of the numbers which may be involved in calculations of this type and to show a general computational procedure which may be used. The calculations of the macroscopic cross sections, however, are properly carried out and should be useful in showing the application of the ideas developed in Chap. 4. Recognizing these deficiencies, we make no claim that these calculations are realistic and therefore can conveniently introduce further approximations which will simplify the computation. The most important of these is the assumption that the source term in the diffusion equation is of the form $\nu\Sigma_f\phi$; thus, in general, we will ignore all "fast losses."

As an example of the application of the one-velocity model to the analysis of nuclear reactors, we will determine the critical fuel mass and maximum flux level in a light-water–moderated reactor which is designed to produce 20 megawatts of heat. Let the core of this reactor be a 60- by 32.3- by 55.4-cm parallelepiped, and for the present analysis assume

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that no reflector is provided; thus the reactor from a purely nuclear viewpoint is a simple bare block geometry of the type studied in Sec. 5.4e. Assume further that there are three basic ingredients in the core, namely, water for moderation, aluminum for structure, and U^{235} for nuclear fuel. We specify that the fuel elements in this system consist of flat plates of aluminum alloyed with U^{235}. Each fuel element is made up of 18 such plates, spaced 0.297 cm apart, and the core contains 28 of these elements, or subassemblies. The light-water moderator occupies the gap between these plates and by the process of natural convection serves in the dual capacity as the coolant for the system. Figure 5.27 shows the core configuration for this system and the arrangement of the fuel elements. Figure 5.28 gives a cross-sectional view of a typical fuel element and shows how the 18 fuel plates in each subassembly are mounted into a single unit by means of side plates.

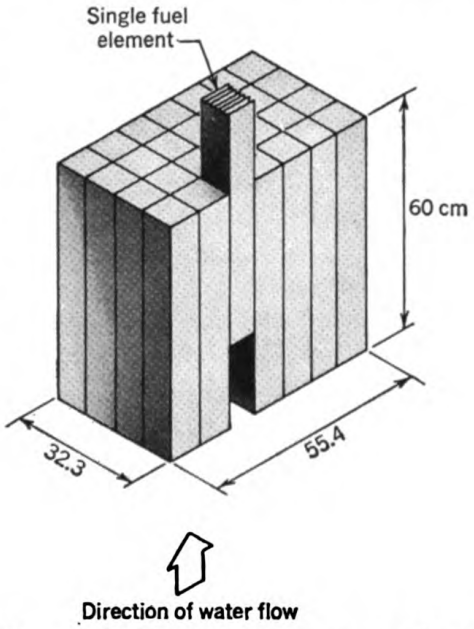

FIG. 5.27 Core configuration of 4 by 7 fuel-element array.

In this calculation we confine our attention to a study of the nuclear features of the proposed system. The only engineering specifications that we require are the total power output of the system, its geometric configuration, and the operating temperature. The core configuration shown in Fig. 5.27 is contained in a suitable tank; we will ignore for the present the effect of this container on the nuclear characteristics of the system. The core and its container are designed to allow water to enter at the bottom of the core and flow upward through the fuel elements in a direction parallel to the fuel plates. We specify the entrant temperature of the water to be 130°F and its exit temperature as 145°F. For our present needs it will suffice to assume that we can use some average temperature, say, 140°F, to describe the complete system. Our nuclear calculations will be based on this figure.

The fuel-element configuration shown in Fig. 5.28 is clearly a heterogeneous arrangement; however, because of the relatively small scale[1] of this heterogeneity, it is a reasonable approximation to assume, at least as a first estimate, that the core consists of a *homogeneous* mixture of water, aluminum, and U^{235}. We will use this approximation in the calculations

[1] The determination of the scale of heterogeneity is discussed in Chap. 10

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which follow. Our study, then, will be based on a bare homogeneous multiplying medium of the configuration shown in Fig. 5.27. The principal specifications and assumptions are as follows: (1) power, 20 megawatts; (2) temperature, 140°F; (3) configuration, bare homogeneous parallelepiped 60 by 32.3 by 55.4 cm; (4) one-velocity model.

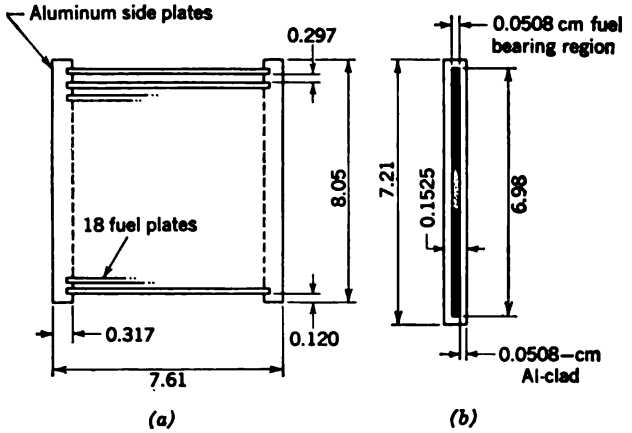

FIG. 5.28 Cross section of fuel element: (*a*) fuel element; (*b*) single fuel plate.

We begin the computation by determining the volume fractions of the principal ingredients of the system. The selected core geometry is uniform in the vertical direction; therefore, a comparison of the cross-sectional areas of its constituents will yield these volume fractions. From the dimensions given in Fig. 5.28 we obtain

1. Cross-sectional area of metal per element:
   a. Aluminum end plates: area = (8.05)(0.317)(2) = 5.10 cm²
   b. Fuel plates: area = (18)(0.1525)(6.98) = 19.2 cm²
2. Cross-sectional area of water per element:
   (6.98)(8.05) − 19.2 = 37.0 cm²
   Total: 61.3 cm²

The appropriate volume fractions are:

$$v_{\text{metal}} = 0.396 \qquad v_{H_2O} = 0.604 \qquad \frac{v_{\text{metal}}}{v_{H_2O}} = 0.656$$

The next step in the calculation is the determination of the thermal-group properties of the core constituents based on the effective neutron temperature T_n [see Eq. (4.211)]. An estimate of this temperature requires, however, a value of the parameter κ which involves the absorption cross section of the mixture. At this stage of the analysis we can only guess at this number inasmuch as the value of Σ_a for the core mixture is a function of the fuel concentration, and this has not as yet been determined. Evidently, the proper procedure to be adopted is an iteration. Thus one should begin with a rough estimate of κ and later check this value on the basis of the resulting critical fuel concentration which is obtained for the

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system. In the present calculation we will assume that $\kappa \equiv 0$, arguing that for a mixture containing large amounts of hydrogen Σ_s will be relatively large, and furthermore we expect that the required fuel concentration will be low so that Σ_a will be small. Both of these effects yield small values of κ [see Eq. (4.116)]. The validity of the assumption $\kappa \equiv 0$ will be examined later.

In the case that $\kappa \equiv 0$, the effective neutron temperature $T_n \equiv T_N$, the moderator temperature. We will base our estimates of the nuclear properties, then, on a neutron temperature of 140°F (333°K). Consider first the water. At 20°C its properties are $\sigma_a = 0.66$ barn, $\lambda_{tr} = 0.48$ cm, $L = 2.85$ cm, and $D = 0.16$ cm (see Tables 2.2 and 5.1). For an effective neutron temperature $T_n = 333$°K, these yield for the thermal absorption cross section[1] $\bar{\sigma}_a$ [see Eq. (4.245)].

$$\bar{\sigma}_a^{(H_2O)}(333) = \sigma_a(T_*)\left(\frac{\pi T_*}{4T_n}\right)^{\frac{1}{2}} = 0.66\left[\frac{\pi(293)}{(4)(333)}\right]^{\frac{1}{2}} = 0.549 \text{ barn}$$

wherein we have tacitly assumed that the absorption cross section for water is $1/v$. For convenience, we will extend this assumption to the aluminum as well. The macroscopic absorption cross section for pure water based on the above figure is[2]

$$\Sigma_a^{(H_2O)}(333) = \bar{\sigma}_a^{(H_2O)}(333)\rho(333)\mathcal{A}/A_{H_2O} = (0.549)(0.985)(0.6025)/18 = 0.0181 \text{ cm}^{-1}$$

Note that the mass density of pure water at 140°F (333°K) has been used as the average value for the reactor. The transport cross section, which is required for the calculation of the diffusion coefficient for the mixture, is found to be

$$\Sigma_{tr}^{(H_2O)}(333) = \frac{\Sigma_{tr}^{(H_2O)}(293)\rho(333)}{\rho(293)} = \frac{0.985}{(0.48)(1)} = 2.05 \text{ cm}^{-1}$$

In this calculation we have assumed that the transport cross section is affected by changes in temperature only through the effect on the density of the material.

The microscopic cross sections for U^{235} and aluminum are determined next. For U^{235}, we have from Eq. (4.247)

$$\bar{\sigma}_a(333) = \left(\frac{\pi}{4}\right)^{\frac{1}{2}}\sigma_a(333)f_a(333) = (0.886)(650)(0.974) = 561 \text{ barns}$$

$$\bar{\sigma}_f(333) = \left(\frac{\pi}{4}\right)^{\frac{1}{2}}\sigma_f(333)f_f(333) = (0.886)(540)(0.976) = 467 \text{ barns}$$

$$\sigma_s = 10 \text{ barns} \qquad \eta = 2.08 \qquad \nu = 2.46$$

[1] In order not to complicate the notation, we have used $\bar{\sigma}$ to represent the thermal-group properties usually given by σ^{th}.

[2] See, for example, U.S. Atomic Energy Commission, "Reactor Handbook," vol. 2, p. 39, AECD-3646, May, 1955, for density of water as a function of temperature.

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The f function, which corrects for the non-$1/v$ part of the cross section, is obtained from Fig. 4.30. The appropriate thermal-group data for aluminum are (refer to Table 2.2)

$$\overline{\sigma_a}(333) = \sigma_a(293)\left[\frac{\pi 293}{(4)(333)}\right]^{\frac{1}{2}} = (0.23)(0.831) = 0.191 \text{ barn}$$
$$\sigma_s = 1.4 \text{ barns}$$

These data may now be used to compute the macroscopic nuclear properties of the core constituents in terms of the volume fractions with which they appear in the core mixture (namely, $v_{H_2O} = 0.604$ and $v_{metal} = 0.396$). In the case of the water

$$\Sigma_a^{(H_2O)} = (0.0181)(0.604) = 0.01092 \text{ cm}^{-1}$$
$$\Sigma_{tr}^{(H_2O)} = (2.05)(0.604) = 1.24 \text{ cm}^{-1}$$

And for the aluminum (where we assume that in first approximation $v_F = 0$)

$$\Sigma_a^{(Al)} = N_{Al}\overline{\sigma_a}^{(Al)} = \frac{\rho_{Al}\mathcal{A}\overline{\sigma_a}^{(Al)}}{A_{Al}} = \frac{(2.7)(0.396)(0.6025)(0.191)}{27}$$
$$= 0.00456 \text{ cm}^{-1}$$
$$\Sigma_{tr}^{(Al)} \simeq \Sigma_s^{(Al)} = \frac{(2.7)(0.396)(0.6025)(1.4)}{27} = 0.0334 \text{ cm}^{-1}$$

The transport cross sections for the water and aluminum may be used to determine the diffusion coefficient of the core mixture. This is an acceptable approximation in the event that the fuel concentration is relatively low; this is the case here. Thus we compute for the mixture

$$D = \frac{1}{3\Sigma_{tr}} \simeq \frac{1}{3[\Sigma_{tr}^{(H_2O)} + \Sigma_{tr}^{(Al)}]} = \frac{1}{(3)(1.24 + 0.0334)} = 0.262 \text{ cm}$$

The critical fuel mass is computed from the criticality condition for the one-velocity model. The appropriate relation is (5.181) with $k = 1$. If we solve this equation for $\Sigma_a^{(F)}$, the macroscopic absorption cross section of the U^{235}, we obtain [cf. (5.185)]

$$\Sigma_a^{(F)} = \frac{\Sigma_a^{(O)} + DB^2}{\eta - 1}$$

where $\Sigma_a^{(O)}$ denotes the absorption cross section for the nonfuel components in the mixture, the aluminum and water. The quantity B^2 is obtained from (5.194). If we neglect the extrapolation dimensions for all three surfaces (i.e., if we use $\tilde{a} \simeq a$, for example), then for $a = 27.7$, $b = 16.15$, and $c = 30$ cm,

$$B^2 \simeq \left(\frac{\pi}{2a}\right)^2 + \left(\frac{\pi}{2b}\right)^2 + \left(\frac{\pi}{2c}\right)^2 = 0.0152$$

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If we introduce this result, along with $D = 0.262$ cm, $\Sigma_a^{(O)} = 0.0155$ cm^{-1}, and $\eta = 2.08$ into (5.185), we obtain

$$\Sigma_a^{(F)} = \frac{0.0155 + (0.262)(0.0152)}{1.08} = 0.01808 \text{ cm}^{-1}$$

The corresponding value of the critical fuel mass M_F is obtained from the relation (where $V = 107{,}000$ cm^3, the volume of the core)

$$M_F = V\rho_F = \frac{V\Sigma_a^{(F)}A_F}{\mathcal{A}\bar{\sigma}_a^{(F)}} = \frac{(107{,}000)(0.01808)(235)}{(0.6025)(561)} = 1.346 \text{ kg} \qquad (5.250)$$

At this point we interrupt the calculation to check our assumed value for the effective neutron temperature, using the absorption cross sections computed above. The expression for κ required for this calculation is given by

$$\kappa = \left(\frac{2T_*}{3T_N}\right)^{\frac{1}{2}} \frac{\Sigma_a(T_*)}{\Sigma_s} \qquad (4.116)$$

where it will be recalled that $T_* \equiv 293°$K, Σ_a corresponds to the 2,200 m/sec value ($T_* = 293°$K), and Σ_s refers to the high-energy value of the scattering cross section. It should be recognized that the above relation for κ was based on a medium consisting of a single nuclear species and therefore not properly applicable to the present situation. However, in the absence of more complete information it will be adequate for making an estimate of T_n. We argue, furthermore, that the concept of an effective neutron temperature is merely a useful tool for estimating the distributions of neutrons in the low-energy range and should not be taken literally. Also, Eq. (4.211) for T_n is only an approximate relation, and therefore a crude estimate for κ should suffice. For this purpose we use the following data:

$$\Sigma_a(293) = \Sigma_a^{(H_2O)}(293) + \Sigma_a^{(Al)}(293) + \Sigma_a^{(F)}(293)$$

where

$$\Sigma_a^{(H_2O)}(293) = \frac{\sigma_a^{(H_2O)}(293)\rho(333)\mathcal{A}}{A_{H_2O}} = \frac{(0.66)(0.985)(0.604)(0.6025)}{18} = 0.01314 \text{ cm}^{-1}$$

$$\Sigma_a^{(Al)}(293) = \frac{(0.23)(2.7)(0.396)(0.6025)}{27} = 0.00549 \text{ cm}^{-1}$$

$$\Sigma_a^{(F)}(293) = \frac{(0.01808)\sigma_a^{(F)}(293)}{\bar{\sigma}_a^{(F)}(333)} = \frac{(0.01808)(687)}{561} = 0.0222 \text{ cm}^{-1}$$

thus $\Sigma_a(293) = 0.0408$ cm^{-1}. For the scattering cross section in (4.116) we will use the value for hydrogen at the high-energy range,

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$\sigma_s^{(H)} = 20.36$ barns. With the volume fraction $v_{H_2O} = 0.604$, this gives

$$\Sigma_s = N_H \sigma_s^{(H)} = \frac{(2)(0.985)(0.604)(0.6025)(20.36)}{18} = 0.811 \text{ cm}^{-1}$$

Thus
$$\kappa = \left[\frac{(2)(293)}{(3)(333)}\right]^{\frac{1}{2}} \left(\frac{0.0408}{0.811}\right) = 0.0386$$

and (using $A = 1$)

$$T_n = T_N(1 + 1.1A\kappa) = 347°\text{K}$$

Thus our initial assumption was somewhat in error.

We compute, next, the maximum neutron flux at the center of the proposed reactor during operation at maximum power. We begin with an expression for the total reactor power P in terms of an average thermal flux $\bar{\phi}$.

$$P = \varepsilon \Sigma_f \bar{\phi} V \tag{5.251}$$

where $\varepsilon = 3.17 \times 10^{-11}$ watt-sec is the energy release per fission.[1] The average flux is obtained from the relation

$$\bar{\phi} = \frac{1}{V} \int \phi(x,y,z)\, dx\, dy\, dz \tag{5.252}$$

If we use for $\phi(x,y,z)$ the result (5.193), and assume that $\tilde{a} \simeq a$, etc., then it is easy to show that

$$\bar{\phi} = \frac{\mathfrak{C}}{V} \left(\frac{4}{\pi}\right)^3 abc \tag{5.253}$$

The substitution of this result into the power relation yields

$$\mathfrak{C} = \frac{P(\pi/4)^3}{\varepsilon \Sigma_f abc} \tag{5.254}$$

For $P = 20$ megawatts, and

$$\Sigma_f = \Sigma_a^{(F)} \left(\frac{\sigma_{ff}}{\sigma_{af a}}\right)_F = 0.01808 \frac{(540)(0.976)}{(650)(0.974)} = 0.01505 \text{ cm}^{-1}$$

we have

$$\mathfrak{C} = \frac{(2)(10^7)(\pi/4)^3}{(3.17)(10^{-11})(0.01505)(13{,}420)} = 1.513 \times 10^{15} \text{ neutrons/cm}^2\text{/sec}$$

which is a close approximation to the flux at the center of the core $\phi(0,0,0)$ [see Eq. (5.193)].

In the event that this reactor were used as a research instrument, it would be of interest to determine the neutron current at the outer surfaces of the system. Presumably, in such an application, the test specimen or sample would be placed as near to the core as possible so as to expose it to the highest neutron density. One can estimate the neutron current at

[1] Based on a figure of 197 Mev per fission (see Table 1.1).

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the outer surface of the core by means of the diffusion-theory model. The appropriate relation is (5.48). In the present case, this may be written as

$$j_+(a,0,0) = \frac{1}{4}\phi(a,0,0) - \frac{D}{2}\left[\frac{\partial}{\partial x}\phi(x,y,z)\right]_{a,0,0} \tag{5.255}$$

wherein we have selected the center of the $x = a$ face of the core as the test point. Note that the above expression yields the maximum current for any point on that face. If (5.193) is used in this relation, it may be shown that

$$j_+(a,0,0) = \mathcal{C}\left(\frac{1}{4}\cos\frac{\pi a}{2\tilde{a}} + \frac{D\pi}{4\tilde{a}}\sin\frac{\pi a}{2\tilde{a}}\right)$$

We compute $\tilde{a}$, using (5.71). This yields for the extrapolation distance $0.7104\lambda_{tr} = (0.7104)3D = 0.558$ and $\tilde{a} = 28.26$ cm. If we use for $\mathcal{C} = 1.513 \times 10^{15}$ neutrons/cm^2/sec, then

$$j_+(a,0,0) = 2.27 \times 10^{13} \text{ neutrons/cm}^2\text{/sec}$$

It should be noted in closing that the flux level and current obtained in this sample calculation are somewhat large. Better estimates will be obtained in later examples which include the effects of fast-neutron losses.

5.7 Applications of the Kernel Method

The diffusion of neutrons from elementary sources was discussed in Secs. 5.2 and 5.3. The direct application of these results to *thermal* neutrons is acceptable in many situations of practical importance; however, in some cases wherein the actual source yields fast neutrons, the source term used in the one-velocity equation for the thermal neutrons must be altered to account for the behavior of the fast neutrons prior to reaching thermal energy. Some aspects of this problem were discussed in Sec. 5.4g in connection with the selection of a suitable fission-source term for the diffusion equation [see Eq. (5.211)]. In this latter application it was observed that fission neutrons appear with large kinetic energies and that during the subsequent slowing-down process some of these fast neutrons are lost to the system by resonance capture and leakage. The discussion of Sec. 5.4g included a few remarks about estimating some of these fast losses, and it was noted that a more complete study of this aspect of the problem would be considered in Chap. 6.

a. First-scattering Source. Another feature of the fast-neutron source which must be given careful consideration is the *first flight*. When a fast neutron first appears in a medium it travels outward from its source point until it makes a collision. At relatively high energies the absorption cross section is small (varies as $1/v$) so that this collision, as well as a number of those following, is likely to be a scattering event. In many

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instances, the largest portion of the total distance traveled by a neutron during slowing down is due to the first flight. Although on the average many additional scatterings are experienced by the neutron, these scatterings occur within a short distance of the point of first collision. As a result, the neutron reaches thermal energy in the vicinity of the first-collision point. We can draw, then, the following crude picture of the slowing-down process: a first long flight which accounts for most of the distance traveled in the fast range, followed by thermalization at the point of the first-scattering collision.

This rough model of the slowing-down process may be used as a first approximation in accounting for some of the wanderings experienced by a fast neutron prior to reaching thermal energy. It is clear from the discussion above that in this model the source strength of the one-velocity thermal neutrons at any given point in the medium is simply the first-scattering-collision density of the fast neutrons at that point. This behavior is quite different from that due to the point source studied in Sec. 5.2b. In the present case a fast source neutron does not join the thermal population until it has suffered a first-scattering collision, and it then does so instantly. The analysis of Sec. 5.2b, on the other hand, assumes that all neutrons which appear from the source are already at the one-velocity of diffusion (i.e., are thermal) and enter into the diffusion process immediately. It is to be expected therefore that the thermal flux distribution due to the present model will be more "spread out" (i.e., it will have larger first and second moments) than the flux given by Eq. (5.84). Let us now determine this effect.

Consider, therefore, an isotropic point source of fast neutrons in an infinite medium. If Σ_s denotes the macroscopic scattering cross section of the fast source neutrons, and the source releases q_0 neutrons per unit time, then the density of first-scattering collisions at radial distance r from the source is given by

$$\begin{bmatrix}\text{Number of neutrons}\\ \text{available at surface}\\ \text{of sphere of radius } r\end{bmatrix}\begin{bmatrix}\text{probability that first}\\ \text{scattering occurs in}\\ \text{shell of radius } r \text{ and}\\ \text{thickness unity}\end{bmatrix} = \left(\frac{q_0}{4\pi r^2}\right)(\Sigma_s e^{-\Sigma_s r}) \tag{5.256}$$

We use this quantity as the source term in the diffusion equation; thus, for a spherically symmetric system,

$$-D\nabla^2\phi(r) + \Sigma_a\phi(r) = \frac{q_0\Sigma_s e^{-\Sigma_s r}}{4\pi r^2} \tag{5.257}$$

The solution to this equation may be obtained by the application of the kernel method. The general solution is given by (5.97). In the present case the source distribution $S(\mathbf{r})$ is given by the right-hand term of

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(5.257); it follows, therefore, that

$$\phi(r) = \frac{q_0\Sigma_s}{D(4\pi)^2}\int_\infty \frac{e^{-\Sigma_s\rho-\kappa|\mathbf{r}-\boldsymbol{\rho}|}\,d\boldsymbol{\rho}}{\rho^2|\mathbf{r}-\boldsymbol{\rho}|} \tag{5.258}$$

with $\kappa^2 \equiv \Sigma_a/D$. The integration is readily carried out by choosing a coordinate system, as shown in Fig. 5.29, where the x axis is taken along the field vector $\mathbf{r}$; then,

$$\phi(r) = \frac{q_0\Sigma_s}{D(4\pi)^2}\int_0^\infty e^{-\Sigma_s\rho}\,d\rho \times \int_0^\pi \frac{\exp\left[-\kappa(r^2+\rho^2-2r\rho\cos\theta)^{\frac{1}{2}}\right]\sin\theta\,d\theta}{(r^2+\rho^2-2r\rho\cos\theta)^{\frac{1}{2}}}\int_0^{2\pi} d\psi$$

and

$$\phi(r) = \frac{q_0\Sigma_s}{4\pi D\kappa r}\left[e^{-\kappa r}\int_0^r \frac{\sinh\kappa\rho\, e^{-\Sigma_s\rho}\,d\rho}{\rho} + \sinh\kappa r\int_r^\infty \frac{e^{-(\Sigma_s+\kappa)\rho}\,d\rho}{\rho}\right] \tag{5.259}$$

It is difficult to compare (5.259) with (5.84) directly. However, a measure of the relative spread of the two functions may be obtained by comparing in the two cases the mean square distance from the source point to the point of absorption (as thermal neutrons); that is, we calculate in the two cases

$$\overline{r^2} \equiv \frac{\int_0^\infty r^2\Sigma_a\phi(r)4\pi r^2\,dr}{\int_0^\infty \Sigma_a\phi(r)4\pi r^2\,dr} \tag{5.260}$$

If we use for $\phi(r)$ the result (5.84) (thermal neutrons appear directly at the point source), (5.260) yields for $\overline{r^2}$

$$\overline{r^2} = 6L^2 \tag{5.227}$$

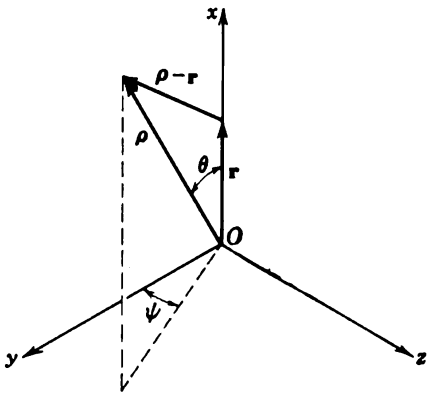

FIG. 5.29 Coordinate system.

For the flux given in (5.259) we compute, on the other hand,

$$\overline{r^2} = \frac{2/\kappa\Sigma_s^3 + 6/\kappa^3\Sigma_s}{1/\kappa\Sigma_s} = 2\lambda_s^2 + 6L^2 \tag{5.261}$$

where $\lambda_s \equiv 1/\Sigma_s$, the scattering mean free path for the fast source neutrons. Thus the mean square distance to absorption in the case of the first-scattering source exceeds that for the point source of Eq. (5.84) by the term $2\lambda_s^2$.

b. Gaussian Source. Although the concept of a first-scattering source described above is useful for a number of situations in reactor physics, it

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does not provide a very satisfactory treatment of the slowing-down process. A superior model is developed in Chap. 6. This model too has limitations, but for many reactor applications it will give adequate results. In order to provide a more complete treatment of the thermal-neutron source problem considered in this section, we will use some of the results of Chap. 6 in our analysis and leave until later the derivation of the appropriate relations.

We recognize, then, that the assumption that a fast neutron is thermalized very near the point at which it suffers its first scattering collision is extremely crude. As noted previously, a neutron may experience considerable additional travel during later stages of slowing down as it passes from one scattering collision to the next. For a medium of the heavier nuclei, the total distance traversed by a fast neutron can in fact become quite large in comparison to the distance it covers as a thermal neutron. The treatment of Chap. 6, which is based on the Fermi age model, shows that this wandering about during slowing down produces a highly spread-out distribution of thermal neutrons. In the case that the source of fast neutrons is a point source, this dispersion of thermal neutrons is given by the relation

$$S(r) = \frac{q_0 p_{\text{th}} e^{-r^2/4\tau_{\text{th}}}}{(4\pi\tau_{\text{th}})^{\frac{3}{2}}} \tag{5.262}$$

where p_{th} is the resonance-escape probability from the source energy to thermal energy and τ_{th} is the *Fermi age* from this source energy to thermal (see Sec. 6.2a). The expression in Eq. (5.262) is developed in Sec. 6.2b.

If we use the relation (5.262) as the source function for the thermal-diffusion equation, then the thermal flux may again be found by using the kernel method. Thus if (5.262) is substituted into (5.97), it can be shown that

$$\phi(r) = \frac{q_0 p_{\text{th}}}{D\kappa r(4\pi\tau_{\text{th}})^{\frac{3}{2}}} \left(e^{-\kappa r} \int_0^r r_0 e^{-r_0^2/4\tau_{\text{th}}} \sinh \kappa r_0 \, dr_0 + \sinh \kappa r \int_r^\infty r_0 e^{-r_0^2/4\tau_{\text{th}} - \kappa r_0} \, dr_0 \right) \tag{5.263}$$

The mean square distance traveled by the neutrons from the source to the point of absorption as thermal neutrons may be obtained by using Eq. (5.260) and the flux given by (5.263); the result for the Gaussian slowing-down source (5.262) is

$$\overline{r^2} = \frac{3\sqrt{\pi}\,(4\tau_{\text{th}})^{\frac{3}{2}}/8\kappa + 6\sqrt{\pi}\,(4\tau_{\text{th}})^{\frac{3}{2}}/4\kappa^3}{\sqrt{\pi}\,(4\tau_{\text{th}})^{\frac{3}{2}}/4\kappa} = 6\tau_{\text{th}} + 6L^2 \tag{5.264}$$

As compared with Eq. (5.227), this result shows the greater spreading out of the neutrons which occurs during the slowing-down process.

c. Isotropic Flux in Slab from First-scattering Collisions. We demonstrate some of the ideas developed above by computing the isotropic flux

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in a slab due to first-scattering collisions from a collimated beam of neutrons. Consider, therefore, an infinite slab of nonmultiplying material one face of which is exposed to a collimated beam of thermal neutrons of strength q_0 neutrons per unit area per unit time. We specify the isotropic flux as the quantity of interest because the effect of the slab is to convert the parallel beam of neutrons into an isotropic distribution. This effect would be most pronounced in slabs of high-nuclear-mass materials because of the fact that a beam neutron scattered by the slab would emerge with a nearly isotropic distribution (in the laboratory system of coordinates). Further scatterings within the slab would increase this tendency; it might be expected, then, that the steady-state distribution of the neutron population could be well described by means of an isotropic flux function.

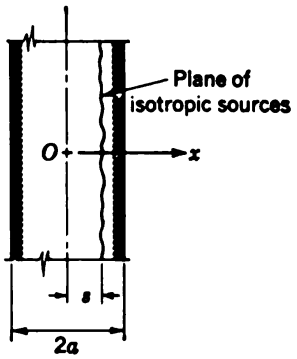

FIG. 5.30 Slab with plane isotropic source.

We begin the analysis by developing a suitable kernel for the specific geometry in question. Consider, therefore, the system shown in Fig. 5.30. The immediate problem is to determine the flux distribution in the slab due to an isotropic plane source of unit strength placed at $x = s$. This result, which is the kernel for the slab problem, may be used directly in the computation of the isotropic flux due to the collimated beam.

The appropriate differential equation and boundary conditions for this system are

$$\phi''(x) - \kappa^2\phi(x) = 0$$

and

$$\begin{aligned} &(1)\ \phi_+(a) = 0 \qquad \phi_-(-a) = 0 \\ &(2)\ J^+(s) = J^-(s) + 1 \\ &(3)\ \phi_+(s) = \phi_-(s) \end{aligned} \tag{5.265}$$

where we have used the approximation $a \simeq \tilde{a}$ for the extrapolated boundary conditions and have denoted by $\phi_+(x)$ the flux in the slab for $x > s$ and by $\phi_-(x)$ the flux for $x < s$. A similar notation is used for the net current J. The current relation (2) follows directly from (5.59), and the flux condition (3) from (5.60). It can be shown that the solution for this system is given by the functions

$$\begin{aligned} \phi_+(x;s) &= \frac{\sinh \kappa(a + s) \sinh \kappa(a - x)}{D\kappa \sinh 2\kappa a} \qquad x > s \\ \phi_-(x;s) &= \frac{\sinh \kappa(a - s) \sinh \kappa(a + x)}{D\kappa \sinh 2\kappa a} \qquad x < s \end{aligned} \tag{5.266}$$

where $\phi(x;s)$ denotes the flux at x due to a plane source of unit strength at $x = s$. The subscripts used in (5.266) refer to the notation mentioned above. These expressions give a complete description of the kernel for the slab geometry throughout the region $-a \leq x \leq a$.

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The application of the result (5.266) to the calculation of the isotropic flux at any point in the slab due to an arbitrary distribution[1] $S(x)$ of isotropic sources is straightforward. In the present case the appropriate integral equation is seen to be [cf. Eq. (5.97)]

$$\phi(x) = \int_{-a}^{a} S(s)\, \phi(x;s)\, ds \tag{5.267}$$

Because the kernel $\phi(x;s)$ has a different form depending on whether $x \gtrless s$, it is convenient to write (5.267) in terms of two integrals; thus,

$$\phi(x) = \int_{-a}^{x} S(s)\, \phi_+(x;s)\, ds + \int_{x}^{a} S(s)\, \phi_-(x;s)\, ds \tag{5.268}$$

In the present situation we do not count the neutrons in the beam as part of the isotropic flux distribution in the slab until they have first experienced a scattering collision. These first collisions convert the beam neutrons into an isotropic distribution, and the resulting flux is given by the diffusion equation. The general solution to this equation is given by (5.268), and the source distribution $S(x)$ which appears is identified as the density of first scatterings in the slab. Thus we define $S(x)\,dx$ as the number of first-scattering collisions per unit time in dx about x. If the beam strikes the slab at face $x = 0$, then

$$\begin{aligned} S(x)\,dx &= \begin{bmatrix} \text{number of neutrons} \\ \text{which enter slab per} \\ \text{unit time and area} \\ \text{of slab surface} \end{bmatrix} \begin{bmatrix} \text{probability that a beam} \\ \text{neutron makes its first} \\ \text{scattering collision in} \\ dx \text{ about } x \end{bmatrix} \\ &= q_0 \frac{\Sigma_s}{\Sigma_t} \left[\Sigma_t e^{-\Sigma_t(x+a)}\, dx\right] = q_0 \Sigma_s e^{-\Sigma_t(x+a)}\, dx \end{aligned} \tag{5.269}$$

where $\Sigma_t \exp\left[-\Sigma_t(x + a)\right] dx$ is the probability [cf. Eq. (2.17) and related discussion] that a first collision occurs in dx about x and Σ_s/Σ_t is the probability that this collision is a scattering event. The substitution of (5.269) into (5.268) and the use of the results (5.266) yield the equation

$$\begin{aligned} \phi(x) = \frac{q_0 \Sigma_s}{D\varkappa \sinh 2\varkappa a} \Bigg[&\sinh \varkappa(a - x) \int_{-a}^{x} e^{-\Sigma_t(s+a)} \sinh \varkappa(s + a)\, ds \\ &+ \sinh \varkappa(a + x) \int_{x}^{a} e^{-\Sigma_t(s+a)} \sinh \varkappa(a - s)\, ds \Bigg] \end{aligned} \tag{5.270}$$

Our purpose in introducing the approximation $a \simeq \tilde{a}$ is now evident. Note that the flux $\phi(x)$ is given as the integral over the physical volume of the system (from $-a$ to a) of all sources S within this volume, whereas properly, the kernel $\phi(x;s)$ should be given in terms of the extrapolated boundary $\tilde{a}$. By using this approximation we are able to write all the pertinent functions in terms of the boundaries $x = \pm a$. Although this

[1] $S(x)$ neutrons per unit area per unit time.

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causes some error, it does simplify the integration and yields more compact results.

The completion of the integrations indicated in Eq. (5.270) yields the solution

$$\phi(x) = \frac{q_0\Sigma_s e^{-\Sigma_t a}}{D(\Sigma_t^2 - \varkappa^2)\sinh 2\varkappa a}\left[e^{\Sigma_t a}\sinh \varkappa(a - x) + e^{-\Sigma_t a}\sinh \varkappa(a + x) - e^{-\Sigma_t x}\sinh 2\varkappa a\right] \quad (5.271)$$

It is of interest to compare this result with an elementary diffusion-theory calculation based on the assumption that the beam neutrons which strike

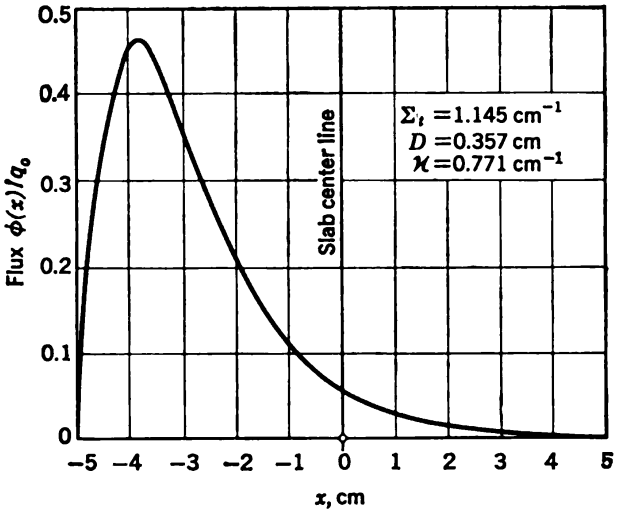

FIG. 5.31 Isotropic flux due to first scattering collisions in iron slab.

the slab enter into a diffusion process immediately at the surface. The resulting flux distribution within the slab is given by the solution to the diffusion equation which satisfies the following boundary conditions:

$$(1)\ j_+(-a) = q_0 \qquad (2)\ \phi(a) = 0 \quad (5.272)$$

This result is

$$\phi(x) = \frac{4q_0\sinh \varkappa(a - x)}{\sinh 2\varkappa a + 2D\varkappa\cosh 2\varkappa a} \quad (5.273)$$

We illustrate the results of this analysis by means of the following numerical example: Consider a 10-cm-thick slab of pure iron for which the microscopic thermal cross sections are taken to be $\sigma_a = 2.53$ barns, and $\sigma_s = 11$ barns, and $\rho = 7.86$ g/cm^3 and $A = 56$. The corresponding macroscopic properties are: $N = 0.0846 \times 10^{24}$ nuclei/cm^3, $\Sigma_s = 0.932$ cm^{-1}, $\Sigma_a = 0.213$ cm^{-1}, $\Sigma_t = 1.145$ cm^{-1}, $\varkappa = 0.771$ cm^{-1}, and $D = 0.357$ cm.

Figure 5.31 shows the isotropic flux distribution $\phi(x)/q_0$ in a 10-cm-thick slab of iron according to the solution (5.271) and using the above data.

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5.8 Problems of Neutron Population in Localized Absorbers

Among the important products of a nuclear reactor are the regions of high neutron density which may be used for a great variety of development testing and fundamental research. The experimental procedure usually involves placing a sample in the neutron field and determining the continuous or integrated change in some property. In the evaluation of the results of such experiments it is usually important to have a quantitative knowledge of some features of the spatial (and energetic) distribution of neutrons inside the sample and in its immediate vicinity. Generally speaking the most important effects are produced by two classes of neutrons: (1) thermal neutrons and (2) very fast neutrons (with kinetic energies of the order of $\frac{1}{2}$ Mev or greater). With respect to the latter group (which produce most of the effects classified under the heading of "radiation damage"), it may be said that the distribution in energy is characteristic of the particular reactor as well as of the specific region of the reactor used. Usually the spectrum of fast neutrons will be little affected by the test sample itself (although auxiliary devices, such as a "converter plate," may be used to affect a major alteration) and, except in the region of absorption resonances, the fast-neutron density may be expected to be fairly uniform over the sample and its immediate vicinity. Thermal neutrons, however, suffer significant variations in density in the region of almost any test sample; this behavior is the subject of discussion for the present section.

In the general case, the conditions of the experiment are poorly amenable to theoretical treatment; the sample will usually be multiregioned and of complex geometry; the site of the experiment will often be a channel in the reactor in which the neutron density varies spatially and in which the streaming of neutrons gives rise to quite different neutron currents incident on the various surfaces of the sample; the immediate surroundings of the sample will usually be a complex mixture of various moderator, structural, coolant, and fuel materials. Consequently, at the very outset, any hope of rigorous treatment of all aspects of the problem is abandoned, and the problem reduces to that of choosing a simplified model, amenable to calculation, which will yield a satisfactory approximation to some feature of the physical situation of particular interest. This section is devoted to a detailed consideration of some of the models and concepts which have been found to be useful.

a. The Distribution of (One-velocity) Neutrons in Absorbing Regions in a Uniform Isotropic Neutron Flux Field. Here we consider an absorber placed in an "isotropic" neutron flux field. By definition, an isotropic flux is such that on an area dS of the surface of the sample there are incident $\frac{1}{2}\phi_0\, dS \cos\theta \sin\theta\, d\theta$ neutrons per unit time whose velocity vectors

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make angles between θ and $\theta + d\theta$ with the normal to the surface; thus [cf. Eq. (5.36)]

$$j(\mu)\, d\mu\, dS \equiv \tfrac{1}{2}\phi_0 \mu\, d\mu\, dS = \begin{array}{l}\text{number of neutrons incident per unit time}\\ \text{on an area } dS \text{ of a surface (in an isotropic}\\ \text{flux) whose angles of attack have cosines}\\ \text{between } \mu \text{ and } \mu + d\mu\end{array} \qquad (5.274)$$

where ϕ_0 is the magnitude of the isotropic flux and $\mu \equiv \cos\theta$. This choice for the angular distribution limits the applicability of this model; in general the angular distribution of neutrons falling on a sample will not be isotropic; even if the flux prior to the introduction of the sample were isotropic, the very presence of the sample would introduce perturbations in the angular distribution. However, in many reactor situations the assumption of an isotropic flux will afford a good approximation to the actual distribution. Precisely, the simple isotropic distribution would be realized near the center of a spherical surface of large radius on which isotropically emitting sources were uniformly and densely packed.

Complete information about this system would involve knowing the neutron density and angular distribution at every point inside the sample. In general, such information requires extensive and laborious calculation; however, in two extreme approximations the problem may be solved with relative ease. The complication in the physical situation is the scattering collisions made by the neutrons before they are absorbed or escape from the sample. If, on the average, this number of scatterings is a small but nonzero number (of the order of 1 to 3), the problem is very difficult, for it is necessary to follow each neutron history in detail. The two extreme cases which are amenable to calculation are: (1) The diffusion model: In this case many scatterings are made, on the average, by the neutron in the sample before it is absorbed or escapes. (2) The first-flight transport model: Here the neutron makes no scattering collisions in the sample, but continues along its original line of flight until it suffers an absorption collision or escapes from the sample.

On the basis of these two models, various quantities of interest may be calculated; those to be discussed here are listed and defined below:

The transmission coefficient:

$$\alpha \equiv \begin{array}{l}\text{fraction of all neutrons incident upon surface}\\ \text{of a lump of material which pass through lump}\\ \text{without being absorbed}\end{array} \qquad (5.275)$$

We will demonstrate later that the transmission coefficient (or transmittance, cf. Sec. 5.3a) is identical, from a physical standpoint, with our previous definition of the albedo [cf. Eq. (5.111)]. It is appropriate, then, to use the same symbol.

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The self-shielding factor:

$$f_s \equiv \frac{\text{average flux throughout volume of a lump}}{\text{average flux at surface of a lump (in the presence of the lump)}} \tag{5.276}$$

The over-all self-shielding factor:[1]

$$f_0 \equiv \frac{\text{average flux throughout volume of a lump}}{\text{value of flux at position of lump before it was introduced}} \tag{5.277}$$

These quantities will, of course, be related for any given physical situation, but there is no universal relation among them. However, there are derived quantities directly related:

The effective absorption cross section:

$$\sigma_{eff} \equiv \text{that microscopic cross section which, if assigned to material of absorber lump, would allow a correct calculation of total absorptions in lump in the form given in Eq. (5.279)} \tag{5.278}$$

$$\text{Total absorptions per unit time in absorber lump} \equiv A_L \equiv N\sigma_{eff}\phi_0 V_L \tag{5.279}$$

where N is the number of atoms per unit volume in the absorber lump, V_L is the volume of the lump, and ϕ_0 is the flux in the system in the absence of the absorber lump. From (5.277) and (5.278) we have

$$\sigma_{eff} = \sigma_a f_0 \tag{5.280}$$

where σ_a is the true microscopic absorption cross section of the absorber material.

The flux correction factor:

$$f_c \equiv \text{that factor by which the quantity } A_L/\Sigma_a V_L \text{ must be multiplied in order to give flux in absence of absorber lump} \tag{5.281}$$

From (5.279) and (5.280)

$$f_c = \frac{\sigma_a}{\sigma_{eff}} = \frac{1}{f_0} \tag{5.282}$$

The general principles for the calculation of the quantities defined in Eqs. (5.275) through (5.281) in the two extreme approximations may now be presented for the case of an absorber of general geometric configuration placed in an isotropic flux.

1. *The Diffusion Model.* The diffusion equation is applied in the form

$$\nabla^2\phi(\mathbf{r}) - \varkappa^2\phi(\mathbf{r}) = 0 \tag{5.283}$$

[1] See also Sec. 7.4b, particularly Eq. (7.227).

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where $\kappa^2 \equiv \Sigma_a/D$, $\Sigma_a = N\sigma_a$ (the macroscopic absorption cross section of the material of the lump), and $D = \lambda_{tr}/3$ (the diffusion coefficient of the material of the lump). (The method presented here is for a single-material lump, but is easily extended to multiregion lumps by application of the diffusion equation in each region and the continuity of flux and neutron current conditions at the interfaces.)

The two boundary conditions required on the flux given by Eq. (5.283) are:

(1) The flux must be a symmetric or nonsingular function (depending on the particular geometry)
(2) The total number of neutrons incident upon unit area of the sample[1] per unit time is $\phi_0/4$ (5.284)

$$\frac{1}{4}\phi(\mathbf{R}) + \frac{D}{2}[\mathbf{n}\cdot\nabla\phi(\mathbf{R})] = \frac{1}{4}\phi_0 \tag{5.285}$$

where $\mathbf{n}$ is the normal vector to the surface and $\mathbf{R}$ represents the coordinates of the surface S; thus, $\phi(\mathbf{r})$ is completely specified. From $\phi(\mathbf{r})$ we calculate

$$\alpha = \int_{S_L} \frac{j(\mathbf{R})\,dS}{\frac{1}{4}\phi_0 S_L} \tag{5.286}$$

where S_L is the total surface area of the sample, and

$$j(\mathbf{R}) = \frac{1}{4}\phi(\mathbf{R}) - \frac{D}{2}[\mathbf{n}\cdot\nabla\phi(\mathbf{R})] \tag{5.287}$$

also

$$f_s = \frac{1}{V_L\phi_{SL}}\int_{V_L}\phi(\mathbf{r})\,d\mathbf{r} \tag{5.288}$$

$$f_0 = \frac{1}{V_L\phi_0}\int_{V_L}\phi(\mathbf{r})\,d\mathbf{r} \tag{5.289}$$

where ϕ_{SL} is the average value of the flux over the surface S_L of the lump.

2. *The First-flight Transport Model.* The calculation of the spatial variation of the flux in this approximation is not so straightforward as in the diffusion model, and no fixed expression can be given. In general, the procedure for finding the flux at any point inside the lump is to choose an infinitesimal volume element dV at that point and calculate the total track length per unit time in dV due to all neutrons impinging on the lump which succeed in passing through dV before being captured. The

[1] Note that the condition at the surface is not taken to be continuity of the flux with ϕ_0. Near the lump the *total* flux will be reduced from ϕ_0, but the part of the flux directed toward the lump, by assumption, will not be changed. Physically, this means that the presence of the lump is assumed not to alter the effective neutron sources.

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flux is given by the limit [cf. Eqs. (3.39) and (3.40)]

$$\lim_{dV\to 0}\left(\frac{\text{total track length/time in } dV}{dV}\right) \equiv \phi \qquad (5.290)\star$$

On the other hand, the transmission coefficient is more readily obtained. For this application we will require the complete directional specification of the current and therefore select the form

$$j(\mu,\psi)\,d\mu\,d\psi = \frac{1}{4\pi}\,\phi_0\mu\,d\mu\,d\psi \qquad (5.291)$$

as given in the first term of (5.32). Here, μ is the cosine of the angle which the direction of motion of the incident neutron makes with the normal to the surface of the lump and ψ is the azimuth of the neutron direction about the normal. Expression (5.291) is a consistent order of approximation with (5.274). The integral of (5.291) over all directions into the lump and over the complete surface area S_L of the lump is $\frac{1}{4}S_L\phi_0$. Since the probability that a neutron will pass through a distance s of material without being absorbed is $\exp(-\Sigma_a s)$, we have for the transmission coefficient

$$\begin{aligned}\alpha &= \frac{4}{S_L\phi_0}\int_{S_L} dS(\mathbf{R}) \int_0^1 d\mu \int_0^{2\pi} j(\mu,\psi)e^{-\Sigma_a s(\mathbf{R},\mu,\psi)}\,d\psi \\ &= \frac{1}{\pi S_L}\int_{S_L} dS(\mathbf{R}) \int_0^1 \mu\,d\mu \int_0^{2\pi} e^{-\Sigma_a s(\mathbf{R},\mu,\psi)}\,d\psi \end{aligned} \qquad (5.292)$$

where $dS(\mathbf{R})$ is the surface element at surface coordinate $\mathbf{R}$ and $s(\mathbf{R},\mu,\psi)$ is the straight-line distance across the lump for a neutron incident at $\mathbf{R}$ traveling in direction (μ,ψ). For several elementary cases, s does not depend upon the position $\mathbf{R}$ of the surface; so $s(\mathbf{R},\theta,\psi) \to s(\theta,\psi)$, and (5.292) reduces to

$$\alpha = \frac{1}{\pi}\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta \int_0^{2\pi} e^{-\Sigma_a s(\theta,\psi)}\,d\psi \qquad (5.293)$$

Of course, the form of $s(\theta,\psi)$ is sufficiently simple so that the integral of Eq. (5.293) may be performed in closed form in only a few cases.

For the cases to which the first-flight transport model is applicable, there is a relation between the transmission coefficient α and the over-all self-shielding factor f_0; this is

$$1 - \alpha = \left(\frac{4\Sigma_a V_L}{S_L}\right) f_0 = \Sigma_a \bar{s} f_0 \qquad (5.294)$$

where [cf. Eq. (7.232)]

$$\bar{s} \equiv \frac{4V_L}{S_L} \qquad (5.295)$$

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That $\bar{s}$, as given by (5.295), is indeed the average path length of neutrons through the volume of the lump (with the lump nonabsorbing and an incident isotropic distribution) may be seen by calculating the flux in the nonabsorbing lump and equating this to the undisturbed flux. (All this is for lumps with nonreentrant surfaces.)

For the case of very slight absorption, Eq. (5.293) reduces to a simple form. If we take, then, $\Sigma_a s(\theta,\psi) \ll 1$, the exponential may be represented by the first two terms of a Taylor series, and we obtain from (5.293), with the aid of (5.295),

$$\alpha \simeq 1 - \Sigma_a \bar{s} \qquad \textit{for small absorption} \tag{5.296}\star$$

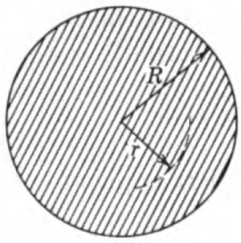
The application of the general methods discussed above to several specific geometries is carried out below and the results are summarized.

The solid sphere:

1. Diffusion approximation:

$$\phi(r) = \left\{ \frac{\phi_0 R}{\sinh \varkappa R \left[1 + \frac{2D}{R} (\varkappa R \coth \varkappa R - 1) \right]} \right\} \frac{\sinh \varkappa r}{r} \tag{5.297}$$

$$\alpha = \frac{1 - 2D(\varkappa R \coth \varkappa R - 1)/R}{1 + 2D(\varkappa R \coth \varkappa R - 1)/R} \tag{5.298}$$



$$f_s = \frac{3}{\varkappa^2 R^2} (\varkappa R \coth \varkappa R - 1) \tag{5.299}$$

$$f_0 = \frac{f_s}{1 + 2D(\varkappa R \coth \varkappa R - 1)/R} \tag{5.300}$$

2. First-flight transport approximation:

$$\phi(r) = \frac{\phi_0}{4r_0} \{2e^{-R_0}(1 + R_0) \sinh r_0 + 2e^{-R_0} r_0 \cosh r_0 - (R_0^2 - r_0^2)[E_1(R_0 - r_0) - E_1(R_0 + r_0)]\} \tag{5.301}$$

where $R_0 \equiv \Sigma_a R$ and $r_0 \equiv \Sigma_a r$. Also,

$$s(\theta,\psi) = 2R \cos \theta \tag{5.302}$$

$$\alpha = \frac{2}{\rho_0^2} [1 - (\rho_0 + 1)e^{-\rho_0}] \qquad \rho_0 \equiv 2\Sigma_a R \tag{5.303}$$

$$f_s = \frac{3}{\rho_0^2} \left(\frac{\rho_0^2 + 2\rho_0 e^{-\rho_0} + 2e^{-\rho_0} - 2}{\rho_0 - e^{-\rho_0} + 1} \right) \tag{5.304}$$

$$f_0 = \frac{3}{2} \left(\frac{1 - \alpha}{\rho_0} \right) \tag{5.305}$$

and[1]

$$E_n(x) \equiv x^{n-1} \int_x^\infty \frac{e^{-y}\, dy}{y^n} \tag{5.306}$$

[1] The $E_n(x)$ functions are defined according to K. M. Case, F. de Hoffmann, and G. Placzek in "Introduction to the Theory of Neutron Diffusion," vol. I, p. 153, Los Alamos Scientific Laboratory, 1953 (U.S. Government Printing Office). The functions E_0 to E_4 are tabulated on pp. 155–161.

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Figure 5.32 gives a comparison of the two approximations for the calculation of α, for the case of a 10-cm sphere with $\lambda_{tr} = 3$ cm, as a function of absorber loading (Σ_a). The diffusion approximation should be valid for

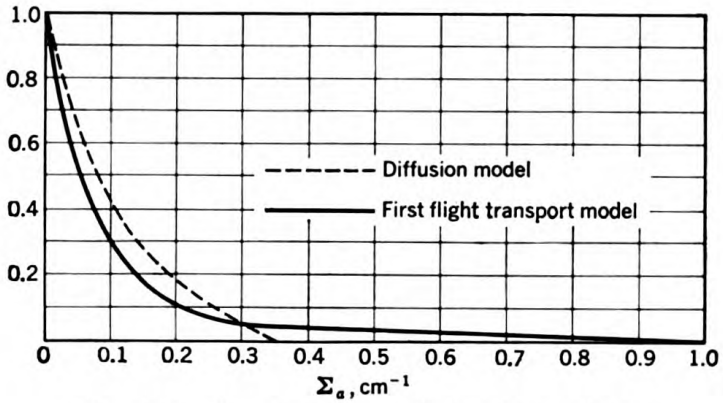


FIG. 5.32 Transmission coefficient for a sphere.

small absorption, while the transport results should hold at large absorption. Note that the diffusion result is totally incorrect beyond a certain value of Σ_a, actually giving negative values of α.

The infinite slab:

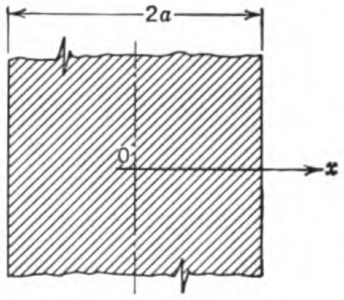
1. Diffusion approximation:

$$\phi(x) = \frac{\phi_0 \cosh \kappa x}{\cosh \kappa a + 2D\kappa \sinh \kappa a} \tag{5.307}$$

$$\alpha = \frac{1 - 2D\kappa \tanh \kappa a}{1 + 2D\kappa \tanh \kappa a} \tag{5.308}$$

$$f_s = \frac{1}{\kappa a \coth \kappa a} \tag{5.309}$$

$$f_0 = \frac{f_s}{1 + 2D\kappa \tanh \kappa a} \tag{5.310}$$



2. First-flight transport approximation (with x measured from left face):

$$\phi(x) = \frac{1}{2}\phi_0\{E_2(\Sigma_a x) + E_2[\Sigma_a(2a - x)]\} \qquad x \geq 0 \tag{5.311}$$

$$s(\theta,\psi) = \frac{2a}{\cos\theta} \tag{5.312}$$

$$\alpha = e^{-x_0} - x_0 e^{-x_0} + x_0^2 E_1(x_0) \qquad x_0 \equiv 2\Sigma_a a \tag{5.313}$$

$$f_0 = \frac{1 - \alpha}{2x_0} \tag{5.314}$$

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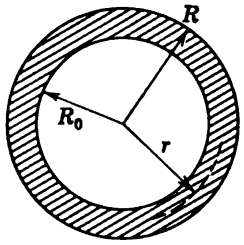
The spherical shell:

1. Diffusion approximation:



$$\phi(r) = \frac{A}{r}\left[\varkappa R_0 \cosh \varkappa(r - R_0) + \sinh \varkappa(r - R_0)\right] \tag{5.315}$$

$$A \equiv R\phi_0 \left\{\cosh \varkappa T\left[R_0\varkappa - 2D\varkappa\frac{R_0}{R} + 2D\varkappa\right] + \sinh \varkappa T\left[1 - \frac{2D}{R} + 2D\varkappa^2 R_0\right]\right\}^{-1} \tag{5.316}$$

with $T \equiv R - R_0$, and

$$\alpha = \frac{1 + 2D/R - 2D\varkappa^2 R_0 + (\varkappa R_0 + 2D\varkappa R_0/R - 2D\varkappa)\coth \varkappa T}{1 - 2D/R + 2D\varkappa^2 R_0 + (\varkappa R_0 - 2D\varkappa R_0/R + 2D\varkappa)\coth \varkappa T} \tag{5.317}$$

$$f_0 = \frac{3R}{\varkappa^2(R^3 - R_0^3)}\left[\frac{\varkappa T \cosh \varkappa T + (\varkappa^2 R R_0 - 1)\sinh \varkappa T}{(\varkappa R_0 - 2D\varkappa R_0/R + 2D\varkappa)\cosh \varkappa T + (1 - 2D/R + 2D\varkappa^2 R_0)\sinh \varkappa T}\right] \tag{5.318}$$

The average flux inside the shell

$$\phi_i \equiv \varkappa A \tag{5.319}$$

2. First-flight transport approximation:

$$\begin{aligned}\alpha = \frac{1}{\rho_0}\Big\{&2 - \frac{3}{2}(1 + \rho_0\sqrt{1-\gamma^2})\exp(-\rho_0\sqrt{1-\gamma^2}) \\ &- \frac{1}{2}[1 + (1-\gamma)\rho_0]e^{-\rho_0(1-\gamma)}\Big\} \\ &- \frac{1}{4}(1-\gamma^2)\exp(-\rho_0\sqrt{1-\gamma^2}) + \frac{(1-\gamma^2)^2 e^{-\rho_0(1-\gamma)}}{4(1-\gamma)^2} \\ &+ \frac{\rho_0}{4}(1-\gamma)^{\frac{1}{2}}\exp(-\rho_0\sqrt{1-\gamma^2}) - \frac{\rho_0(1-\gamma^2)^2 e^{-\rho_0(1-\gamma)}}{4(1-\gamma)} \\ &+ \frac{\rho_0^2}{4}(1-\gamma^2)^2[E_1(\rho_0 - \rho_0\gamma) - E_1(\rho_0\sqrt{1-\gamma^2})]\end{aligned} \tag{5.320}$$

with

$$\rho_0 \equiv 2\Sigma_a R \qquad \gamma \equiv \frac{R_0}{R} \tag{5.321}$$

$$\phi_i = \frac{3\phi_0}{8\gamma^3}\int_{1-\gamma}^{\sqrt{1-\gamma^2}} y^{-4}[(1-\gamma^2)^2 - y^4][(1-\gamma^2) - y^2]e^{-\rho_0 y/2}\,dy \tag{5.322}$$

for a very thin shell:

$$\phi_i = \phi_0(1 - \tfrac{3}{2}\Sigma_a T + \cdots) \tag{5.323}$$

$$\alpha = 1 - 4\Sigma_a T + \cdots \tag{5.324}$$

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The infinite solid cylinder:

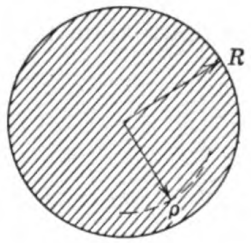
1. Diffusion approximation:

$$\phi(\rho) = \frac{\phi_0 I_0(\varkappa\rho)}{I_0(\varkappa R) + 2D\varkappa I_1(\varkappa R)} \tag{5.325}$$

$$\alpha = \frac{1 - 2D\varkappa I_1(\varkappa R)/I_0(\varkappa R)}{1 + 2D\varkappa I_1(\varkappa R)/I_0(\varkappa R)} \tag{5.326}$$

$$f_s = \frac{2I_1(\varkappa R)}{\varkappa R I_0(\varkappa R)} \tag{5.327}$$

$$f_0 = \frac{f_s}{1 + 2D\varkappa I_1(\varkappa R)/I_0(\varkappa R)} \tag{5.328}$$



2. First-flight transport approximation. The integrals involved here cannot be performed in closed form. However, some results have been

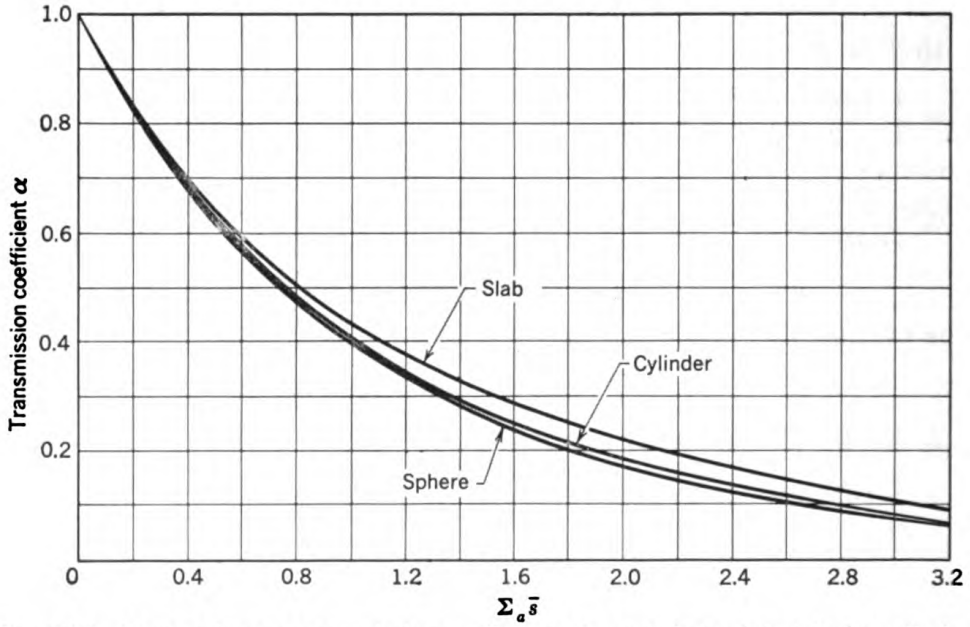


FIG. 5.33 Comparison of transmission coefficients for an infinite slab, infinite cylinder, and sphere in the first-flight transport approximation. $\Sigma_a \bar{s} \equiv 4\Sigma_a V_L/S_L$.

obtained by numerical calculation.[1] The form of the integral for the transmission coefficient is

$$\alpha = \frac{4R^3}{\pi} \int_0^\infty dz \int_0^\pi \rho^{-4}(1 - \cos\beta)^2 e^{-\Sigma_a \rho}\, d\beta \tag{5.329}$$

with $\rho \equiv [z^2 + 2R^2(1 - \cos\beta)]^{\frac{1}{2}}$

$$f_0 = \frac{1 - \alpha}{2\Sigma_a R} \tag{5.330}$$

The transmission coefficient α is given in Fig. 5.33 as a function of

[1] W. J. C. Bartels, "Self-absorption of Monoenergetic Neutrons," KAPL-336, May 1, 1950; R. J. Royston, "The Absorption of Neutrons by a Small, Strongly Absorbing Rod," AERE T/M 117, November, 1954.

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dimension in the first-flight transport approximation for slab, cylinder, and sphere. The values were obtained for the slab from Eq. (5.313), for the sphere from Eq. (5.303), and for the cylinder from the paper by Bartels.

The approximations to α for lightly absorbing slabs, cylinders, and spheres are given below:

Slab: $\Sigma_a a \ll 1$ $\quad \alpha \simeq 1 - 4\Sigma_a a^{-(2\Sigma_a a)^2} \ln \gamma 2\Sigma_a a - \frac{8}{3}(2\Sigma_a a)^2 + \cdots$ with $\gamma = 1.781072 \ldots$ (5.331)

Cylinder: $\Sigma_a R \ll 1$ $\quad \alpha \simeq 1 - 2\Sigma_a R + \frac{8}{3}\Sigma_a^2 R^2 + \cdots$ (5.332)

Sphere: $\Sigma_a R \ll 1$ $\quad \alpha \simeq 1 - \frac{2}{3}(2\Sigma_a R) + \frac{1}{4}(2\Sigma_a R)^2 + \cdots$ (5.333)

Infinitely long cylindrical shell. This case will not be discussed here, but numerical calculations in the first-flight transport approximation have been performed by Dwork, Hofmann, Hurwitz, and Clancy.[1]

b. Flux Distribution in Vicinity of Localized Absorbers in Diffusion Media. If a localized absorber is placed in a diffusion medium, the neutron flux in the vicinity of the absorber is reduced from its value in the absence of the absorber; the effect of the absorber falls off with increasing distance from the absorber. Since there is always a net current of neutrons into the absorber, the angular distribution of neutrons incident upon the surface of the absorber is no longer isotropic, but, in general, has additional terms depending upon the derivatives of the flux at the surface. For example, in the diffusion approximation the angular distribution is [cf. Eq. (5.36)]

$$j(\mu)\, d\mu = \tfrac{1}{2}\phi_0 \mu\, d\mu + \tfrac{3}{2} D \phi_0' \mu^2\, d\mu \tag{5.334}$$

where ϕ_0 and ϕ_0' are the neutron flux and its derivative in the direction in question, respectively. The treatment of the preceding part of this section is rigorously correct where all derivatives of the flux vanish; while this condition can never be met near an absorber, it may be true that the second term of (5.334) is negligible in comparison with the first. In the following presentation we first neglect the second term in (5.334), then try to take it into account in simple cases. In both calculations, however, we retain the assumption that diffusion theory holds outside the absorber; for small, highly absorbing lumps this assumption is in error.

1. *Assumption of an Isotropic Flux Incident on the Absorber from the Moderator.* As a first model for the calculation of the flux depression near an absorber we assume that (*a*) diffusion theory holds everywhere in the moderator, (*b*) an isotropic flux of neutrons is incident upon the surface of the absorber. Both of these assumptions are closely met if the flux is nearly isotropic everywhere in the diffusion medium. The second assumption allows the use of the quantities calculated for the

[1] J. Dwork, P. L. Hofmann, H. Hurwitz, Jr., and E. F. Clancy, "Self-shielding Factors for Infinitely Long, Hollow Cylinders," KAPL-1262, Jan. 10, 1955.

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absorber in Sec. 5.8a; for example, the fraction of the incident neutrons transmitted through the absorber will, on this basis, be the α calculated in Sec. 5.8a, which is appropriate for the geometry and neutron properties of the absorber. Thus the absorber may be replaced, in so far as the diffusion medium is concerned, by a boundary condition to be applied at the surface of the absorber which states that the fraction of all neutrons incident upon the absorber which return to the diffusion medium is the appropriate α (see also discussion in Sec. 5.3c).

The additional boundary condition needed for the flux in the diffusion medium will be found from the external conditions, including the neutron-source distribution, at some distance from the absorber. Such boundary conditions may be chosen to fit the particular case, but a very useful one, which may be presented here as an illustration of the procedure, is obtained by imagining that the neutron sources are uniformly distributed throughout an infinite medium of the diffusion material and that the introduction of the absorber does not alter these sources. Then, in the presence of the absorber, it must be required that the flux at great distances from the absorber approach its value in the absence of the absorber. Such an assumption might very well represent the case of an absorber placed in a large moderator region in which neutrons are being thermalized fairly uniformly over the region. However, the results which follow have a wider applicability than might be suspected from the restrictive choice for the neutron-source function. For any case in which the absorber is placed in a large moderator region not near any strong localized source of neutrons and in which the introduction of the absorber does not alter the distribution or strength of the sources, the flux depression will not differ appreciably from the forms given.

The mathematical formulation of this model is, for the flux in the moderator:

$$\nabla^2\phi(\mathbf{r}) - \kappa^2\phi(\mathbf{r}) = -\frac{S_0}{D} \tag{5.335}$$

and

$$\frac{j_+(\mathbf{R})}{j_-(\mathbf{R})} = \frac{\phi(\mathbf{R}) - 2D\phi'(\mathbf{R})}{\phi(\mathbf{R}) + 2D\phi'(\mathbf{R})} = \alpha \tag{5.336}$$

$$\lim_{\mathbf{r}\to\infty} \phi(\mathbf{r}) = \phi_0 = \frac{S_0}{D\kappa^2} = \frac{S_0}{\Sigma_a^{(M)}} \tag{5.337}$$

where S_0 is the uniform source intensity of neutrons, ϕ_0 the flux in the region of the absorber prior to its introduction, and α the transmission coefficient calculated on the basis of the diffusion-theory estimate for the partial currents [see Eqs. (5.49)]. We find then, as indicated earlier, that the transmission coefficient takes the functional form of the albedo as defined in Eqs. (5.111) and (5.117) in the case that the current relations at the surface of the lump are based on the use of the diffusion-theory model in the adjoining regions. Thus, in a real physical sense the trans-

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mission coefficient and the albedo serve to measure the same effect, namely, the capability of a particular region to return neutrons to the surrounding medium, be it by "transmission" or "reflection." It is important to recognize, however, that the precise analytical form of the albedo and the transmission coefficient will not in general be the same. For example, when computing the albedo of a reflector in a reactor wherein the origin of the coordinate system is located at the "center" of the reactor, the correct form to use is j_-/j_+, as shown in Eq. (5.117). In the present applications, on the other hand, the coordinate system is conveniently centered within the transmitting (or absorbing) region, and this leads to the form j_+/j_- for the transmission coefficient, as indicated in Eq. (5.336). These differences are due entirely to the choice of coordinate systems and should not be attributed to the physical interpretations.

The use of this model involves the choice of the proper α from Sec. 5.8a, on the basis of either the diffusion approximation or the first-flight transport approximation, depending on the nature of the absorber. It should not be expected to give good results in cases where the angular distribution in the moderator, near the lump, differs appreciably from isotropic, as it would, for example, near a small, highly absorbing lump.

A useful measure of the effect of the lump on the flux in the moderator is the *flux depression factor* defined below:

$$\text{Flux depression factor} \equiv \mathfrak{D} \equiv \frac{\phi_0 - \phi(\mathbf{R})}{\phi_0} \tag{5.338}$$

where ϕ_0 is the original flux and $\phi(\mathbf{R})$ is the flux at the surface of the lump.

The results of the application of this model for three elementary geometrical forms for the absorber lump, all immersed in an infinite diffusion medium, are presented below. In the following, the quantities ϕ, $\varkappa$, and D are, respectively, the neutron flux, the reciprocal diffusion length, and the diffusion coefficient for the diffusion medium.

Infinite slab:

$$\phi(x) = \phi_0[1 - \mathfrak{D}e^{-\varkappa(x-a)}] \qquad x \geq 0 \tag{5.339}$$

$$\mathfrak{D} = \left[1 + 2D\varkappa\left(\frac{1 + \alpha^{Sl}}{1 - \alpha^{Sl}}\right)\right]^{-1} \tag{5.340}$$

where α^{Sl} (for the uniform slab) is obtained from Eq. (5.308) or (5.313), depending on the neutron properties of the slab.

Infinitely long cylinder (ρ is radial distance from cylinder axis):

$$\phi(\rho) = \phi_0\left[1 - \mathfrak{D}\frac{K_0(\varkappa\rho)}{K_0(\varkappa R)}\right] \tag{5.341}$$

$$\mathfrak{D} = \left[1 + 2D\varkappa\frac{K_1(\varkappa R)}{K_0(\varkappa R)}\left(\frac{1 + \alpha^C}{1 - \alpha^C}\right)\right]^{-1} \tag{5.342}$$

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where α^C (for the uniform cylinder) is obtained from Eq. (5.326) or (5.329), depending on the neutron properties of the cylinder.

Sphere (r is distance from center of sphere):

$$\phi(r) = \phi_0\left[1 - \frac{\mathfrak{D}R}{r}\, e^{-\varkappa(r-R)}\right] \tag{5.343}$$

$$\mathfrak{D} = \left[1 + \frac{2D}{R}(\varkappa R + 1)\left(\frac{1 + \alpha^{Sp}}{1 - \alpha^{Sp}}\right)\right]^{-1} \tag{5.344}$$

where α^{Sp} (for the uniform sphere) is obtained from (5.298) or (5.303), depending on the neutron properties of the sphere.

Of course, Eqs. (5.339) through (5.344) are applicable to slabs, cylinders, and spheres of any internal structure if the appropriate α can be determined and if the simple one-dimensional variation of properties is maintained. For example, (5.344) gives the flux depression for a hollow sphere of outer radius R if the α^{Sp} is chosen from (5.317) or (5.320).

The model, as presented in Eqs. (5.335) through (5.337), may be applied to other geometries, the limitation being the difficulty of solution of the mathematical problem. Two special cases of great practical interest, the finite cylinder and the finite slab, are omitted from the discussion here since no general analytical solution is known (to the authors); each case is treated by approximational methods deemed appropriate.

As an example of the use of these concepts, suppose that the (thermal) absorption cross section of some material were to be measured by putting a solid sphere of the material into a moderating region supporting a known neutron flux (thermal) ϕ_0. By the neutron activation of the material and subsequent determination of the rate of radioactive decay, a value may be obtained for the total neutron absorptions occurring in the sphere during the time t of exposure to the flux. The total number of absorptions A_L is now to be calculated in terms of the quantities discussed above. If the flux at the surface of the sphere during the exposure is $\phi(R)$, the number of neutrons incident upon unit area of the surface in unit time (under the isotropic assumption) is $\phi(R)/4$; since $(1 - \alpha)$ of these are absorbed, we have

$$A_L = \tfrac{1}{4}\phi(R)(1 - \alpha)4\pi R^2 t \tag{5.345}$$

From the definition of Eq. (5.344), the flux at the surface of the sphere is related to the flux depression factor $\mathfrak{D}$ and the flux in the absence of the absorber ϕ_0 by

$$\phi(R) = \phi_0(1 - \mathfrak{D}) \tag{5.346}$$

If it is appropriate to use the first-flight transport model for the material of this sphere, then Eq. (5.303) may be used for α, and we have finally [using Eqs. (5.338) and (5.344) for $\mathfrak{D}$, and denoting moderator quantities by the index M and absorber quantities by the index A]

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$$A_L = \pi R^2 \phi_0 l \left[\frac{2D_M(\varkappa_M R + 1)(1 + \alpha_A)/R}{1 + 2D_M(\varkappa_M R + 1)\left(\frac{1 + \alpha_A}{1 - \alpha_A}\right)\frac{1}{R}} \right] \tag{5.347}$$

$$\alpha_A = \frac{2}{[2R\Sigma_a^{(A)}]^2}\{1 - e^{-2R\Sigma_a^{(A)}}[2R\Sigma_a^{(A)} + 1]\} \tag{5.348}$$

Since A_L is known, the absorption cross section of the material of the sphere may be determined from (5.347) and (5.348).

2. *Application of Diffusion Approximation for Current from Moderator to Absorber.* (*a*) Diffusion theory applicable to both moderator and absorber: Here we drop the restrictive assumption of an isotropic flux incident upon the absorber and consider the angular distribution given in Eq. (5.334) which is appropriate to the diffusion approximation. We consider first the results when diffusion theory is also assumed to hold in the absorber region and subsequently discuss the first-flight transport model for the absorber.

In the case that diffusion theory holds in both the moderator and the absorber, and assuming a uniform distribution of neutron sources as before, the mathematical formulation of the problem is:

In the moderator: $$\nabla^2\phi_M(\mathbf{r}) - \varkappa_M^2\phi_M(\mathbf{r}) = -\frac{S_0}{D_M} \tag{5.349}$$

In the absorber: $$\nabla^2\phi_A(\mathbf{r}) - \varkappa_A^2\phi_A(\mathbf{r}) = 0 \tag{5.350}$$

where $$\varkappa_M^2 \equiv \frac{\Sigma_a^{(M)}}{D_M} \equiv \frac{1}{L_M^2} \qquad \varkappa_A^2 \equiv \frac{\Sigma_a^{(A)}}{D_A} \equiv \frac{1}{L_A^2} \tag{5.351}$$

At the moderator-absorber interface (coordinate **R**), the boundary conditions to be applied are

$$(1)\ \phi_M(\mathbf{R}) = \phi_A(\mathbf{R}) \tag{5.352}$$

$$(2)\ -D_M[\mathbf{n} \cdot \nabla\phi_M(\mathbf{R})] = -D_A[\mathbf{n} \cdot \nabla\phi_A(\mathbf{R})] \tag{5.353}$$

where **n** is the outwardly directed normal to the absorber surface. In addition we require

$$(3)\ \text{Symmetry or nonsingularity on } \phi_A(\mathbf{r}) \tag{5.354}$$

$$(4)\ \lim_{\mathbf{r}\to\infty} \phi_M(\mathbf{r}) = \phi_0 = \frac{S_0}{\Sigma_a^{(M)}} \tag{5.355}$$

This model gives precisely the same results for the flux depression as that presented in the preceding discussion [Eqs. (5.335) through (5.337)] if the α is calculated on the basis of the diffusion model [Eqs. (5.283) through (5.285)]. In the following, these quantities are given with the dependence on the absorber properties explicitly shown along with the spatial dependence of the flux for the three simple geometries previously used. In this case, of course, the results are applicable only to uniform absorbers.

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Infinite slab:

$$\phi_M(x) = \phi_0[1 - \mathfrak{D}e^{-\kappa_M(x-a)}] \qquad x \geq 0 \tag{5.356}$$

$$\phi_A(x) = \phi_0\left[1 + \frac{D_A\kappa_A \tanh \kappa_A a}{D_M\kappa_M}\right]^{-1} \frac{\cosh \kappa_A x}{\cosh \kappa_A a} \tag{5.357}$$

$$\mathfrak{D} = \left(1 + \frac{D_M\kappa_M \coth \kappa_A a}{D_A\kappa_A}\right)^{-1} \tag{5.358}$$

Infinite cylinder:

$$\phi_M(\rho) = \phi_0\left[1 - \frac{K_0(\kappa_M\rho)}{K_0(\kappa_M R) + \dfrac{D_M\kappa_M K_1(\kappa_M R) I_0(\kappa_A R)}{D_A\kappa_A I_1(\kappa_A R)}}\right] \tag{5.359}$$

$$\phi_A(\rho) = \phi_0\left[1 + \frac{D_A\kappa_A I_1(\kappa_A R)K_0(\kappa_M R)}{D_M\kappa_M I_0(\kappa_A R)K_1(\kappa_M R)}\right]^{-1} \frac{I_0(\kappa_A\rho)}{I_0(\kappa_A R)} \tag{5.360}$$

$$\mathfrak{D} = \left[1 + \frac{D_M\kappa_M I_0(\kappa_A R)K_1(\kappa_M R)}{D_A\kappa_A I_1(\kappa_A R)K_0(\kappa_M R)}\right]^{-1} \tag{5.361}$$

Sphere:

$$\phi_M(r) = \phi_0\left[1 - \frac{\mathfrak{D}R}{r} e^{-\kappa_M(r-R)}\right] \tag{5.362}$$

$$\phi_A(r) = \phi_0\left[1 + \frac{D_A}{D_M}\left(\frac{\kappa_A R \coth \kappa_A R - 1}{\kappa_M R + 1}\right)\right]^{-1} \frac{R \sinh \kappa_A r}{r \sinh \kappa_A R} \tag{5.363}$$

$$\mathfrak{D} \equiv \left[1 + \frac{D_M}{D_A}\left(\frac{\kappa_M R + 1}{\kappa_A R \coth \kappa_A R - 1}\right)\right]^{-1} \tag{5.364}$$

(*b*) Diffusion theory applicable in the moderator and the first-flight transport model applicable in the absorber: In the case of a small, fairly heavily absorbing lump placed in a diffusion medium, the most appropriate model might be diffusion theory in the moderator and first-flight transport in the absorber; however, there is a restriction on the applicability of this model. The angular distribution of the neutrons emerging from the absorber will be far from isotropic since those neutrons moving in directions for which the path length in the absorber is short will be preferentially transmitted. Further, if the sample is small in comparison with a diffusion length in the moderator, the rapid spatial variation of the neutron flux in the vicinity of the sample may make the omission of higher derivatives than the first (the basic approximation of diffusion theory) invalid. Thus, even though the medium surrounding the absorber is a "good" diffusion medium in the sense that $D/\lambda_a \ll 1$, diffusion theory will be a poor approximation if the absorber is very heavily absorbing. Thus the model to be discussed here is applied best to an absorber whose dimensions are comparable with a diffusion length (or greater) and whose absorption cross section is greater than its scattering cross section but not so large as greatly to alter the angular distribution of the neutrons in passage.

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For the absorber, the general form for the transmission coefficient (for such geometries that the path length through the absorber is a function only of the angular coordinate of the direction of motion with the normal to the surface, and not of the particular position on the surface) is, in the notation of Eq. (5.293),

$$\alpha = \frac{\int_0^{2\pi} d\psi \int_0^1 j(\mu)\, e^{-\Sigma_{as}(\mu,\psi)}\, d\mu}{\int_0^{2\pi} d\psi \int_0^1 j(\mu)\, d\mu} \tag{5.365}$$

with $\mu \equiv \cos\theta$. For the distribution $j(\mu)$ given by (5.334), appropriate for the diffusion approximation, α may be put into the form

$$\alpha = \frac{\phi_0 f_1 + 2D\phi_0' f_2}{\phi_0 + 2D\phi_0'} \tag{5.366}$$

where we define

$$f_1 \equiv \frac{1}{\pi} \int_0^{2\pi} d\psi \int_0^1 \mu e^{-\Sigma_{as}(\mu,\psi)}\, d\mu \tag{5.367}$$

$$f_2 \equiv \frac{3}{2\pi} \int_0^{2\pi} d\psi \int_0^1 \mu^2 e^{-\Sigma_{as}(\mu,\psi)}\, d\mu \tag{5.368}$$

The boundary condition to be applied on the flux in the diffusion medium at the absorber surface is that given in (5.336) and, using (5.366) for α, this condition may be expressed as a relationship between the flux and its first derivative at the absorber surface. Thus

$$2D\phi'(\mathbf{R}) = \left(\frac{1 - f_1}{1 + f_2}\right) \phi(\mathbf{R}) \tag{5.369}$$

For comparison, Eq. (5.336) gives for this relationship

$$2D\phi'(\mathbf{R}) = \left(\frac{1 - \alpha}{1 + \alpha}\right) \phi(\mathbf{R}) \tag{5.370}$$

Since the model under discussion here is the same in the diffusion medium as that presented in Eqs. (5.335) through (5.337), except for the replacement of the boundary condition of (5.336) by that of (5.369), it is clear from a comparison of Eqs. (5.369) and (5.370) that the results for the flux in the moderator and the flux depression are as presented for the three simple geometries in Eqs. (5.339) through (5.344), except that the ratio $(1 - \alpha)/(1 + \alpha)$ is to be replaced by the ratio $(1 - f_1)/(1 + f_2)$.

The quantity f_1 defined in Eq. (5.367) is just the α previously used and defined in Eq. (5.293). However, f_2 is a new quantity which must be computed for each geometry of interest. Note that f_2 is the fraction of neutrons transmitted by a sample upon which is incident a flux of neutrons whose angular distribution is an isotropic distribution weighted with a cosine factor which makes it more "forward." For the cases of an infinite slab and a sphere, the quantities f_2 are presented below; no

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simple solutions and no numerical work for other cases are known to the authors.

Slab, half-width a:

$$f_2 = e^{-x_0} - \tfrac{1}{2}x_0[e^{-x_0}(1 - x_0) + x_0^2 E_1(x_0)] \qquad (5.371)$$
$$x_0 \equiv 2\Sigma_a a$$

Sphere, radius R:

$$f_2 = \frac{3}{\rho_0^3}[2 - 2(1 + \rho_0)e^{-\rho_0} - \rho_0^2 e^{-\rho_0}] \qquad (5.372)$$
$$\rho_0 \equiv 2\Sigma_a R$$

If a sizable fraction of the neutrons incident upon an absorber are nonisotropic in distribution, the expressions developed in the first part of this section for determining the flux inside the absorber, and therefore also the self-shielding factors, will be invalid. The correct calculation of these quantities, although straightforward, is laborious.

(*c*) The foil problem: Considerable effort has been directed toward improving the models presented above for the particular case of an absorbing foil in a diffusion medium, i.e., geometrically, a slab whose thickness is small in comparison with its extent, but whose extent is comparable to or less than a diffusion length of the diffusion medium. The corrections involve accounting for the "end effects" and attempting to take into account higher-order angular terms (than the two considered by the diffusion approximation) in the diffusion medium near the foil. Experiments have been performed in an attempt to check the value of the proposed corrections. Some references on this subject are given below.[1]

c. Internal Heating in Localized Absorbers. There are several mechanisms by which the existence of a neutron flux in the interior of a body may give rise to heating; of these the most important is the fission process. In case the absorber consists of a fissionable material in whole or in part, and if the material is such that the fission fragments do not escape from the sample, the experiment must be designed so that available cooling will be sufficient to keep the maximum temperature inside the sample below some specified value.

Let us examine the case of the conduction of heat in a uniform body in which the sources of heat are everywhere proportional to the neutron flux (restricting attention to steady-state problems). If $G(\mathbf{r})$ is the heat generated per unit volume per unit time around point $\mathbf{r}$, then

$$G(\mathbf{r}) \equiv \beta\phi(\mathbf{r}) \qquad (5.373)$$

[1] W. Bothe, *Z. Physik*, **120**, 437 (1943); E. Corinaldesi, *Nuovo cimento*, **3**, 131 (1946); E. Klema and R. H. Ritchie, *Phys. Rev.*, **87**, 167 (1952); C. W. Tittle, *Nucleonics*, **9**, 60 (1951); R. H. Ritchie et al., "Thermal Neutron Depression," Health Physics Division Annual Progress Report for Period Ending July 31, 1959, ORNL-2806, pp. 133–136, Oct. 29, 1959.

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where β is some constant, and the temperature distribution $T(\mathbf{r})$ is to be determined from

$$-\mathfrak{k}\nabla^2 T(\mathbf{r}) = G(\mathbf{r}) = \beta\phi(\mathbf{r}) \tag{5.374}$$

where $\mathfrak{k}$ is the thermal conductivity. The solution to Eq. (5.374) is to satisfy also certain boundary conditions. The solution of the mathematical problem posed by (5.374) can be very laborious and usually requires machine computations; however, a simple case will be considered here as an example.

If the absorber is such that diffusion theory is applicable in its interior, then the flux $\phi(\mathbf{r})$ satisfies Eq. (5.283), and it may be seen that $T(\mathbf{r})$ must be of the form

$$T(\mathbf{r}) = c_1 F_1(\mathbf{r}) + c_2 F_2(\mathbf{r}) - \frac{\beta}{\mathfrak{k}\varkappa^2}\phi(\mathbf{r}) \tag{5.375}$$

where

$$\nabla^2 F_1(\mathbf{r}) = \nabla^2 F_2(\mathbf{r}) = 0 \tag{5.376}$$

i.e., F_1 and F_2 are independent solutions of Laplace's equation for the particular geometry.

Suppose, as a particular example, that a very long cylinder (radius R) of a dilute uranium alloy is placed in a channel of water which supports a neutron flux ϕ_0. If the water is flowing along the cylinder, and if we neglect the rise in temperature of the water axially, the boundary condition at the absorber-water interface may be taken to be

$$T(R) = T_0 = \text{water temperature} \tag{5.377}$$

If we assume the applicability of diffusion theory for the flux inside the absorber, then from equations (5.360) and (5.375)

$$T(\rho) = c_1 + c_2 \ln \rho - \left(\frac{\beta\phi_0}{\mathfrak{k}\varkappa_A^2}\right)\left\{\frac{I_0(\varkappa_A\rho)}{I_0(\varkappa_A R)\left[1 + \dfrac{D_A\varkappa_A}{D_M\varkappa_M}\dfrac{I_1(\varkappa_A R)}{I_0(\varkappa_A R)}\dfrac{K_0(\varkappa_M R)}{K_1(\varkappa_M R)}\right]}\right\} \tag{5.378}$$

where subscript A refers to the absorber and subscript M to the water; the functions $F_1(\rho)$ and $F_2(\rho)$ have been chosen appropriately for cylindrical geometry. We discard the term $\ln \rho$ in (5.378) inasmuch as it is singular at the axis of the cylinder (i.e., we take $c_2 \equiv 0$). The application of the boundary condition (5.377) yields, finally,

$$T(\rho) - T_0 = \frac{\beta\phi_0}{\mathfrak{k}\varkappa_A^2}\left[1 + \frac{D_A\varkappa_A I_1(\varkappa_A R)K_0(\varkappa_M R)}{D_M\varkappa_M I_0(\varkappa_A R)K_1(\varkappa_M R)}\right]^{-1}\left[1 - \frac{I_0(\varkappa_A\rho)}{I_0(\varkappa_A R)}\right] \tag{5.379}$$

In this result, we use

$$\beta \equiv \varepsilon\Sigma_f^{(A)} \tag{5.380}$$

where $\Sigma_f^{(A)}$ is the macroscopic fission cross section of the material of the

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absorber and ε is the energy release per fission (7.53×10^{-12} calories[1]). Of course, the usual experimental situation cannot be represented by any such simple model; besides the commonly met geometrical and material complications in engineering heat-flow problems, there is the additional burden of the form of $\phi(\mathbf{r})$, which may be known only by numerical calculation [as, for example, in the first-flight transport model for the solid cylinder, Eq. (5.329)]. A further, extreme, complication may arise if the neutron properties of the material of the absorber (for example, the density) depend on the temperature; then the neutron and heat-flow problems are interdependent and some sort of iterative technique may be needed to obtain a solution.

PROBLEMS

5.1 Consider a sphere of radius R which has at its center a point source which is emitting neutrons of one velocity at a rate q_0 neutrons per unit time.

a. Find the flux distribution within the sphere, using the boundary condition that no neutrons return from the surrounding vacuum (evaluate all constants).

b. Show that the flux computed by *a* vanishes at approximately $r = R + 2D$.

c. Find the flux distribution within the sphere, using the boundary condition that $\phi(r)$ vanishes at $r = R + 2D$.

d. Compare the two expressions obtained from *a* and *c* for $\phi(r)$, and show that the expression of part *c* is approximately equal to that of part *a* when D/L and $L/R \ll 1$.

e. Using $D/L = \frac{1}{2}$ and $R/L = 2$, compute the net current of neutrons at the surface of the sphere for cases *a* and *c*, and find

$$\Delta \equiv \frac{J^{(c)} - J^{(a)}}{J^{(a)}}$$

f. Compute the absorptions in the extrapolated region of the sphere, using $\phi(r)$ from part *c*, and compare this number to the total absorptions in the sphere. Show that the ratio of absorptions in the extrapolated region to the total absorption is very small compared with unity when D/L and $L/R \ll 1$.

5.2 If radiation from infinitely removed sources is allowed to pass through a sphere of radius R, it may be shown directly that the average chord length of radiation path in the sphere is given by $\bar{s} = \frac{4}{3}R$.

Consider an infinite medium in which there is a spatially constant neutron flux; introduce a spherical cavity of radius R and use the theorem stated above, along with the definition of neutron flux (i.e., the flux is the neutron track length per unit volume per unit time), to show that the first term in the expression for the partial current in terms of the flux is $\phi/4$.

5.3 Consider an infinite slab (width $2a$) of uniform material described by the one-velocity constants Σ_a and D.

a. Determine the general expression for the flux distribution throughout the slab arising from a continuous distribution of plane isotropic sources defined by $S(x) = S_0 \cos(\pi x/2a)$.

b. The general requirement for criticality of a reactor is that the total number of neutrons in the reactor in each generation be constant. Show that the application of this definition to the slab reactor described above yields the correct statement of the

[1] Based on 197 Mev per fission (see Table 1.1).

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criticality equation for this configuration. It should be noted that this result demonstrates that a flux distribution which is proportional to the source distribution is equivalent to a system of uniformly distributed fissionable material.

5.4 A point source is emitting thermal neutrons isotropically at a rate q_0 neutrons per unit time. A thermal neutron detector is placed at a distance l from the point source. With the distance l held fixed, a multiplying spherical shell (inner radius R_1, outer radius R_2 and $R_2/l \ll 1$) is placed around the source so that the source is the geometric center of the shell. Find the ratio of the counting rates at the detector, with and without the multiplying shell, under the following assumptions:

1. Diffusion theory holds in the material of the shell; the spherical hole and the space outside the shell are taken to be void of material.

2. The material of the shell is such that all fissions are caused by thermal neutrons; fast neutrons from fission become thermal with the same spatial distribution as that in which they were born as fast neutrons. However, in slowing down to thermal there are losses by absorption and leakage.

3. With the spherical-shell geometry, the assembly would not operate in steady state without the source; however, the composition of the multiplying shell is such that $g_{th}k_\infty > 1$ (where g_{th} is the probability that a fast neutron will not escape from the shell before it reaches thermal).

5.5 A spherical shell of material has inner radius R_1 and outer radius R_2. (There is a vacuum both inside and outside the spherical shell.) The material of the shell is described by neutron cross sections Σ_a, Σ_f, and Σ_{tr} (one-velocity model).

a. Find the spatial distribution of the steady-state neutron flux.

b. Find the critical equation.

Assume that the neutron flux satisfies the one-velocity diffusion equation in the material of the shell and that appropriate boundary conditions are to be applied.

5.6 Consider a sphere of radius R composed of material with the one-velocity constants D, Σ_a, and Σ_f. The fissionable material is distributed uniformly throughout the sphere. At the center of the sphere is an isotropic source of neutrons of strength q_0 neutrons per unit time. Take as the source term $S(r) = k_\infty \Sigma_a \phi(r)$, and use the boundary condition that the flux vanish at the extrapolation radius $\tilde{R}$.

a. Find the general expression for the flux distribution $\phi(r)$ in terms of the parameter B, where

$$B^2 \equiv \frac{1}{L^2}(k_\infty - 1)$$

Note that the flux may have a singularity as $r \to 0$ because of the uniqueness of the center point of the sphere.

b. Determine the expression for $\phi(r)$ in the following special cases:

$$B^2 = \left(\frac{\pi}{\tilde{R}}\right)^2 \qquad B^2 = 0 \qquad k_\infty = 0$$

c. Describe the physical situation represented by the values of B^2 listed below. Indicate whether the system would be supercritical, critical, or subcritical on the basis of the fission sources alone.

$$B^2 > \left(\frac{\pi}{\tilde{R}}\right)^2 \qquad 0 < B^2 < \left(\frac{\pi}{\tilde{R}}\right)^2 \qquad -L^{-2} < B^2 < 0$$

d. Sketch the family of functions given by $\phi(r)/(q_0/4\pi D)$ in terms of the parameter B. In particular, pick out the three special functions described in part *b*.

5.7 What fraction of all neutrons produced per unit time in a finite, critical, one velocity, cylindrical, bare reactor escape across the curved surface?

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5.8 A fixed volume V of a homogeneous multiplying material is to be put into the form of a right circular cylinder.

a. Find the condition on the height h (in terms of the infinite multiplication constant k_∞, the diffusion length L, and the volume V of the material) such that the cylinder be subcritical.

b. Discuss briefly the physical meaning of results in part *a* with the aid of a graph of k versus the height h.

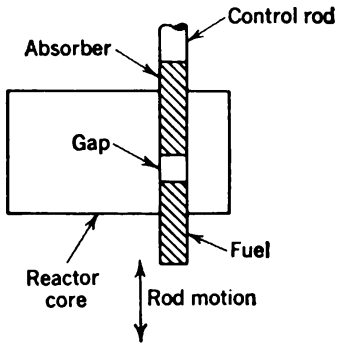
5.9 Find the fuel distribution and the corresponding flux that give a constant power distribution in an infinite-slab reactor. Assume (1) a bare one-velocity reactor, (2) the extrapolation boundary condition, and (3) that the diffusion coefficient is unaffected by local variations in fuel concentration.

5.10 The one-velocity neutron parameters Σ_a and D are constant over an infinite medium. Outside a spherical region of radius R, the absorption cross section is made up solely of radiative-capture cross section; inside the spherical region, the absorption cross section has the same value as outside but is made up of fission cross section as well as radiative capture. For every neutron which is absorbed inside the spherical region, $\nu\Sigma_f/\Sigma_a$ neutrons are born from fission; there are no other sources of neutrons.

a. Find the one-velocity neutron flux both for the regions inside and outside the active core. Give the condition for criticality.

b. Set up the integral equation satisfied by the neutron flux, using as a kernel the solution of the diffusion equation for a point source in an infinite medium. Show that the solution obtained in part *a* satisfies this integral equation.

5.11 A control rod has been designed for an MTR-type reactor in which the lower portion of the rod consists of fuel and the upper portion of high thermal absorber (nonfuel). The two sections of the rod are separated by a gap as shown below. When the control rod is inserted, the gap in the rod will be occupied by water (moderator). The problem is to determine whether the presence of this cavity of pure moderator in the core will cause excessive peaking of the thermal flux (and therefore increased



Note: Reactor core consists of fuel plates immersed in water

energy deposition) in the vicinity. Examine this problem on the basis of the following model: (1) represent the core (moderator plus fuel) by an infinite homogeneous medium and the gap by a spherical cavity of radius a which contains only moderator; (2) assume that the one-velocity approximation applies for the thermal flux; and (3) let the source of thermal neutrons be uniform throughout the core and cavity regions and take it to be some fixed number S neutrons/time/vol.

a. What are the differential equations and boundary conditions which apply for the thermal flux?

b. Write down the general solution for the thermal-flux distribution throughout the entire space.

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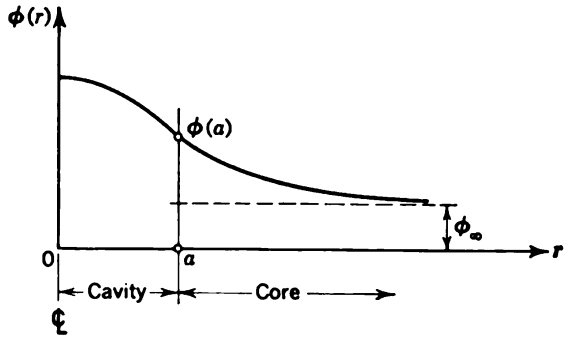
c. Apply the boundary conditions to eliminate one of the arbitrary constants in the general solution for each region. How would you evaluate the others?

d. If all the constants are evaluated (not required here), one can show that the flux has the form (Σ_a moderator $< \Sigma_a$ core) shown in the sketch.



Determine the effect of the cavity by computing the ratio $\phi(a)/\phi_\infty$ (allow only one arbitrary constant in the final result). Note that this ratio is a measure of the increase in ϕ at the gap boundaries over the flux computed without the cavity.

5.12 Using the cross sections and assumptions of Prob. 4.8, calculate the critical radius of a spherical homogeneous reactor for five ratios of moderator to fuel: N_M/N_F = 100, 500, 1,000, 5,000, and 10,000. Assume that neutrons are born at E_0 and slow down, reaching thermal energy at the same space point at which they were born (i.e., no fast leakage). Calculate also

a. The nonleakage probability for thermal neutrons p_{NL}.

b. The critical mass (and plot versus moderator-fuel ratio).

c. Mean lifetime of a neutron in the reactor.

Do *not* neglect extrapolation distance, work to three-figure accuracy, and take the moderator density to be 1.65 g/cm³.

5.13 The feasibility of a gaseous reactor can be examined on the basis of a relatively simple model. The present problem is to determine the important features and dimensions of such a system on the basis of some reasonable specifications. For this purpose the following simplified model will suffice:

1. The reactor is a bare gaseous sphere of radius R_0.
2. The reactor is critical and at steady state.
3. Energy sources arise from fission reactions only.
4. The outer boundary of the sphere is at absolute temperature $T = T(R_0) \equiv T_0$.
5. The gas mixture is a stagnant system at some fixed pressure p.
6. Energy losses in the gas are due to conduction only; therefore, neglect radiation, convection, and gravitational forces.
7. One-velocity diffusion equation gives an adequate representation of the neutron physics.
8. The extrapolation boundary condition is applicable.
9. The diffusion coefficient D is spatially invariant (assume some suitable average value for the mixture).
10. The coefficient of thermal conduction can be represented by some suitable average value $\mathfrak{k}$.

The problem is to find the flux, temperature, and nuclear-density distributions in this reactor and its power-generating capacity as a function of the reactor size R_0, the surface temperature T_0, the pressure p, and the various nuclear and physical constants of the gas.

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Part A. General Solution

1. Set up the appropriate equations which describe the neutron-balance, energy-balance, and the equation of state for the gas mixture which consists of a nonfuel and a fuel component (fissionable material). List all boundary conditions.

2. Show that the neutron-balance and energy-balance equations can be reduced to the form

Neutron:
$$\nabla^2\phi(r) + \frac{\phi(r)\alpha^2}{T(r)} = 0 \tag{P5.1}$$

Energy:
$$\nabla^2 T(r) + \frac{\phi(r)\beta^2}{T(r)} = 0 \tag{P5.2}$$

where
$$\alpha^2 \equiv \frac{p}{kD(n+1)}\{\nu\sigma_f - [n\sigma_a^{(O)} + \sigma_a^{(F)}]\} \tag{P5.3}$$

$$\beta^2 \equiv \frac{p\varepsilon\sigma_f}{kk(n+1)} \tag{P5.4}$$

and $\phi(r)$ is the flux, $T(r)$ the absolute temperature, k Boltzmann's constant, ε the energy release per fission, and $n \equiv N_O/N_F$ (the ratio of concentration of nonfuel to fuel). Let

$$\mu \equiv \frac{akT_0}{R_0 p\sigma_s} \qquad \tilde{R}_0 \equiv R_0 + \epsilon \qquad \epsilon \equiv a\lambda_{tr} = \frac{a}{\sigma_s N(R_0)} \tag{P5.5}$$

Here, $N(r)$ denotes the total number of nuclei (of all kinds) in unit volume at r, σ_s is a suitable microscopic scattering cross section (may be approximated by nonfuel components), and a is some constant (say 0.7104).

3. With these definitions,

a. Show that $\tilde{R}_0 = R_0(1+\mu)$

b. Find the criticality condition

c. Determine the temperature function and show that it may be written ($x \equiv r/R_0$)

$$T(x) \equiv \frac{T_0}{\mu(2+\mu)}[(1+\mu)^2 - x^2] \tag{P5.6}$$

d. Find also the expressions for the flux $\phi(x)$ and the nuclear density $N(x)$, eliminating all arbitrary constants by means of the boundary conditions.

4. Sketch the functions $T(x)$, $\phi(x)$, and $N(x)$.

a. What is the condition on μ if the diffusion equation is to yield a reasonable representation of the neutron physics?

b. Find an approximate expression for the temperature at the center of the sphere $T(0)$ which is valid when $\mu \ll 1$.

5. Critical fuel concentration. Let $D \equiv \lambda_{tr}/3 \equiv 1/3\sigma_s N_{av}$ with $N_{av} \equiv p/kT_{av}$ and $T_{av} \equiv bT_0$. Thus we compute a diffusion coefficient in terms of some average temperature of the system T_{av} which is defined in terms of T_0. If we call

$$\theta \equiv \frac{2b\sigma_s}{a^2\sigma_a^{(F)}} \tag{P5.7}$$

then

a. Show that the critical fuel concentration ratio n is

$$n = n(\mu;\theta) = \frac{(2+\mu)(\eta-1) - \theta\mu}{(2+\mu)\sigma_a^{(O)}/\sigma_a^{(F)} + \theta\mu} \tag{P5.8}$$

where $\eta \equiv \nu\sigma_f/\sigma_a^{(F)}$.

b. Sketch n as a function of μ for various θ.

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6. Derive expressions for the total power output of the reactor by two separate approaches.

Part B. Computations

7. Assume that a gaseous reactor is a sphere of the type analyzed in Part A, and let the material be UF_6 so that the fluorine acts as the moderator. Find the critical enrichment of U^{235}.

a. Derive expressions for σ_s and $\sigma_a^{(O)}$ on the premise that the U^{235} enrichment is negligibly low.

b. Compute these cross sections using the following data:

	σ_s	σ_a	σ_f
U^{238}	8.2*b*	2.75*b*	0
F^{19}	3.9	0.01	0
U^{235}	10.0	687	580

8. Compute μ, $T(0)$, θ, n and the enrichment N_{235}/N_{238} for the critical system, using $T_0 = 1000°K$, $R_0 = 10$ m, and $p = 100$ atm. Use $\mathfrak{k} = 1{,}260$ ergs/(cm)(sec)(°C), $\varepsilon = 3.2 \times 10^{-4}$ ergs/fission, $a = 0.7$, and $b = 2.5$.

9. Compute the total power output of the reactor.

10. Compute and plot $\phi(x)$, $N(x)$, and $T(x)$.

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CHAPTER 6

FERMI AGE THEORY: THE HOMOGENEOUS MULTIVELOCITY REACTOR

6.1 The Combined Slowing-down and Diffusion Equation

a. Assumptions and Limitations. The two elementary models developed in the preceding chapters allowed us a limited picture of the neutron population in a nuclear reactor. The first model, which was considered in Chap. 4, confined attention to the slowing-down process, and all calculations were based on the infinite-medium concept. The second model was developed in Chap. 5, and in this approach the energy dependence of the neutron flux was removed by the introduction of the one-velocity approximation. The use of this model allowed attention to be focused on the diffusion process, and the application of the model to the determination of the spatial distribution of neutrons in a finite system was demonstrated.

In an actual reactor the two phenomena slowing down and diffusion cannot in fact be separated if a complete description of the neutron population is to be achieved. As we previously noted, the separation was made in the preceding work so as to simplify the presentation by limiting the number of analytical methods and physical concepts considered at any one time. It should be recognized, however, that although each model may give a poor representation of the general reactor problem, there are numerous special situations in which either the slowing-down phenomenon or neutron diffusion is the essential feature, and a direct application with minor adjustments can be made. This was the case when the one-velocity diffusion model was used to perform a reactor calculation. In this application it was assumed that the system was highly thermal so that the one-velocity model gave a good description of the spatial distribution of the thermal flux.

The purpose of the present chapter is to develop a more general model for describing the neutron population in a reactor, a model which combines the principal features of the two already presented. That this is indeed a prerequisite for treating the general reactor problem may be

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appreciated from a consideration of the various physical phenomena which occur in a chain-reacting system. In an actual reactor the neutron sources are the fission reactions which take place throughout the fuel component of the core. The neutrons produced by fission are distributed over the entire energy range (see Fig. 4.24) and have an average energy in the order of 2 Mev. These fast neutrons from fission are subsequently slowed down by various scattering collisions with the nuclei of the reactor medium. This sequence of scattering collisions, which results in a continual degradation of the neutron energy, is accompanied by a spatial wandering as the neutron progresses from one scattering center to the next. Thus intimately associated with the slowing-down process is a spatial movement or "diffusion."

The present diffusion process differs somewhat, however, from the previous definition of the word. The previous definition referred to the spatial wanderings of a particle which suffered little, if any, change in its speed as it passed from one collision to another. It was first used to describe the motion of thermal neutrons. It will be useful in the work which follows to extend this concept to include the motion of the fast neutrons as well. By limiting our study to systems in which the slowing down is relatively gradual, we can use the idea of a diffusion process to describe the motion of fast neutrons in any small interval of energy. This would be the case in reactor systems which consist primarily of the heavier nuclear-mass materials. For these materials the energy change per collision is relatively small so that a given neutron will suffer many collisions in a specified small-energy band. Note that this extension of the diffusion concept is in accord with the essential feature of the thermal diffusion process, namely, that the neutron experiences many small-energy change collisions while in the thermal range.

The combined process of slowing down and diffusion is correctly treated by means of the Boltzmann equation, which takes into account the angular distribution of the neutron population as well as its energy and spatial dependence. This more general approach will not be applied in the present analysis but will be deferred until the next chapter. We will use instead a less rigorous derivation of the combined slowing-down diffusion relations which we will later show to be a first approximation to the Boltzmann equation. As in the development of the one-velocity model of Chap. 5, the present model will possess all the pertinent features which are allowed by the various approximations and constraints imposed on the physical system, but we will be unable at this time to demonstrate all the motivations or justifications for establishing these conditions. Thus we begin our analysis by defining a set of assumptions which will form the basis of the neutron-balance relation. All the analyses which follow, then, as well as each system to which the general equations are applied, must satisfy the conditions given below:

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(1) The medium is isotropic and homogeneous
(2) The macroscopic absorption cross section of the medium is much less than the macroscopic scattering cross section over large energy-intervals
(3) The general dimensions of the system are much larger than characteristic neutron lengths such as the diffusion length
(4) The regions within the reactor which are to be described by the combined slowing-down diffusion model are free of localized neutron sources and sinks
(5) These regions do not contain large voids
(6) Neutron scattering is isotropic in the center-of-mass system (C)
(7) The mass number of the moderator is sensibly larger than unity

(6.1)

Except for statements (6) and (7) we offer no explanation in support of this particular choice of assumptions. Statements (1) through (5) will be justified in Chap. 7 wherein the Boltzmann equation is reduced to the present model. It is appropriate to remark on the validity and usefulness of assumptions (6) and (7) here inasmuch as we will use the continuous slowing-down model to describe the trajectory of the fast neutrons in energy space. This model was introduced in Sec. 4.2b in connection with the slowing-down process in an infinite medium. It will be recalled from that application that the actual trajectory in energy space is a discontinuous function; however, if the decrements in energy per collision are small, the slowing-down track may be approximated by means of a continuous function (see Fig. 4.11). It is clear that this would be a good approximation to the physical system only if the slowing-down medium consists primarily of large nuclear-mass materials. Thus in the present study we exclude hydrogenous substances. Furthermore, the requirement that $A \gg 1$ will also tend to satisfy condition (6) [see Eq. (4.30)].

It is known from experiment that the scattering in the (C) system is nearly isotropic for materials with large A, provided that the neutron speeds are in the ev to Mev range. Below the ev range, thermal effects become important; above the Mev range, scatterings tend to be forward. In the work which follows we will assume that all collisions are isotropic in (C) and moreover are purely elastic. We will not consider inelastic collisions in the present treatment in the interests of simplicity. Actually, those applications of the combined slowing-down diffusion model which are most useful involve highly thermal systems. Such systems include relatively small nuclear-mass materials and, except for the structural members within the reactor, do not serve as a significant source of inelastic scatterings (see Sec. 4.1a).

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b. Neutron Balance in Space and Lethargy. We begin our study of the slowing-down diffusion model by establishing the neutron-balance condition in terms of the lethargy and space variables. For this purpose we introduce the following two functions:

$$\phi(\mathbf{r},u)\,du \equiv \begin{array}{l}\text{track length per unit volume per unit time around}\\ \mathbf{r} \text{ due to neutrons in that volume whose lethargies}\\ \text{lie in } du \text{ about } u\end{array}$$

$$q(\mathbf{r},u) \equiv \begin{array}{l}\text{number of neutrons slowing down past lethargy } u\\ \text{per unit time per unit volume around } \mathbf{r}\end{array}$$

Consider a small volume $d\mathbf{r}$ of material in a medium supporting a multivelocity neutron flux at steady state, and let us confine attention to those neutrons whose lethargies lie in du about u. It is possible to write an expression for the neutron balance in the differential volume $d\mathbf{r}\,du$ in the four-dimensional space defined by the coordinates $\mathbf{r}$ and u (see Fig. 6.1). This balance is obtained by accounting for all the processes which can add or remove a neutron from either $d\mathbf{r}$ or du. At steady state, neutron gains into $d\mathbf{r}\,du$ consist of (1) neutrons which slow past lethargy u, thus entering du from lower lethargies, and (2) neutrons with lethargies in du about u which appear directly from sources in $d\mathbf{r}$. Neutron losses are due to (3) neutrons with lethargy in du which are scattered while in $d\mathbf{r}$ and thereby slowed down out of du, (4) neutrons which are absorbed by nuclei while in $d\mathbf{r}\,du$, and (5) neutrons of lethargy in du which escape across the boundaries of $d\mathbf{r}$.

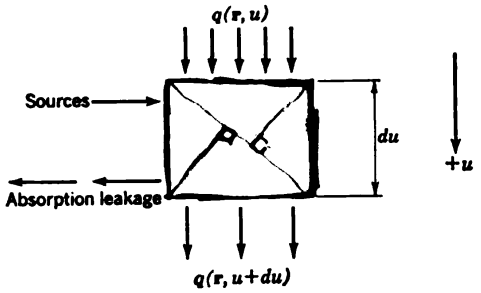

FIG. 6.1 Differential volume $d\mathbf{r}\,du$.

By calculating each of the above quantities on the basis of events per unit time, we may write the balance equation as follows:

$$q(\mathbf{r},u)\,d\mathbf{r} + S(\mathbf{r},u)\,d\mathbf{r}\,du = q(\mathbf{r},u+du)\,d\mathbf{r} + \Sigma_a(u)\phi(\mathbf{r},u)\,d\mathbf{r}\,du + \mathfrak{L}(\mathbf{r},u)\,d\mathbf{r}\,du \quad (6.2)$$

where $S(\mathbf{r},u)\,du \equiv$ number of neutrons with lethargies in du about u which are born per unit time per unit volume around $\mathbf{r}$ from sources

$\mathfrak{L}(\mathbf{r},u)\,du \equiv$ number of neutrons with lethargies in du about u which escape per unit time from unit volume around $\mathbf{r}$

All the terms in Eq. (6.2) except the last are readily specified by means of the concepts and functions already developed. The leakage term $\mathfrak{L}$ causes some difficulty in the present formulation and can be properly

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identified only through the use of the Boltzmann relation. This complication is similar to that encountered in selecting a suitable leakage term for the one-velocity model (see Sec. 5.1c). In fact, it is from the one-velocity result that a hint is obtained as to the form to be chosen for the leakage in the multivelocity case. If the system were such that neutrons spent a great deal of time in the lethargy interval du and made many scattering collisions before being reduced to higher lethargies (lower energies), there would be some justification for treating all the neutrons with lethargies in du as a group of diffusing neutrons of a single speed. Then from results such as Eq. (5.51b) for the one-velocity case, it might be assumed that the net current of neutrons in du and $d\mathbf{r}$ around $\mathbf{r}$ would be proportional to the gradient of the neutron density. Further, the constant of proportionality would be identified as in (5.51c) so that, finally, we could write for the leakage

$$\mathfrak{L}(\mathbf{r},u) \equiv -D(u)\,\nabla^2\phi(\mathbf{r},u) \tag{6.3}$$

This relation can, in any case, be made as an assumption which is to be tested by application; however, the approach here indicates that the assumed form (6.3) would be especially applicable to media composed chiefly of heavy nuclei for which the average gain in lethargy per collision would be small, and, consequently, the neutrons would make many collisions in the neighborhood of a given lethargy (i.e., the scattering process in du could be described as a "diffusion").

If we use (6.3) in (6.2), along with the identification

$$\lim_{du\to 0}\,[q(\mathbf{r},u+du) - q(\mathbf{r},u)] = \frac{\partial}{\partial u}\,q(\mathbf{r},u)\,du \tag{6.4}$$

Then (6.2) may be written

$$-D(u)\,\nabla^2\phi(\mathbf{r},u) + \Sigma_a(u)\,\phi(\mathbf{r},u) = S(\mathbf{r},u) - \frac{\partial}{\partial u}\,q(\mathbf{r},u) \tag{6.5}\star$$

It is reasonable to assume, under the same conditions, that the relationship between the flux and the slowing-down density obtained for the infinite medium in (4.112) would be a good approximation. Thus we write

$$q(\mathbf{r},u) = \xi\Sigma_s(u)\,\phi(\mathbf{r},u) \tag{6.6}\star$$

In treating the interval du as infinitesimal, we are requiring for the validity of (6.4) that the neutrons make infinitely many collisions in passing through a finite lethargy interval. In the limit this implies the "continuous slowing-down" model suggested in Sec. 6.1a; thus, neutrons take on every value of lethargy in traversing the interval du, since in the limit the average gain per collision must be zero. In an actual case, of course, the slowing-down process is not continuous; a given neutron takes on only a limited number of lethargy values in slowing from fission energy to thermal. For a small (but finite) lethargy interval, it may be

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true that most neutrons which slow down through that interval never take on lethargies in that interval. From this viewpoint it is rather surprising that (6.5) gives good results in application. That the result (6.5) is justifiable to a given approximation by consideration of the energy-dependent Boltzmann equation indicates that (6.5) is useful because we are usually dealing with very large numbers of neutrons. Thus while the slowing down and diffusion of any one neutron are by jumps, the behavior of the total neutron population closely approximates that of a continuous flow (in space and lethargy).

We conclude the present remarks on the combined slowing-down diffusion model with the reminder that the set of equations (6.5) and (6.6) are to be applied to those situations which satisfy the conditions (6.1). Equations (6.5) and (6.6), along with appropriate boundary conditions, have been used in situations wherein one or more of these requirements have been violated. Although good results can be obtained in some such instances, usually requiring considerable modification to the original equations, such applications generally yield erroneous results and must be used with caution.

6.2 Fermi Age Theory and Elementary Solutions

a. The Fermi Age Equation. The purpose of the present section is to demonstrate the reduction of the combined slowing-down diffusion equation (6.5) to the Fermi age relation (which is mathematically identical to the standard form of the heat-conduction equation) and to display the solutions of this equation for several elementary source distributions. The reduced equation will be given in terms of the spatial coordinates of the position vector $\mathbf{r}$ and the lethargy u. It will be convenient in carrying out this derivation to treat simultaneously the analytically equivalent problem of the time-dependent one-velocity flux equation which is the generalization of the steady-state relations (5.51). The time-dependent equation describes the distribution of neutrons in space and time, whereas the slowing-down relations give the distribution in space and lethargy. However, the lethargy and time variables are mathematically similar quantities, each being defined for the real, positive range. Furthermore, the neutron-balance relations for these two situations are mathematically identical. Thus the two problems may be easily treated together.

The slowing-down diffusion relations which are to be reduced are given by (6.5) and (6.6). The corresponding relations for the time-dependent one-velocity flux distribution are

$$-D\nabla^2\phi(\mathbf{r},t) + \Sigma_a\phi(\mathbf{r},t) = -\frac{\partial}{\partial t}n(\mathbf{r},t) \tag{6.7}$$

$$\phi(\mathbf{r},t) = vn(\mathbf{r},t) \tag{6.8}$$

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The neutron-absorption cross section Σ_a and the diffusion coefficient D are defined for the one-velocity v specified by the problem. The function $n(\mathbf{r},t)$ is the corresponding neutron density and is expressed in neutrons per unit volume. The two differential equations (6.5) and (6.7) may be written in terms of a single function by applying the coupling relations (6.6) and (6.8). If we omit the source term, the results are

$$-\frac{D(u)}{\xi\Sigma_t(u)}\nabla^2 q(\mathbf{r},u) + \frac{\Sigma_a(u)}{\xi\Sigma_t(u)} q(\mathbf{r},u) = -\frac{\partial}{\partial u} q(\mathbf{r},u) \tag{6.9}$$

$$-Dv\nabla^2 n(\mathbf{r},t) + \Sigma_a v n(\mathbf{r},t) = -\frac{\partial}{\partial t} n(\mathbf{r},t) \tag{6.10}$$

We eliminate the linear terms in these equations by introducing integrating factors in the variables u and t, respectively, i.e., by defining two new functions $Q(\mathbf{r},u)$ and $N(\mathbf{r},t)$ which are related to $q(\mathbf{r},u)$ and $n(\mathbf{r},t)$ by

$$q(\mathbf{r},u) \equiv Q(\mathbf{r},u) \exp\left[-\int_0^u \frac{\Sigma_a(u')\,du'}{\xi\Sigma_t(u')}\right] \tag{6.11}$$

$$n(\mathbf{r},t) \equiv N(\mathbf{r},t) \exp\left[-\int_0^t \Sigma_a v\,dt'\right] \tag{6.12}$$

Note that $q(\mathbf{r},u)$ is the slowing-down density for a system which includes absorption represented by the cross section Σ_a and that the exponential

$$\exp\left[-\int_0^u \frac{\Sigma_a(u')\,du'}{\xi\Sigma_t(u')}\right] \equiv p(u)$$

is the resonance-escape probability for the lethargy interval $(0,u)$ according to the previous definition (4.113); thus $Q(\mathbf{r},u)$ must be the slowing-down density for this system in the special case that $\Sigma_a(u) = 0$. We observe, furthermore, that $n(\mathbf{r},t)$ is the neutron density for a system with absorption and

$$\exp\left[-\int_0^t \Sigma_a v\,dt'\right] = e^{-\Sigma_a vt}$$

is the fraction of neutrons which traverse the path vt without being absorbed. If $\Sigma_a = 0$, $n(\mathbf{r},t) \equiv N(\mathbf{r},t)$; thus we identify the function $N(\mathbf{r},t)$ as the neutron density for the same system with the absorption cross section zero.

The substitution of (6.11) and (6.12) into (6.9) and (6.10) yields

$$\frac{D(u)}{\xi\Sigma_t(u)}\nabla^2 Q(\mathbf{r},u) = \frac{\partial}{\partial u} Q(\mathbf{r},u) \tag{6.13}$$

$$Dv\nabla^2 N(\mathbf{r},t) = \frac{\partial}{\partial t} N(\mathbf{r},t) \tag{6.14}$$

These equations reduce to the standard form of the heat-conduction equa-

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tion by changing from the variable u to τ, a function of u, by

$$\frac{d\tau}{du} \equiv \frac{D(u)}{\xi\Sigma_t(u)} \qquad \tau(0) = 0 \tag{6.15}\star$$

and from t to the function η of t by

$$\frac{d\eta}{dt} \equiv Dv \tag{6.16}\star$$

For the transformation $u \to \tau$ we write $Q(\mathbf{r},u) \to Q(\mathbf{r},\tau)$, and for $t \to \eta$, $N(\mathbf{r},t) \to N(\mathbf{r},\eta)$. (This is a commonly used notational convenience which, if misunderstood, can lead to mischief.) The application of these transformations to (6.13) and (6.14) gives the relations

$$\nabla^2 Q(\mathbf{r},\tau) = \frac{\partial}{\partial\tau} Q(\mathbf{r},\tau) \tag{6.17}\star$$

$$\nabla^2 N(\mathbf{r},\eta) = \frac{\partial}{\partial\eta} N(\mathbf{r},\eta) \tag{6.18}\star$$

which are in the standard form of the heat-conduction equation. The first of these equations (6.17) is called the *Fermi age* equation; the variable τ is called the Fermi age and will be discussed in greater detail later in this section. Each of the above relations is expressed in terms of the spatial coordinates given by $\mathbf{r}$ and a timelike variable τ or η. The solutions to this class of partial differential equations are well known in the case of several elementary source distributions. However, in order to provide a greater insight into the physical situations involved, we will develop in detail the solutions to the slowing-down problem for the cases of the plane and line-source distributions. The corresponding solution to the point-source problem is easily obtained by the same methods. These methods can also be applied to the three elementary solutions to the time-dependent problem.

b. Elementary Sources in Infinite Media. Our purpose here is to display the solutions to Eqs. (6.17) and (6.18) for the three elementary-source distributions, namely, the plane, the line, and the point. In the case of the slowing-down problem, we define the physical system by specifying the "initial condition" that neutrons of lethargy $u = 0$ are released steadily according to some known spatial distribution. These high-energy source neutrons subsequently enter a slowing-down process and wander through the system until they are absorbed. The problem is to determine the spatial distribution of these neutrons for all $u > 0$; that is, we should like to know how the neutrons spread out through the medium as they progress from collision to collision, each time losing a fraction of their energy. The corresponding space-time problem defined by (6.18) gives the distribution of neutrons which first appear in the medium from prescribed sources according to some "initial condition,"

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beginning at say $t = 0$. This condition gives the number of neutrons released into the medium at that time. The solution to (6.18) which satisfies this initial condition describes how these neutrons are distributed throughout the medium for all subsequent time.

The calculation to be performed first is the determination of the slowing-down density of neutrons that originate from a plane isotropic source. We specify that the source emits neutrons of lethargy zero (corresponding to some high energy) at the rate q_0 neutrons per unit area per unit time. This problem is clearly one-dimensional, and, for convenience, we place the source plane at the origin of the x axis. The appropriate differential equation is obtained from (6.17).

$$\frac{\partial^2}{\partial x^2} Q(x,\tau) = \frac{\partial}{\partial \tau} Q(x,\tau) \qquad x \neq 0,\ \tau > 0 \qquad -\infty < x < \infty \tag{6.19}$$

The conditions which describe the source are

(1) q_0 neutrons per unit area per unit time appear at $x = 0, \tau = 0$
(2) No sources other than those at $x = 0$ (6.20)
(3) $\int_0^\infty Q(x,\tau)\, dx = \frac{1}{2} q_0 \qquad Q(-x,\tau) = Q(x,\tau) \qquad \tau > 0$

Note that condition (3) follows immediately from the definition of Q as the slowing-down density in the absorber-free medium. The actual slowing-down density is obtained from the relations

$$q(x,\tau) = Q(x,\tau)\, p(u) \qquad \text{with } \tau = \tau(u) \tag{6.21}$$

as given by (6.11) and (6.15).

The differential equation (6.19) may be reduced to a more elementary form by applying a Laplace transform $\mathcal{L}\{\ \}$ to the Fermi age variable τ. We define

$$\Theta(x,s) \equiv \mathcal{L}\{Q(x,\tau)\} \equiv \int_0^\infty e^{-\tau s} Q(x,\tau)\, d\tau \tag{6.22}$$

where the transform variable s is real and positive. The initial conditions (1) and (2) require that for all $x \neq 0$

$$Q(x, +0) \equiv \lim_{\substack{\tau \to 0 \\ \tau > 0}} Q(x,\tau) = 0 \tag{6.23}$$

Thus the Laplace transform of (6.19) is

$$\frac{\partial^2}{\partial x^2} \Theta(x,s) - s\, \Theta(x,s) = 0 \tag{6.24}$$

which has the general solution

$$\Theta(x,s) = A(s) \exp(-x\sqrt{s}) + B(s) \exp(x\sqrt{s}) \tag{6.25}$$

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We have left the arbitrary constants A and B as general functions of the transform variable s inasmuch as s appears as a parameter in (6.24). These constants are evaluated by the application of condition (3), the Laplace transform of which is

$$\int_0^\infty \Theta(x,s)\,dx = \frac{q_0}{2s} \qquad \Theta(-x,s) = \Theta(x,s) \tag{6.26}$$

The substitution of (6.25) into this integral requires that

$$A(s)\int_0^\infty \exp(-x\sqrt{s})\,dx + B(s)\int_0^\infty \exp(x\sqrt{s})\,dx = \frac{q_0}{2s}$$

When x is positive, $B(s)$ must be taken identically zero in order that the

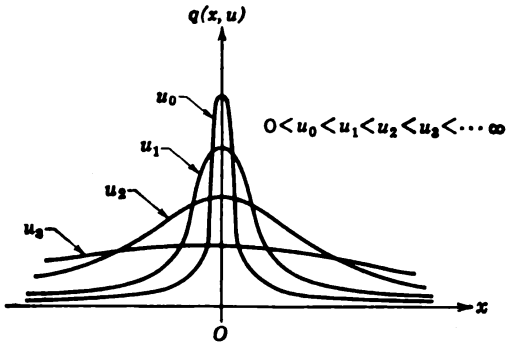

FIG. 6.2 Slowing-down density for a plane source.

slowing-down density remain finite (for all positive values of x). With this choice the above relation yields the result $A(s) = q_0/2\sqrt{s}$, and (6.25) may be written

$$\Theta(x,s) = \frac{q_0 \exp(-x\sqrt{s})}{2\sqrt{s}} \qquad x > 0$$

The Laplace inversion $\mathfrak{L}^{-1}\{\Theta(x,s)\}$ of this transform yields the solution to the differential equation (6.19) which satisfies the conditions (6.20). For both positive and negative values of x the result is

$$Q(x,\tau) = \mathfrak{L}^{-1}\{\Theta(x,s)\} = \frac{q_0 e^{-x^2/4\tau}}{(4\pi\tau)^{\frac{1}{2}}} \tag{6.27}$$

Recalling (6.21), the slowing-down density with absorption is obtained from (6.27) and is

$$q(x,u) = \frac{q_0 p(u)\, e^{-x^2/4\tau(u)}}{[4\pi\tau(u)]^{\frac{1}{2}}} \tag{6.28}\star$$

In Fig. 6.2, $q(x,u)$ is plotted as a function of x for several values of u. It is helpful to remember that τ as defined by (6.15) is a monotonically increasing function of u.

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We observe from this figure that $q(x,u)$ has two important features. First, we note that for any given value of lethargy the slowing-down density is greatest at the source plane $x = 0$; and second, the effect of the resonance-escape probability $p(u)$ is always to lower the magnitude of $q(x,u)$ for given x. It is clear from the figure that the high-energy neutrons which have suffered, on the average, only a few collisions are concentrated in the vicinity of the source. This effect is in keeping with the continuous slowing-down model which allows the neutron a degradation of energy that is continuous in time as it wanders through the medium.[1] It is to be expected, then, that the neutrons of lowest energy

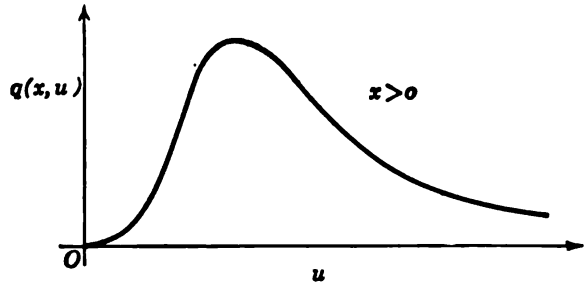

FIG. 6.3 Slowing-down density at a given space point as a function of lethargy.

will be found farthest from the source (see, for example, the curve $u = u_3$ in Fig. 6.2) and will have distributed themselves more uniformly throughout the medium. However, regardless of the neutron energy in question, the greatest density of neutrons of any given energy is at the source proper.

Consider also $q(x,u)$ as a function of the lethargy. At some space point $|x| > 0$, the slowing-down density will have the form shown in Fig. 6.3. The slowing-down density, and therefore also the flux, of high-energy neutrons (u near zero) is low because most neutrons which have migrated as far as x will have also suffered several scattering collisions and thereby had their kinetic energies reduced. The density of low-energy neutrons at x is also relatively low because of two factors: (1) neutrons of low energy have diffused far from the source and are spread out over such a large volume that their density in any given locality is low, and (2) the presence of neutron-absorbing materials causes the continual removal of neutrons from the slowing-down stream and progressively fewer neutrons are available in the lower-energy regions.

Let us examine now the analogous problem of the distribution of one-velocity neutrons in the space-time system. Consider, then, an isotropic plane source of neutrons in an infinite medium. The initial condition is that a burst of q_0 neutrons per unit area is released from the source (placed for convenience at the origin) at time, say, $t = 0$. These neutrons have speed v, and they retain this speed for all subsequent time;

[1] Actually, one also expects this to happen in the discontinuous model.

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that is, no energy loss results from collision. The differential equation which describes the distribution of the source neutrons in space and time is obtained from (6.18); thus,

$$\frac{\partial^2}{\partial x^2} N(x,\eta) = \frac{\partial}{\partial \eta} N(x,\eta) \qquad \begin{matrix} \eta > 0 \\ -\infty < x < \infty \end{matrix} \tag{6.29}$$

The conditions which must be satisfied by $N(x,\eta)$ are

$$\begin{aligned} &(1)\ q_0 \text{ neutrons per unit area at } \eta = 0,\ x = 0 \\ &(2)\ \text{No sources in the medium except at } x = 0,\ \eta = 0 \\ &(3)\ \int_0^\infty N(x,\eta)\,dx = \tfrac{1}{2}q_0 \qquad \begin{matrix} N(-x,\eta) = N(x,\eta) \\ \eta > 0 \end{matrix} \end{aligned} \tag{6.30}$$

Again we find it convenient to specify the source strength in terms of a function defined for an absorber-free medium, $N(x,\eta)$. This function is related to the actual neutron density through Eq. (6.12), namely,

$$n(x,\eta) \equiv N(x,\eta)e^{-\kappa^2\eta} \tag{6.31}$$

where $\kappa^2 \equiv \Sigma_a/D$ and $\eta = \eta(t) = Dvt$. The solution to (6.29) which satisfies conditions (6.30) can be obtained by the procedure used for the lethargy-space problem. The result is easily shown to be

$$n(x,t) = \frac{q_0 e^{-\Sigma_a vt} e^{-x^2/4Dvt}}{(4\pi Dvt)^{\frac{1}{2}}} \tag{6.32}\star$$

The x dependence of this function has a form similar to $q(x,u)$ shown in Fig. 6.2. In the present case the curves would be drawn for fixed values of time t as a parameter. At small values of t, $n(x,t)$ would show a sharply peaked form much like the shape of $q(x,u_0)$. This curve would represent the spatial distribution of neutrons soon after their release from the source plane at $x = 0$. As t increases, the density function $n(x,t)$ flattens much like the curves for u_1, u_2, etc., of Fig. 6.2, indicating that as time progresses the neutrons wander farther from the source plane and tend to spread more evenly throughout the medium. One can also sketch from Eq. (6.32) a density-time function for given x analogous to the curve of Fig. 6.3. Such a curve would show how the density at a specified station would vary as the initial neutron burst passes by. In this case the interpretation of the curve would be as follows: At short times after the burst, the neutrons have not yet had time to reach point x, and therefore the density would be low; very long after the burst, the neutron density has fallen everywhere because of spreading out and absorption. Thus from the viewpoint of an observer at x, a pulse of neutrons passes sometime after the initial burst at the source. The time at which the maximum neutron density occurs at x is

$$t = \frac{x}{2v(3D\Sigma_a)^{\frac{1}{2}}} \tag{6.33}$$

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The calculations of the neutron distributions in both the space-lethargy and space-time problems for the line- and point-source distributions are quite similar and will not be considered here in any detail. However, a few of the essential features of these two problems differ markedly from the preceding ones, and in order to demonstrate them more clearly, we will outline briefly the analysis of the lethargy problem for a line source. This system is axially symmetric, and the appropriate differential equation is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}Q(\rho,\tau)\right] = \frac{\partial}{\partial\tau}Q(\rho,\tau) \qquad \rho > 0, \tau > 0 \tag{6.34}$$

where ρ denotes the radial distance from the line source. The function $Q(\rho,\tau)$ must satisfy the conditions

(1) q_0 neutrons per unit time per unit length of source line released at $\rho = 0, \tau = 0$
(2) No sources other than at $\rho = 0, \tau = 0$, i.e., for $\rho > 0$, $Q(\rho, +0) = 0$ (6.35)
(3) $\int_V Q(\mathbf{r},\tau)\,d\mathbf{r} = q_0$

$S(\rho,\tau) = q_0 \delta(\rho)\delta(\tau)$

The volume integral specified in (3) is over a cylindrical region of unit height and infinite radial extent. Equation (6.34) may be reduced to the modified Bessel equation of order zero by the application of the Laplace transform on the age variable τ. If as before Θ is the transform of Q, then the transform of (6.34) is

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\Theta(\rho,s)\right] - s\,\Theta(\rho,s) = 0$$

and has the solution

$$\Theta(\rho,s) = A(s)\,I_0(\rho\sqrt{s}) + B(s)\,K_0(\rho\sqrt{s})$$

The coefficients $A(s)$ and $B(s)$ are obtained from condition (3). The appropriate relation is found to be

$$A(s)\int_0^\infty I_0(\rho\sqrt{s})\rho\,d\rho + B(s)\int_0^\infty K_0(\rho\sqrt{s})\rho\,d\rho = \frac{q_0}{2\pi s} \tag{6.36}$$

The first of these integrals is divergent; therefore $A(s) \equiv 0$. The value of the second integral[1] is $1/s$. Thus from (6.36), $B(s) \equiv q_0/2\pi$, and

$$\Theta(\rho,s) = \frac{q_0 K_0(\rho\sqrt{s})}{2\pi}$$

[1] See, for example, G. N. Watson, "Theory of Bessel Functions," 2d ed., p. 388, Cambridge University Press, London, 1952.

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The Laplace inversion yields the function $Q(\rho,\tau)$ from which we obtain

$$q(\rho,u) = \frac{q_0 p(u) e^{-\rho^2/4\tau(u)}}{4\pi\tau(u)} \tag{6.37}\star$$

It is easily shown that in the case of the point source at $r = 0$, and of strength q_0 neutrons per unit time, the solution is given by

$$q(r,u) = \frac{q_0 p(u) e^{-r^2/4\tau(u)}}{[4\pi\tau(u)]^{\frac{3}{2}}} \tag{6.38}\star$$

The solution for the corresponding space-time problems may be obtained in the same way. These results are, for a line source (q_0 neutrons per unit length at $t = 0$),

$$n(\rho,t) = \frac{q_0 e^{-\Sigma_a v t} e^{-\rho^2/4Dvt}}{4\pi Dvt} \tag{6.39}$$

and for a point source (q_0 neutrons at time $t = 0$),

$$n(r,t) = \frac{q_0 e^{-\Sigma_a v t} e^{-r^2/4Dvt}}{(4\pi Dvt)^{\frac{3}{2}}} \tag{6.40}$$

c. Kernel Method and Distributed Sources. It will be convenient to summarize here the general expressions for the neutron density and the slowing-down density due to point sources. These relations will be useful in treating the problems encountered in reactor calculations which have more general source distributions. From Eq. (6.38) we obtain, for an infinite medium, the slowing-down density, past lethargy u at $\mathbf{r}$, that results from a unit point source of neutrons of lethargy u_0 at $\mathbf{r}_0$, $(u_0 < u)$,

$$q(\mathbf{r},u;\mathbf{r}_0,u_0) = \frac{p(u;u_0) e^{-|\mathbf{r}-\mathbf{r}_0|^2/4\tau(u;u_0)}}{[4\pi\tau(u;u_0)]^{\frac{3}{2}}} \tag{6.41}$$

where the age $\tau(u;u_0)$, in accordance with (6.15), is defined by

$$\tau(u;u_0) \equiv \int_{u_0}^{u} \frac{D(u')\,du'}{\xi\Sigma_t(u')} = \tau(u;0) - \tau(u_0;0) \equiv \tau(u) - \tau(u_0) \tag{6.42}$$

In the same way, we generalize the one-velocity time-dependent problem by specifying the neutron density at $\mathbf{r}$ and t in an infinite medium due to the release of one neutron at time t_0 from the point $\mathbf{r}_0$,

$$n(\mathbf{r},t;\mathbf{r}_0,t_0) = \frac{\exp\left[-\Sigma_a v(t-t_0) - \dfrac{|\mathbf{r}-\mathbf{r}_0|^2}{4Dv(t-t_0)}\right]}{[4\pi Dv(t-t_0)]^{\frac{3}{2}}} \tag{6.43}$$

In terms of the position vectors $\mathbf{r}$ and $\mathbf{r}_0$, these functions depend only on the length $|\mathbf{r} - \mathbf{r}_0|$, the distance of the field point $\mathbf{r}$ from the source point $\mathbf{r}_0$, which replaces the length r in the right-hand members of Eqs. (6.38) and (6.40), i.e., the distance of the field point $\mathbf{r}$ from the source

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point 0. This simplification arises, of course, from the assumptions of an isotropic source and a homogeneous medium.

The general expressions for the slowing-down density (6.41) and the neutron density (6.43) due to a point source may be utilized to derive expressions for the corresponding functions in systems with distributed sources. Both of these relations represent solutions to linear partial differential equations; thus the principle of superposition is applicable to situations which involve either. A procedure which may be adopted for this purpose will be demonstrated in detail for the lethargy-space problem only. The technique presented below is readily applicable to the one-velocity time-dependent problems.

Equation (6.41) gives the slowing-down density at $\mathbf{r}$ past u due to a point source of unit strength releasing neutrons at $\mathbf{r}_0$ with lethargy u_0. We now suppose that $S(\mathbf{r}_0,u_0)$ specifies a neutron source that is distributed both in space and in lethargy, that is, the source emits $S(\mathbf{r}_0,u)$ neutrons per unit time per unit volume per unit lethargy at $\mathbf{r}_0$ with lethargy u_0. By (6.41), the slowing-down density, at $\mathbf{r}$ past u, due to the neutrons appearing at $\mathbf{r}_0$, u_0 is

$$q(\mathbf{r},u;\mathbf{r}_0,u_0) = \frac{S(\mathbf{r}_0,u_0)\ p(u;u_0)\ e^{-|\mathbf{r}-\mathbf{r}_0|^2/4\tau(u;u_0)}}{[4\pi\tau(u;u_0)]^{\frac{3}{2}}} \tag{6.44}$$

for $u_0 < u$. However, all neutrons which appear from sources at lethargies less than u can contribute to the slowing-down density past u. The total contribution of all these sources is obtained by integrating the expression (6.44) over the lethargy range $(0,u)$ and over all space containing sources. For convenience we select an origin for the lethargy scale such that no neutrons appear in the system with $u < 0$. If $S(\mathbf{r}_0,u_0)$ were, for example, the fission sources in a reactor, then $u = 0$ would be so selected that the fission spectrum would show negligibly few neutrons of energy greater than the energy corresponding to lethargy zero. In general notation, the expression for $q(\mathbf{r},u)$ may be written

$$\begin{aligned} q(\mathbf{r},u) &= \int_0^u du_0 \int_V q(\mathbf{r},u;\mathbf{r}_0,u_0)\, d\mathbf{r}_0 \\ &= \int_0^u du_0 \int_V \frac{S(\mathbf{r}_0,u_0)\ p(u;u_0) \exp\left[-\dfrac{|\mathbf{r}-\mathbf{r}_0|^2}{4\tau(u;u_0)}\right] d\mathbf{r}_0}{[4\pi\tau(u;u_0)]^{\frac{3}{2}}} \end{aligned} \tag{6.45}$$

The application of this result to a specific calculation will be demonstrated for the case of a line source of neutrons of lethargy zero. If we assume the source strength to be q_0 neutrons per unit time per unit length of line, then the final expression for $q(\mathbf{r},u)$ should agree, of course, with the result (6.37). The appropriate statement of the source distribution for this case is

$$S(\mathbf{r},u) \equiv q_0\, \delta(x)\, \delta(y)\, \delta(u) \tag{6.46}$$

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where we have selected the z axis as the source line. $S(\mathbf{r},u)\,dx\,dy\,dz\,du$ is the number of neutrons which appear in the volume element $dx\,dy\,dz$ per unit time and which have lethargies in du about u. The number of neutrons emitted by a unit length of source line is easily computed by integrating the expression (6.46) over the appropriate space and lethargy range:

$$q_0 \int_0^1 dz \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(u)\,du = q_0 \text{ neutrons/time/length}$$

Note that the lethargy integration is formally written for the interval $(-\infty,\infty)$. In point of fact, the variable u is defined for $u \geq 0$; thus we have implied that

$$\int_{-\infty}^{\infty} \delta(u)\,du = \int_0^{\infty} \delta(u)\,du = 1$$

The proposed calculation for the slowing-down density, in an infinite medium, due to a uniform line source of zero lethargy neutrons may be computed from (6.45), using the source (6.46):

$$q(x,y,z,u) = q_0 \int_{-\infty}^{\infty} dz_0 \int_{-\infty}^{\infty} dy_0 \int_{-\infty}^{\infty} dx_0 \int_0^u \frac{\delta(u_0)\,\delta(x_0)\,\delta(y_0)\,p(u;u_0)e^{-r_0^2/4\tau(u;u_0)}\,du_0}{[4\pi\tau(u;u_0)]^{\frac{3}{2}}}$$

where $r_0^2 = (x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2$. The lethargy integration yields

$$q(x,y,z,u) = \frac{q_0 p(u)}{[4\pi\tau(u)]^{\frac{3}{2}}} \int_{-\infty}^{\infty} dz_0 \int_{-\infty}^{\infty} dy_0 \int_{-\infty}^{\infty} \delta(x_0)\,\delta(y_0)e^{-r_0^2/4\tau(u)}\,dx_0$$

The x_0 and y_0 integrations reduce this relation to

$$q(x,y,z,u) = \frac{q_0 p(u)e^{-\rho^2/4\tau(u)}}{[4\pi\tau(u)]^{\frac{3}{2}}} \int_{-\infty}^{\infty} e^{-(z-z_0)^2/4\tau(u)}\,dz_0$$

where we have defined $\rho^2 \equiv x^2 + y^2$. The z_0 integration may be written in the form of the error integral, and it is easily shown that the final expression for $q(\mathbf{r},u)$ reduces to the expression (6.37).

d. The Slowing-down Length. We conclude this section with some remarks on the physical significance of the Fermi age. The age was previously defined by means of the differential equation (6.15). In integral form this variable is given by

$$\tau(u;0) \equiv \tau(u) = \int_0^u \frac{D(u')\,du'}{\xi\Sigma_t(u')} \qquad (6.47)\star$$

An alternate expression which is commonly seen is

$$\tau(u) = \int_0^u \frac{D(u')\,du'}{\xi\Sigma_s(u')} \qquad (6.48)$$

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The presence of Σ_t instead of Σ_s in (6.47) results from the particular choice of the relationship between the slowing-down density and the flux made in (6.6). It should be remembered that this relationship is justifiable only in a medium of light absorption, that is, one in which $\Sigma_t \simeq \Sigma_s$ [see (4.112) and related discussion]. Thus in most cases in which the present theory is expected to be useful, the two choices of τ give nearly the same result; however, for consistency, the definition (6.47) will be used here and in the remainder of this work.

It is possible to draw some conclusions about the relationship between the age and the slowing-down process by inquiring into the average distance traveled by a neutron in slowing down. The calculation which follows is analogous to the one presented in Sec. 5.5a in connection with the average distance traveled by a thermal neutron from the point at which it enters the thermal range to the point at which it is absorbed. Here the problem is to compute the average distance traveled by a neutron while it slows through a given lethargy range. Consider an infinite medium in which we have a point source of unit strength which emits neutrons of zero lethargy. The slowing-down density due to the neutrons released by this source (arbitrarily placed at $r = 0$) was found to be

$$q(r,u) = \frac{p(u)e^{-r^2/4\tau(u)}}{[4\pi\tau(u)]^{\frac{3}{2}}} \equiv G(u)e^{-r^2/4\tau(u)} \tag{6.49}$$

Let us compute the average value of the radial distance squared $\overline{r^2}$ at which neutrons from this source acquire lethargy u by means of the relation

$$\overline{r^2(u)} \equiv \int_0^\infty r^2 f(r;u)\, dr \tag{6.50}$$

where $f(r;u)\, dr$ = probability that a neutron acquires lethargy u in dr about r

This probability is obtained by taking the ratio of the number of neutrons slowing past u in the spherical shell dr to the total number slowing past u throughout the entire medium. Thus

$$f(r;u)\, dr = \frac{q(r,u)4\pi r^2\, dr}{\int_0^\infty q(r,u)4\pi r^2\, dr} \tag{6.51}$$

The substitution of this relation along with (6.49) into (6.50) yields the result

$$\tfrac{1}{6}\overline{r^2(u)} = \tau(u) \tag{6.52}\star$$

The linear dependence of the average $r^2(u)$ on $\tau(u)$ is a general result. The constant of proportionality is determined by the source distribution. Thus, in general, the square root of the Fermi age is a measure of the *slowing-down length* of a neutron.

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It is seen from Eq. (6.48) that roughly the square of the scattering mean free path appears in the numerator of the integral for the age and ξ appears in the denominator. This leads us to the gross observation that a large age, and therefore a large slowing-down length, is associated with a large scattering mean free path and a large nuclear mass. A large scattering mean free path clearly implies long free flights between scattering collisions; thus for a given energy loss per collision, a neutron in a medium of large λ_s need travel a greater distance, on the average, to slow

TABLE 6.1 AGE OF FISSION NEUTRONS IN VARIOUS MODERATORS[a]

Material	Age to indium resonance, cm²	Estimated age from indium to thermal, cm²	Age to thermal, cm²
H_2O[b]	26.7	1	27.7
D_2O[c] (0.2% H_2O)	.....	....	109.
(4.7% H_2O)	.....	....	93.
(8.2% H_2O)	.....	....	78.
Be(1.85 g/cm³)	80. ± 2	17.2	97.2
BeO(3.0 g/cm³)	.....	....	105 ± 10
C(1.60 g/cm³)	311 ± 3	53	364

[a] See also U.S. Atomic Energy Commission, "Reactor Handbook: Physics," p. 485, McGraw-Hill Book Company, Inc., New York, 1955.

[b] This figure is reported by T. V. Blosser and D. K. Trubey, "An Investigation of the Slowing Down of Fission Neutrons in Water," Neutron Physics Division Annual Progress Report for Period Ending Sept. 1, 1959, Oak Ridge National Laboratory, ORNL-2842, pp. 109–112, Nov. 9, 1959; a summary of the most recent measurements is given in U.S. Atomic Energy Commission, *Power Reactor Technology*, **2** (4), 31–33 (1959).

[c] James W. Wade, "Neutron Age in Mixtures of Light and Heavy Water," E. I. du Pont de Nemours & Co., DP-163, June, 1956.

through a given energy interval than it must in a medium with a small λ_s. Consequently, the neutrons slowing down in the former system will tend to spread out more than those in the latter, hence, a larger $\overline{r^2(u)}$ and a larger age. Likewise, if we fix the scattering cross section, then a neutron slowing down in a medium with a large nuclear mass (small ξ) must make many more collisions (travel farther) to slow to a given lethargy than it would have to make in a medium of small nuclear mass. Thus any effect which tends to lengthen the slowing down process will yield a larger Fermi age.

These rather elementary conclusions have considerable significance in the selection of suitable moderators for a given type of reactor. Presuming that a moderator is desired (i.e., the chain reaction is dependent on the thermal, or low-energy, properties of the fuel), the choice of materials may in some instances be limited by over-all size considerations. For example, if for some reason a small thermal reactor is desired, it is obvious

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from the results obtained above that the choice of moderators is limited to those materials which have the smaller values of the Fermi age (from fission energy to thermal). The intention here, of course, is to slow down the fission neutrons in as small a volume as practical; this can be accomplished only if the characteristic dimension of the moderator is much larger than the slowing-down length (i.e., if $\sqrt{\tau} \ll$ moderator size). Table 6.1 lists the Fermi age (to thermal[1]) of some materials of interest to reactor technology.

6.3 The Unreflected Reactor in Age Theory

a. Monoenergetic Fission Neutrons. The general results obtained in the two preceding sections may be extended to systems of practical importance. In particular, we examine here the application of the combined slowing-down and diffusion equations to the analysis of systems which include fission sources. We begin the analysis by developing the appropriate relations for a reactor in which the neutrons produced by fission appear at some fixed energy $E_0 \gg E_{th}$. As in previous calculations, E_0 may be selected to correspond to some average energy computed from the fission spectrum. The techniques used in this calculation will then be applied to the analysis of a reactor in which the fission neutrons appear according to some spectrum $\chi(u)$. It is clear that the first model based on the monoenergetic source is a special case of the second if $\chi(u)$ is defined in terms of the delta function $\delta(u)$. However, both calculations will be performed in detail so as better to demonstrate the methods used.

The simple reactor model we propose for these analyses may be defined in terms of the following assumptions:

$$\left.\begin{array}{l}(1)\ \text{The reactor is bare, finite, and homogeneous}\\ (2)\ \text{The medium is multiplying; neutrons appear only from fission sources}\\ (3)\ \text{The nuclei of the medium are at rest}\\ (4)\ \text{The slowing-down process terminates at thermal energy}\\ (5)\ \text{Neutrons in the thermal range can be treated as a single group}\end{array}\right\} \quad (6.53)$$

As mentioned, our first model is based on the assumption of monoenergetic fission sources. The use of a monoenergetic spectrum for the fission neutrons as a first model seems justified in view of the fact that the actual spectrum is a sharply peaked function (see Fig. 4.25). Thus a delta function placed in the vicinity of the peak will yield a reasonable description of the fission spectrum, especially in regards to its influence

[1] "Age to thermal" usually implies the age of fission neutrons which have slowed to thermal energy.

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on the neutron population at the higher end of the lethargy scale, i.e., the thermal group. Practice has shown that this model is entirely adequate for many bare-reactor calculations.

The combined slowing-down diffusion equations which will be used for the study of the multiplying medium are

$$-D(u)\,\nabla^2\phi(\mathbf{r},u) + \Sigma_a(u)\,\phi(\mathbf{r},u) = S(\mathbf{r},u) - \frac{\partial}{\partial u}\,q(\mathbf{r},u) \qquad 0 \le u \le u_{th} \quad (a)$$

$$q(\mathbf{r},u) = \xi\Sigma_t(u)\,\phi(\mathbf{r},u) \qquad 0 \le u < u_{th} \quad (b) \qquad (6.54)\star$$

$$-D_{th}\nabla^2\phi_{th}(\mathbf{r}) + \Sigma_a^{th}\phi_{th}(\mathbf{r}) = q(\mathbf{r},u_{th}) \qquad (c)$$

Note that the coupling equation which involves the total cross section has been selected. Experience shows that this relation gives satisfactory results for the systems of interest here. The last of these equations (c) describes the spatial distribution of the neutrons in the thermal range by means of the one-velocity model. It is clear that the source term[1] in this equation is indeed the rate at which neutrons slow down into the thermal range from above (direct contributions to the thermal group from the fission process are negligibly few). For the case of the monoenergetic fission neutrons, the source term in the slowing-down relation may be written in the following form:

$$S(\mathbf{r},u)\,du = \left[\nu\int_0^{u_{th}} \Sigma_f(u')\,\phi(\mathbf{r},u')\,du' + \nu\Sigma_f^{th}\phi_{th}(\mathbf{r})\right]\delta(u)\,du \quad (6.55)$$

This relation gives the number of neutrons which appear per unit volume and time around $\mathbf{r}$ with lethargies in du about u. Note that we have used the delta function $\delta(u)$ to represent the fission spectrum; the origin $u = 0$ may be selected so that it corresponds to the peak of the $\chi(u)$ curve. The factor in the bracket gives the spatial distribution of fission neutrons; the first term of this expression represents the number of neutrons produced by fast fissions, the second term, the number produced by thermal fissions. The delta function $\delta(u)$ gives the lethargy dependence of $S(\mathbf{r},u)$ and states that neutrons which appear from fission enter the system only at $u = 0$. Note that the above description assumes that the space and lethargy distributions of the source neutrons are independent; thus all fissions in this reactor model have the same energy spectrum, regardless of where they occur in the system.

The distribution of fissions in the reactor as a function of lethargy, at a given space point, is given by the function $f(\mathbf{r},u)$, see illustrative sketch, Fig. 6.4. Thus $f(\mathbf{r},u)\,du$ gives the number of fissions, per unit volume and time at $\mathbf{r}$, due to neutrons with lethargy in du about u. Evidently, the integral of this function over the entire lethargy range gives the total

[1] Cf. Sec. 5.4g, especially Eq. (5.210).

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number of fissions at $\mathbf{r}$ due to neutrons of all lethargies. Note that the thermal fissions at $\mathbf{r}$ are represented by a "spike" $\Sigma_f^{th}\phi_{th}(\mathbf{r})\,\delta(u - u_{th})$ at u_{th}.[1] Clearly, ν times the area under the curve in Fig. 6.4, including the "area" under the "spike" at u_{th}, is the bracketed factor in (6.55).

The purpose of the present section is to solve the differential equations (6.54) for two cases of the source function $S(\mathbf{r},u)$. For the first case we use the monoenergetic fission source given by (6.55) and look for the solution to (6.54) which satisfies certain boundary conditions. In a system of finite extent these conditions are specified on the outer surface of the medium. An appropriate set of such conditions was discussed in Sec. 5.1e

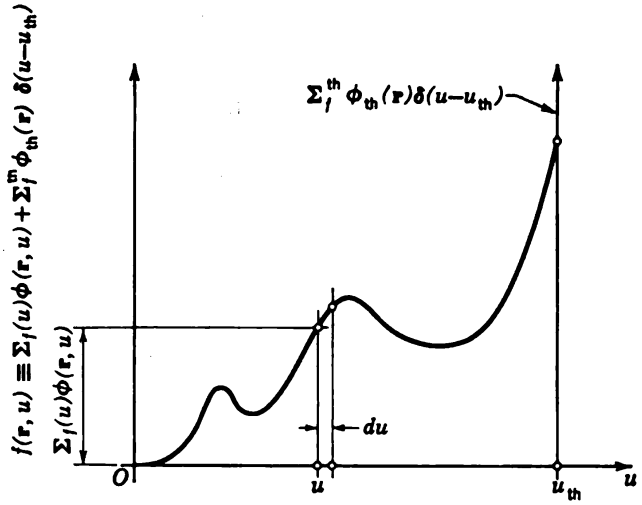

FIG. 6.4 $f(\mathbf{r},u)$, the density of fissions as a function of lethargy.

for the one-velocity model, and suitable approximations for reactor application were presented in Eqs. (5.72). We will adopt this set for use in the combined slowing-down diffusion model. In this application the approximate statement of the outer boundary condition in terms of the extrapolation length must be satisfied for every value of the lethargy. Thus we require that

$$\begin{aligned} \phi(\tilde{\mathbf{r}},u) &= 0 \qquad 0 \le u \le u_{th} \qquad (a) \\ \phi_{th}(\tilde{\mathbf{r}}) &= 0 \qquad\qquad\qquad\quad (b) \end{aligned} \tag{6.56}$$

where $\tilde{\mathbf{r}}$ denotes the position vector to the extrapolated boundary of the reactor. The extrapolation length, and therefore the value of $\tilde{\mathbf{r}}$, however, is a function of the transport mean free path [see (5.71)] which depends on the neutron speed, i.e., lethargy. Thus properly $\tilde{\mathbf{r}}$ is a lethargy-dependent function. However, we will assume here that the extrapolation length is the same for neutrons of all lethargies;[2] in so

[1] The delta function $\delta(u - u_{th})$ is unrelated to the $\delta(u)$ in Eq. (6.55).

[2] The usual procedure is to select one value which represents well the bulk of the neutron population; e.g., in a highly thermal reactor one might use the extrapolation length based on the transport cross section of thermal neutrons.

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doing, we separate the space and energy dependence of our boundary-condition statements. It should be recognized that this is a gross assumption and can be justified only if the predictions of the resulting model agree well with experiment. The motivation in this choice is to develop a simple model to describe the reactor physics which retains the principal features of the physical phenomena and yet is easily applied.

The analytical model we use in this formulation is based on the separation of variables. That is, we look for solutions to Eqs. (6.54) which can be expressed as products of a space-dependent function and an energy-dependent function. This separation of variables can be achieved, however, only if the boundary conditions are stated in a similar manner; hence, the particular choice made above. The resulting model, although simple in form, is very useful for many reactor applications. Serious difficulty is encountered in applying it to actual systems only when the transport cross section (and therefore the extrapolation length) is a highly energy-dependent function. This proves to be the case for hydrogenous media (see Fig. 4.31). Under these circumstances, the selection of a single value of the extrapolation length for the entire lethargy range of interest may yield large errors in the leakage estimates of neutrons whose lethargies differ considerably from the value corresponding to the lethargy average $\bar{\imath}$. Even in these cases, however, this approximation is often used to facilitate the calculation.

It is convenient in developing the solution to Eqs. (6.54) to rewrite equation (a) in terms of the slowing-down density $q(\mathbf{r},u)$ in place of the fast flux $\phi(\mathbf{r},u)$. In the present case of a monoenergetic neutron source, the function $S(\mathbf{r},u)$ in Eq. (6.54a) is properly represented by a delta function in lethargy. We will not use this formulation here but will instead apply the requirements of the source as an initial condition in the variable u when developing the general solution for $q(\mathbf{r},u)$. The appropriate statement of (6.54a) is then

$$-\alpha(u)\,\nabla^2 q(\mathbf{r},u) + \beta(u)\,q(\mathbf{r},u) = -\frac{\partial}{\partial u}\,q(\mathbf{r},u) \qquad 0 < u \leq u_{\text{th}} \quad (6.57)$$

which applies for all points $\mathbf{r}$ inside the reactor. We have used here the abbreviations

$$\alpha(u) \equiv \frac{D(u)}{\xi\Sigma_t(u)} \qquad \beta(u) \equiv \frac{\Sigma_a(u)}{\xi\Sigma_t(u)} \quad (6.58)$$

As indicated above, we look for a solution to (6.57) which can be written as the product of a space-dependent and a lethargy-dependent function. Further, we will assume that this space-dependent function involved in the fast flux describes also the spatial distribution of the thermal flux. We will determine later the consequences of this assumption and examine

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any conditions it may impose on the system. Thus we select the relations

$$q(\mathbf{r},u) \equiv F(\mathbf{r})\, \mathcal{Q}(u) \tag{6.59}$$
$$\phi_{\text{th}}(\mathbf{r}) \equiv F(\mathbf{r})\psi_{\text{th}} \tag{6.60}$$

where the fast flux is given by

$$\phi(\mathbf{r},u) \equiv F(\mathbf{r})\, \psi(u) \qquad \text{with } \psi(u) \equiv \frac{\mathcal{Q}(u)}{\xi\Sigma_t(u)} \tag{6.61}$$

The substitution of (6.59) into (6.57) yields

$$\frac{\nabla^2 F(\mathbf{r})}{F(\mathbf{r})} = \frac{\beta(u)}{\alpha(u)} + \frac{\mathcal{Q}'(u)}{\alpha(u)\,\mathcal{Q}(u)} \equiv -B^2 \tag{6.62}$$

where B^2 is an arbitrary constant to be determined by other requirements. (For example, the constant B has the form $B = \pi/\tilde{R}$ for a sphere.) From this expression we obtain the differential equations

$$\nabla^2 F(\mathbf{r}) + B^2 F(\mathbf{r}) = 0 \qquad \mathbf{r} \text{ inside the reactor} \tag{6.63}$$
$$\mathcal{Q}'(u) + [\alpha(u)B^2 + \beta(u)]\,\mathcal{Q}(u) = 0 \qquad 0 \le u \le u_{\text{th}} \tag{6.64}$$

The solutions to the stationary-wave equation (6.63) which satisfy the extrapolation boundary condition have been discussed in Chap. 5 in considerable detail. In this application, we have from (6.56*a*)

$$F(\tilde{\mathbf{r}})\, \psi(u) = 0 \qquad \text{therefore } F(\tilde{\mathbf{r}}) = 0 \tag{6.65}$$

The functions F and B^2 are determined by the reactor geometry; solutions of (6.63) subject to (6.65) for the three basic shapes have been summarized in Table 5.3.

The solution to Eq. (6.64) is

$$\mathcal{Q}(u) = \mathcal{Q}(0) \exp\left\{ - \int_0^u [\alpha(u')B^2 + \beta(u')]\, du' \right\} \tag{6.66}$$

from which we immediately identify the exponentials [see (6.47)]

$$\exp\left[-B^2 \int_0^u \alpha(u')\, du' \right] \equiv e^{-B^2\tau(u)} \tag{6.67}$$
$$\exp\left[- \int_0^u \beta(u')\, du' \right] \equiv p(u) \equiv \text{resonance-escape probability from zero lethargy to } u \tag{6.68}$$

Thus (6.66) reduces to

$$\mathcal{Q}(u) = \mathcal{Q}(0)\, p(u) e^{-B^2\tau(u)} \tag{6.69}$$

The thermal-flux equation (6.54*c*) may now be written in terms of these results. We have, with the aid of (6.59) and (6.60),

$$[-D_{\text{th}}\nabla^2 F(\mathbf{r}) + \Sigma_a^{\text{th}} F(\mathbf{r})]\, \psi_{\text{th}} = F(\mathbf{r})\, \mathcal{Q}(u_{\text{th}})$$

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If we rearrange and use also (6.69), then

$$\nabla^2 F(\mathbf{r}) + \left[-\frac{\Sigma_a^{th}}{D_{th}} + \frac{\mathcal{Q}(0) p_{th} e^{-B^2\tau_{th}}}{D_{th}\psi_{th}} \right] F(\mathbf{r}) = 0 \tag{6.70}$$

where $p_{th} \equiv p(u_{th})$ and $\tau_{th} \equiv \tau(u_{th})$. However, we have required that the thermal flux and the fast flux have the same spatial form; thus $F(\mathbf{r})$ from either the fast- or thermal-flux equations [(6.63) and (6.70)] must satisfy the same differential equation. This requirement can be met only if the coefficient of the $F(\mathbf{r})$ term in (6.70) is identically B^2. If this condition is imposed, it follows that ψ_{th} is given by

$$\psi_{th} = \frac{\mathcal{Q}(0) p_{th} e^{-B^2\tau_{th}}}{\Sigma_a^{th} + B^2 D_{th}} \tag{6.71}$$

All the functions which appear in Eqs. (6.54) are now completely identified. These results are summarized below:

$$\begin{aligned} q(\mathbf{r},u) &= \mathcal{Q}(0)\, p(u) e^{-B^2\tau(u)} F(\mathbf{r}) && (a) \\ \phi(\mathbf{r},u) &= \frac{\mathcal{Q}(0)\, p(u) e^{-B^2\tau(u)} F(\mathbf{r})}{\xi\Sigma_t(u)} && (b) \\ \phi_{th}(\mathbf{r}) &= \frac{\mathcal{Q}(0) p_{th} e^{-B^2\tau_{th}} F(\mathbf{r})}{\Sigma_a^{th} + B^2 D_{th}} && (c) \end{aligned} \tag{6.72}$$

where $F(\mathbf{r})$ is defined by the relations (6.63) and (6.65). Note that each of the results (6.72) involves a constant $\mathcal{Q}(0)$ which is necessarily established by the initial conditions in the reactor prior to the steady state described by (6.54).

The volume integral over the reactor of the above functions yields the relations

$$\begin{aligned} \int_R q(\mathbf{r},u)\, d\mathbf{r} &= \mathcal{Q}(0)\, p(u) e^{-B^2\tau(u)} \int_R F(\mathbf{r})\, d\mathbf{r} && (a) \\ \int_R \phi(\mathbf{r},u)\, d\mathbf{r} &= \frac{\mathcal{Q}(0)\, p(u) e^{-B^2\tau(u)}}{\xi\Sigma_t(u)} \int_R F(\mathbf{r})\, d\mathbf{r} && (b) \\ \int_R \phi_{th}(\mathbf{r})\, d\mathbf{r} &= \frac{\mathcal{Q}(0) p_{th} e^{-B^2\tau_{th}}}{\Sigma_a^{th} + B^2 D_{th}} \int_R F(\mathbf{r})\, d\mathbf{r} && (c) \end{aligned} \tag{6.73}$$

The physical interpretation of these quantities is obvious; for example, the volume integral of $q(\mathbf{r},u)$ gives the total number of neutrons slowing past lethargy u throughout the entire reactor. This number corresponds to the function $q(u)$ obtained for an infinite system [cf. Eq. (4.151)]. In fact, for the infinite medium $B^2 \to 0$, and (6.73a) yields the appropriate limiting form of the lethargy dependence. It should be noted that in each of the above relations the integral $\int_R F(\mathbf{r})\, d\mathbf{r}$ is some constant which must be computed from the known geometry of the system.

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The critical equation for the bare homogeneous reactor with monoenergetic fission neutrons may be developed from the solutions (6.72). By the usual definition of criticality, we require that the number of neutrons introduced into the system be equal to the number produced by fissions in the subsequent generation. Thus the source function (6.55) may be used to establish the criticality condition. The lethargy integral of (6.55) yields

$$\int_0^{u_{th}} S(\mathbf{r},u)\,du = \nu\left[\int_0^{u_{th}} \Sigma_f(u)\,\phi(\mathbf{r},u)\,du + \Sigma_f^{th}\,\phi_{th}(\mathbf{r})\right]$$

This expression gives the total number of neutrons of all lethargies introduced at point $\mathbf{r}$ in the reactor per unit volume per unit time; but, this is precisely the number slowing past lethargy zero because, by assumption, neutrons from fission appear only at lethargy zero. Thus

$$q(\mathbf{r},0) = F(\mathbf{r})\,\mathcal{Q}(0) = \int_0^{u_{th}} S(\mathbf{r},u)\,du = \nu\left[\int_0^{u_{th}} \Sigma_f(u)\,\psi(u)\,du + \Sigma_f^{th}\psi_{th}\right]F(\mathbf{r}) \tag{6.74}$$

where we have used the relations (6.59) and (6.61). This expression may be further simplified if we use $\mathcal{Q}(u)$ in place of $\psi(u)$, and the definition $\gamma(u) \equiv \Sigma_f(u)/\xi\Sigma_t(u)$. The function $\psi(u)$ is related to $\mathcal{Q}(u)$ by (6.61) and the solution for $\mathcal{Q}(u)$ is given by (6.69). The substitution of these relations into (6.74), along with (6.71) for ψ_{th}, yields

$$1 = \nu\left[\int_0^{u_{th}} \gamma(u)\,p(u)\,e^{-B^2\tau(u)}\,du + \frac{\Sigma_f^{th}\,p_{th}e^{-B^2\tau_{th}}}{\Sigma_a^{th} + D_{th}B^2}\right] \tag{6.75}\star$$

where the common factor $F(\mathbf{r})\,\mathcal{Q}(0)$ has been canceled out. This relation is the critical equation for the bare reactor with monoenergetic fission neutrons.

It is, at times, more convenient to write this relation in an alternate form which involves the fast effect. In the present problem it is seen that the fast effect ϵ is given by

$$\epsilon = \frac{\int_R d\mathbf{r}\int_0^{u_{th}} \Sigma_f(u)\,\phi(\mathbf{r},u)\,du + \int_R \Sigma_f^{th}\,\phi_{th}(\mathbf{r})\,d\mathbf{r}}{\int_R \Sigma_f^{th}\,\phi_{th}(\mathbf{r})\,d\mathbf{r}} \tag{6.76}$$

where the integrations are to be performed over the volume of the reactor. Note that this expression for ϵ is for a *finite* system and therefore differs from the definition developed for the infinite medium in Sec. 4.8b. If we use the results (6.72), (6.76) may be written

$$\epsilon = \frac{\int_0^{u_{th}} \gamma(u)\,p(u)e^{-B^2\tau(u)}\,du + \Sigma_f^{th}\,p_{th}e^{-B^2\tau_{th}}\,(\Sigma_a^{th} + D_{th}B^2)^{-1}}{\Sigma_f^{th}\,p_{th}e^{-B^2\tau_{th}}\,(\Sigma_a^{th} + D_{th}\,B^2)^{-1}} \tag{6.77}\star$$

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Note that this definition of the fast effect differs markedly from the corresponding result (4.298) for the infinite medium. The difference is due to the additional factors in (6.77) which involve the size parameter B^2. This result, however, is entirely consistent with (4.298) since in the limit for infinite media $B^2 \to 0$, and the two expressions become identical.

If the relation (6.77) is used in (6.75), we obtain

$$1 = \frac{\nu\epsilon\Sigma_f^{th} p_{th} e^{-B^2\tau_{th}}}{\Sigma_a^{th}(1 + L^2B^2)} \tag{6.78}$$

where $L^2 \equiv D_{th}/\Sigma_a^{th}$. We recognize the factor $(1 + L^2B^2)^{-1}$ as the nonleakage probability for the neutrons diffusing in the thermal range [see Eq. (5.217)]. The exponential we identify as

$$e^{-B^2\tau_{th}} \equiv g_{th} \equiv \begin{matrix}\text{nonleakage probability during slowing} \\ \text{down from zero lethargy to } u_{th}\end{matrix} \tag{6.79}$$

This identification will be demonstrated in Sec. 6.3e. If we use the notation of (6.79), and call p_{NL} the nonleakage probability of the thermal neutrons, then (6.78) may be abbreviated as

$$1 = \eta\epsilon f p_{th} g_{th} p_{NL} \tag{6.80}\star$$

This relation may be given a physical interpretation. If n_0 neutrons of lethargy zero are introduced into the reactor, then

1. $n_0 p_{th} g_{th}$ neutrons reach $u = u_{th}$ (i.e., do not escape from the reactor as fast neutrons or are absorbed during slowing down).

2. Of these $n_0 p_{th} g_{th}$ thermal neutrons, $n_0 p_{th} g_{th} p_{NL}$ are absorbed by various nuclei (i.e., are not lost by leakage as thermal neutrons).

3. $n_0 p_{th} g_{th} p_{NL} f$ neutrons are absorbed in fuel material.

4. $n_0 p_{th} g_{th} p_{NL} f\eta$ neutrons are produced by these absorptions in the fuel; thus, these neutrons are due to *thermal fissions.*

5. $n_0 p_{th} g_{th} p_{NL} f\eta\epsilon$ neutrons are produced by both *fast* and thermal fissions.

6. For criticality, the resultant number of neutrons produced by these fissions must equal the number introduced into the system; so $n_0 p_{th} g_{th} p_{NL} f\eta\epsilon = n_0$, and Eq. (6.80) results.

It is important to note that (6.80) is the condition on the parameters of the system for criticality. If the system is not critical, then we use instead the relation

$$k = \eta\epsilon f p_{th} g_{th} p_{NL} \tag{6.81}$$

where $k = 1$ at critical [see discussion accompanying Eq. (5.183)]. The multiplication k is different from unity if, for example, the reactor dimensions are too small for a given fuel concentration (i.e., the neutron leakage is excessive). Thus, given a fuel, (6.80) defines the reactor dimension for criticality through the parameter B which appears in p_{NL}, g_{th}, and ϵ. If

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both the size and fuel concentration of a given system are specified, (6.81) gives the effective multiplication of the system.

It is of interest here to identify some limiting cases of (6.81):

1. For the case of an infinite medium, $B^2 \to 0$ and (6.81) reduces to

$$k = \eta\epsilon f p_{\text{th}} \equiv k_\infty \tag{6.82}$$

2. Ignoring the fast leakage is equivalent to saying that fission neutrons slow to thermal at the point where they are produced, in which case $g_{\text{th}} = 1$ and

$$k = \eta\epsilon f p_{\text{th}} p_{\text{NL}} \tag{6.83}$$

Note that, although this expression assumes no losses by escape during slowing down, fast losses by absorption are still possible.

3. For the one-velocity finite reactor,

$$k = \eta f p_{\text{NL}} \tag{6.84}$$

In this case fission neutrons are assumed to appear directly at thermal energy. Thus losses by absorption or leakage during slowing down are not allowed, and therefore we take $g_{\text{th}} = p_{\text{th}} = 1$. Clearly, in the one-velocity model all fissions are due to thermal neutrons; so $\epsilon = 1$.

4. Large, highly thermal reactors ($\epsilon \simeq 1$)

$$k \simeq k_\infty g_{\text{th}} p_{\text{NL}} \tag{6.85}$$

This relation is a good approximation for such systems because, although fissions by fast neutrons may be negligibly few, losses by absorption and leakage during slowing down may be appreciable. The infinite-medium multiplication k_∞ is used to represent the factors $\eta\epsilon f p_{\text{th}}$ since the fast effect ϵ for the large, finite medium (6.77) is very nearly equal to that for the infinite medium [cf. Eq. (4.298) for $B^2 \ll 1$].

5. No fast fissions ($\epsilon = 1$)

$$k = \eta f p_{\text{th}} g_{\text{th}} p_{\text{NL}} \tag{6.86}$$

In this case the solution of Eq. (6.57) for the slowing-down density yields a source term for the thermal-diffusion equation (6.54*c*) which is proportional to the thermal flux. In general, the source term $q(\mathbf{r},u_{\text{th}})$ is obtained from (6.72*a*). The factor $\mathcal{Q}(0)\,F(\mathbf{r})$ which appears in this result is given by (6.74); but, when $\epsilon \equiv 1$ (i.e., no fast fissions) this expression reduces to the form $F(\mathbf{r})\,\mathcal{Q}(0) = \nu\Sigma_f^{\text{th}}\,\phi_{\text{th}}(\mathbf{r})$. If we use this relation in (6.72*a*), then (6.54*c*) may be written

$$-D_{\text{th}}\,\nabla^2\phi_{\text{th}}(\mathbf{r}) + \Sigma_a^{\text{th}}\phi_{\text{th}}(\mathbf{r}) = \nu\Sigma_f^{\text{th}} p_{\text{th}} g_{\text{th}}\phi_{\text{th}}(\mathbf{r}) = k_\infty\Sigma_a^{\text{th}} g_{\text{th}}\phi_{\text{th}}(\mathbf{r}) \tag{6.87}$$

It is important to recognize that this result satisfies the requirements of the one-velocity model developed in Sec. 5.4g, namely, that the neutrons from fission appear in the same space form as the thermal flux. Accord-

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ing to (6.87) the system *acts* as though neutrons slowed down at the same space point at which the parent fission reaction occurred but with losses due to fast leakage and resonance absorption. This is by no means what actually occurs, since individual neutrons migrate from the point of birth during the slowing-down process; but, in a single-medium system, wherein the problem is so defined as to yield a solution by separation of variables, neutrons are found to diffuse in the same space mode at all lethargies, including their behavior in the thermal range.

b. The Use of the Fission Spectrum. With the above model as an elementary but useful approach to the problem of the homogeneous bare reactor, we turn now to the more general systems which include the distributed fission spectrum. All the assumptions established for the analysis of the monoenergetic fission spectrum model will be extended to the present calculation, except, of course, the description of the spectrum itself. We denote this distribution by the symbol $\chi(u)$; this function will be obtained, in practice, from experimental measurements. We choose the origin of the lethargy scale such that $\chi(0) \simeq 0$; thus, the spectrum $\chi(u)$ has the form shown in Fig. 4.25, but with the origin $u = 0$ shifted far to the left. The peak in this function occurs in the vicinity of $u \sim 2$ (if $u = 0$ corresponds to $E = 10$ Mev). As in the calculations of Sec. 4.6c, it will be convenient to normalize the function $\chi(u)$ so that

$$\int_0^{u_{th}} \chi(u)\, du = 1 \tag{6.88}$$

This function is related to the fission spectrum in energy space $z(E)$ through the equation

$$\int_0^{u_{th}} \chi(u)\, du = \int_{E_{th}}^{E_0} z(E)\, dE \qquad \text{or} \qquad |\chi(u)\, du| = |z(E)\, dE| \tag{6.89}$$

Note that $u = 0$ corresponds to E_0, and u_{th} to E_{th}.

If we use $\chi(u)$ as the distribution for fission neutrons, the source function for Eq. (6.54*a*) takes the form [cf. Eq. (6.55)]

$$S(\mathbf{r},u)\, du = \nu \left[\int_0^{u_{th}} \Sigma_f(u')\, \phi(\mathbf{r},u')\, du' + \Sigma_f^{th} \phi_{th}(\mathbf{r}) \right] \chi(u)\, du \tag{6.90}$$

The immediate objective of this calculation is to solve Eqs. (6.54) with the boundary conditions (6.56) using the source (6.90). The procedure is straightforward and follows closely the analysis of Sec. 6.3a. As before, we assume that the solutions for the fast and thermal flux can be written in the forms of (6.59), (6.60), and (6.61); we will determine the consequences of these assumptions later. The substitution of these relations into (6.90) yields

$$S(\mathbf{r},u) = \nu \chi(u) \left[\int_0^{u_{th}} \Sigma_f(u')\, \psi(u')\, du' + \Sigma_f^{th} \psi_{th} \right] F(\mathbf{r}) \tag{6.91}$$

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Again it will be convenient to normalize this function; that is, we assume that one neutron is introduced into the reactor. In order for the chain reaction to perpetuate itself, the yield from the resulting fissions must be exactly one neutron; therefore,

$$\int_R d\mathbf{r} \int_0^{u_{th}} S(\mathbf{r},u)\, du = 1$$
$$= \nu \left[\int_0^{u_{th}} \Sigma_f(u)\, \psi(u)\, du + \Sigma_f^{th} \psi_{th} \right] \int_R F(\mathbf{r})\, d\mathbf{r} \int_0^{u_{th}} \chi(u)\, du \quad (6.92)$$

where $\int_R$ denotes the volume integral over the reactor. Now, the solutions $F(\mathbf{r})$ may be taken as a normalized set, i.e.,

$$\int_R F(\mathbf{r})\, d\mathbf{r} \equiv 1 \quad (6.93)$$

Moreover, the integral of $\chi(u)$ is unity by (6.88); therefore, (6.92) may be reduced to

$$\nu \left[\int_0^{u_{th}} \Sigma_f(u)\, \psi(u)\, du + \Sigma_f^{th} \psi_{th} \right] = 1 \quad (6.94)$$

We identify this relation as the criticality condition and will consider it in greater detail shortly. If we use this result in (6.91), we obtain

$$S(\mathbf{r},u) = F(\mathbf{r})\, \chi(u) \quad (6.95)$$

The combined slowing-down diffusion equation (6.54*a*) for a bare reactor with the fission spectrum $\chi(u)$ may now be written in the notation of (6.57) with the source $S(\mathbf{r},u)$ given by (6.95).

$$-\alpha(u)\, \mathcal{Q}(u)\, \nabla^2 F(\mathbf{r}) + \beta(u)\, \mathcal{Q}(u)\, F(\mathbf{r}) = -F(\mathbf{r})\, \mathcal{Q}'(u) + \chi(u)\, F(\mathbf{r}) \quad (6.96)$$

where we have used (6.59), (6.60), and (6.61). Separation of variables yields

$$\nabla^2 F(\mathbf{r}) + B^2 F(\mathbf{r}) = 0 \qquad \mathbf{r} \text{ inside reactor} \quad (6.63)$$
$$\mathcal{Q}'(u) + [B^2 \alpha(u) + \beta(u)]\, \mathcal{Q}(u) = \chi(u) \qquad 0 \le u \le u_{th} \quad (6.97)$$

The solution to Eq. (6.97) is

$$\mathcal{Q}(u) = e^{-\int_0^u G(u')\, du'} \left[\int_0^u \chi(u') e^{\int_0^{u'} G(w)\, dw}\, du' + \mathcal{Q}(0) \right] \quad (6.98)$$

where $G(u) \equiv B^2 \alpha(u) + \beta(u)$.

The constant $\mathcal{Q}(0)$ is determined from the requirement that [see (6.95)]:

$$S(\mathbf{r},0) \equiv q(\mathbf{r},0) = F(\mathbf{r})\, \mathcal{Q}(0) \quad (6.99)$$

But, by (6.95) and the choice of the lethargy scale, $S(\mathbf{r},0) = 0$; therefore, $\mathcal{Q}(0) = 0$. Equation (6.98) may now be written

$$\mathcal{Q}(u) = \int_0^u \chi(u')\, g(u;u')\, p(u;u')\, du' \quad (6.100)$$

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where we define

$$g(u;u') \equiv \exp\left[-B^2 \int_0^u \alpha(w)\, dw + B^2 \int_0^{u'} \alpha(w)\, dw \right] \equiv e^{-B^2[\tau(u)-\tau(u')]} \quad (6.101)$$

$$p(u;u') \equiv \exp\left[- \int_0^u \beta(w)\, dw + \int_0^{u'} \beta(w)\, dw \right] = \frac{p(u)}{p(u')} \quad (6.102)$$

Note that g denotes the fast nonleakage probability and p the resonance-escape probability in the usual way [see Eqs. (4.260)]. Clearly, the probability that a neutron will not be absorbed in traversing the lethargy interval $(0,u)$ is the product of the probability that it will not be absorbed in the interval $(0,u')$ and the probability that it will not be absorbed in (u',u); thus for $u' < u$,

$$p(u;0) \equiv p(u) = p(u')\, p(u;u') \quad (6.103)$$

from which follows the relation (6.102). Equation (6.103) applies since the resonance-escape probability for any particular interval is independent of the past history of the neutron. The fast nonleakage probabilities are also independent, and it follows that

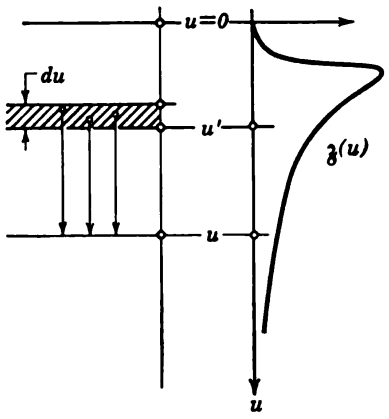

FIG. 6.5 Slowing-down density and fission spectrum.

$$g(u;0) \equiv g(u) = g(u')\, g(u;u') \quad (6.104)$$

Thus we identify the functions (6.101) and (6.102)

$$g(u;u') \equiv \text{probability that a neutron will not escape from the system while slowing through lethargy interval } (u',u) \quad (6.105)$$

$$p(u;u') \equiv \text{probability that a neutron will not be captured while slowing through lethargy interval } (u',u) \quad (6.106)$$

Note that with these definitions one can also obtain (6.100) directly by physical arguments. Thus to find the slowing-down density past u, we compute first the increment dq arising from the neutrons born in du' about u' which slow down successfully as far as u without being captured or escaping from the system (see Fig. 6.5);

$$dq(u;u') = p(u;u')\, g(u;u')\, \chi(u')\, du'$$

Then, to obtain the total slowing-down density past u we sum the contributions from all neutrons born in the range $(u' < u)$. If we call this

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number $\mathcal{Q}(u)$, integration of dq over the interval $(0,u)$ yields

$$\mathcal{Q}(u) = \int_0^u dq(u;u') = \int_0^u \mathfrak{z}(u')\, p(u;u')\, g(u;u')\, du'$$

in agreement with (6.100).

The result (6.100) may be used to solve the thermal-flux equation. The appropriate relation is [see (6.54*c*) and (6.59)].

$$-D_{\text{th}}\nabla^2\phi_{\text{th}}(\mathbf{r}) + \Sigma_a^{\text{th}}\phi_{\text{th}}(\mathbf{r}) = F(\mathbf{r}) \int_0^{u_{\text{th}}} \mathfrak{z}(u)\, g(u_{\text{th}};u)\, p(u_{\text{th}};u)\, du \quad (6.107)$$

We have used the slowing-down density $q(\mathbf{r},u_{\text{th}})$ as the source term since few neutrons are contributed to the thermal group directly by the fission reactions. As in the analysis of Sec. 6.3a, we take the relation (6.60) for $\phi_{\text{th}}(\mathbf{r})$ and determine the condition on ψ_{th} which allows that solution. The substitution of (6.60) into (6.107) yields an equation for $F(\mathbf{r})$, which when compared to (6.63) shows that

$$\psi_{\text{th}} = \frac{1}{\Sigma_a^{\text{th}} + D_{\text{th}}B^2} \int_0^{u_{\text{th}}} \mathfrak{z}(u)\, g(u_{\text{th}};u)\, p(u_{\text{th}};u)\, du \quad (6.108)$$

The solutions to the case of the bare homogeneous reactor with a distributed fission spectrum may be summarized in terms of the above results:

$$q(\mathbf{r},u) = F(\mathbf{r}) \int_0^u \mathfrak{z}(u')\, g(u;u')\, p(u;u')\, du' \quad (a)$$

$$\phi(\mathbf{r},u) = \frac{F(\mathbf{r})}{\xi\Sigma_t(u)} \int_0^u \mathfrak{z}(u')\, g(u;u')\, p(u;u')\, du' \quad (b) \quad (6.109)$$

$$\phi_{\text{th}}(\mathbf{r}) = \frac{F(\mathbf{r})}{\Sigma_a^{\text{th}} + D_{\text{th}}B^2} \int_0^{u_{\text{th}}} \mathfrak{z}(u)\, g(u_{\text{th}};u)\, p(u_{\text{th}};u)\, du \quad (c)$$

The space function $F(\mathbf{r})$ satisfies the differential equation (6.63) and the boundary condition $F(\tilde{\mathbf{r}}) = 0$. Note that, except for a constant, the above relations reduce to the set (6.72) if we take $\mathfrak{z}(u) = \delta(u)$.

We return now to the development of the critical equation which was previously recognized as the relation (6.94). This expression may be expanded by introducing the solution for $\psi(u)$ from (6.100), using (6.61), and ψ_{th} from (6.108). It is easily shown that the critical equation is

$$1 = \nu \left[\int_0^{u_{\text{th}}} \gamma(u)\, du \int_0^u \mathfrak{z}(u')\, g(u;u')\, p(u;u')\, du' + \frac{\Sigma_f^{\text{th}}}{\Sigma_a^{\text{th}}(1 + L^2B^2)} \int_0^{u_{\text{th}}} \mathfrak{z}(u)\, g(u_{\text{th}};u)\, p(u_{\text{th}};u)\, du \right] \quad (6.110)$$

If we introduce the definition of the fast effect ϵ and the thermal utilization f, then this equation may be written in the alternate form:

$$1 = \eta\epsilon f p_{\text{NL}} \int_0^{u_{\text{th}}} \mathfrak{z}(u)\, g(u_{\text{th}};u)\, p(u_{\text{th}};u)\, du \quad (6.111)\star$$

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c. Reactor Calculations. The results obtained by use of the Fermi age model for the flux distributions and the criticality conditions in a bare homogeneous reactor may be useful as first estimates in undertaking a reactor-design study. It is not to be expected, however, that these results will give accurate critical-mass figures for an actual design. Their primary usefulness rests with the fact that they do yield, with an economy of effort, reasonably dependable numbers. These results will suffice, in many cases, to indicate the approximate proportions and other principal features of a proposed configuration. More accurate estimates will, of course, require more exacting techniques and models, such as the multigroup treatments (see Sec. 8.8).

The criticality relations (6.80) and (6.111) will prove to be useful in making quick estimates of fuel concentrations, given a reactor configuration (or vice versa). For calculations of this sort it is entirely within the limitations of the theory to assume that the reactor materials are homogeneously distributed throughout the system, although, in fact, this may not be the case. It should be noted that the above-mentioned equations are transcendental in the parameter B^2 through the fast nonleakage probability g and the fast effect ϵ, so that a trial-and-error procedure is required for solving them. In the situations wherein these relations will be at all useful, such as thermal reactors, the fast effect will be very nearly unity; thus, in making a first estimate, it may be convenient to take $\epsilon \equiv 1$ and then later correct the result when B has been determined from the remaining factors in the critical equation.

Better to demonstrate some of the suggestions on the use of the Fermi age model, let us consider the problem of determining the *critical size* and *minimum fuel mass* of a uniform bare reactor. For simplicity, assume that the basic constituents fuel, moderator, structure, and coolant have all been selected and an estimate of the relative proportions of these materials in the reactor is available. Our problem is to find the dimensions of the reactor with these basic characteristics which has the smallest fuel mass. The procedure is straightforward and involves the calculation of a number of systems each of different size.

The critical-mass calculation for any one of these reactors entails a trial-and-error procedure. One can either assume a size and compute the critical concentration, or vice versa. We will do the latter. The next step then is to evaluate the macroscopic cross sections of the reactor medium for the entire lethargy range, i.e., $\Sigma_a(u)$, $\Sigma_f(u)$, $D(u)$, and ξ, as well as the thermal-group cross sections at the operating temperature of the system. These data may now be applied directly to the computation of the various parameters which appear in the critical equation, i.e., τ_{th}, p_{th}, η, f, and L. In the first trial calculation it is convenient to assume a value for the fast effect (perhaps $\epsilon = 1$) and check it later. All the necessary data having been collected, B^2 may be computed from the appropri-

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ate critical equation by trial and error [see Eqs. (6.80) and (6.111)]. This first estimate of B^2 may be checked by applying it to the calculation of ϵ, which can be obtained from Eq. (6.81), for example. If this value of ϵ agrees well with the assumed value, the first estimate of B^2 will be adequate; if not, the computed value of ϵ may be inserted into the critical equation and a second estimate of B^2 obtained. As a rule ϵ is very nearly unity; consequently, the trial procedure for B^2 should converge very rapidly.

If this procedure is carried out for several fuel concentrations, a family of reactors of various sizes will be obtained. A plot of the critical fuel mass and reactor size as a function of fuel concentration will yield a curve of the shape shown in Fig. 6.6. The symbol N_F^∞ denotes the fuel concentration required for an infinite medium of the constituents used in the reactor study. The curve to the left of the asymptote is a mathematical solution to the critical-mass equation having no physical significance.

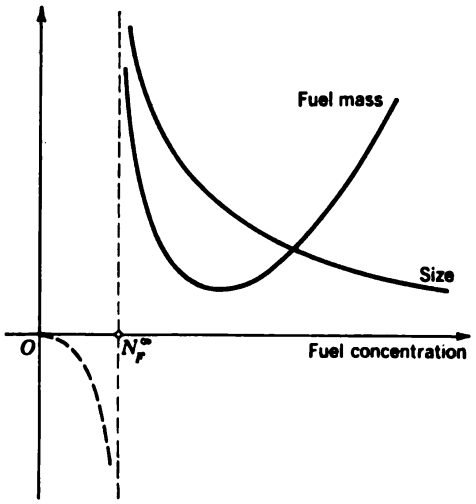

FIG. 6.6 Critical fuel mass and reactor size as a function of fuel concentration.

d. Migration Area. The critical equation (6.80) takes an especially simple form in the case of very large reactors. For these systems $B^2 \ll 1$, and the fast effect ϵ, which is relatively insensitive to B^2, may be computed as though the reactor medium were infinite [i.e., Eq. (6.77) reduces to (4.298)]. Thus (6.80) reduces to

$$1 \equiv \eta f \epsilon p_{\text{th}} g_{\text{th}} p_{\text{NL}} \simeq k_\infty g_{\text{th}} p_{\text{NL}} \tag{6.112}$$

Moreover, if also $\tau_{\text{th}} B^2 \ll 1$, then $g_{\text{th}} \simeq (1 + B^2\tau_{\text{th}})^{-1}$, and (6.112) may be written

$$1 = \frac{k_\infty}{(1 + B^2\tau_{\text{th}})(1 + L^2B^2)} \simeq \frac{k_\infty}{1 + B^2M^2} \tag{6.113}\star$$

where

$$M^2 \equiv \tau_{\text{th}} + L^2 \tag{6.114}\star$$

and the term in B^4 has been neglected. The quantity M^2 is called the *migration area* and M the *migration length*. Thus the factor $(1 + B^2M^2)^{-1}$ is an estimate of the combined leakage losses from the reactor due to escapes which occur during both the slowing-down process and thermal diffusion. The number M, then, is a measure of the combined slowing down and diffusion length so that M^2 is the mean square distance from the point of birth as a fast neutron to death in thermal capture. Note,

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finally, that the migration length can serve as a measure of the *size* of a reactor.

e. Fast Nonleakage Probability. The identification of the fast nonleakage probability $g(u)$ [see Eqs. (6.67) and (6.105)] is demonstrated by means of a simple probability argument. The calculation is carried out by using the technique developed in Sec. 4.3b for computing the resonance-escape probability for hydrogen. As in that calculation, we compute first the probability of a particular event associated with a single lethargy interval Δu_i and then extend this result to the range $(0,u)$. The probability that the neutron will not escape from the system while slowing through the interval Δu_i is denoted as $g(u_i + \Delta u_i;u_i)$; therefore

$$g(u_i + \Delta u_i;u_i) = 1 - (\text{probability of escape in } \Delta u_i) \tag{6.115}$$

The probability that the neutron will escape while traversing Δu_i is given by

$$\frac{\text{Total leakage of neutrons in } \Delta u_i \text{ from system}}{\text{Total neutron population}} = \frac{\Delta u_i \int_S \mathbf{J}(\mathbf{r},u_i) \cdot d\mathbf{S}}{\int_V q(\mathbf{r},u_i)\, d\mathbf{r}} \tag{6.116}$$

where $\int_S$ and $\int_V$ indicate integrations over the surface and volume, respectively. Now

$$\begin{aligned} \mathbf{J}(\mathbf{r},u_i) &= -D(u_i)\, \nabla\phi(\mathbf{r},u_i) = -D(u_i)\, \psi(u_i)\, \nabla F(\mathbf{r}) \\ q(\mathbf{r},u_i) &= \xi\Sigma_t(u_i)\, \phi(\mathbf{r},u_i) = \xi\Sigma_t(u_i)\, \psi(u_i)\, F(\mathbf{r}) \end{aligned}$$

where we have used Eqs. (6.59) through (6.61). We substitute these expressions into (6.116) and apply Gauss's theorem to obtain

$$\frac{-D(u_i)\, \Delta u_i \int_S \nabla\phi(\mathbf{r},u_i) \cdot d\mathbf{S}}{\xi\Sigma_t(u_i) \int_V \phi(\mathbf{r},u_i)\, d\mathbf{r}} = \frac{-D(u_i)\, \psi(u_i)\, \Delta u_i \int_V \nabla \cdot \nabla F(\mathbf{r})\, d\mathbf{r}}{\xi\Sigma_t(u_i)\, \psi(u_i) \int_V F(\mathbf{r})\, d\mathbf{r}}$$

The application of (6.63) to the fraction at the right yields $B^2D(u_i)\, \Delta u_i / \xi\Sigma_t(u_i)$. Thus the probability of nonleakage while the neutron is in Δu_i is

$$g(u_i + \Delta u_i;u_i) = 1 - \frac{B^2D(u_i)\, \Delta u_i}{\xi\Sigma_t(u_i)} \tag{6.117}$$

The probability that the neutron will not escape in traversing the interval $(0,u)$ is obtained by taking the product of the individual probabilities for each Δu_i; thus

$$g(u;0) \equiv g(u) = \prod_i \left[1 - \frac{B^2D(u_i)\, \Delta u_i}{\xi\Sigma_t(u_i)}\right] \tag{6.118}$$

$$\text{and} \quad \ln g(u) = \sum_i \ln\left[1 - \frac{B^2D(u_i)\, \Delta u_i}{\xi\Sigma_t(u_i)}\right] \simeq -\sum_i \frac{B^2D(u_i)\, \Delta u_i}{\xi\Sigma_t(u_i)} \tag{6.119}$$

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(We have assumed here that the neutrons lost by escape from the system are few in comparison to the total population.) In the limit as $\Delta u_i \to 0$, the summation goes over into an integral, and we obtain

$$g(u) = \exp\left[-B^2 \int_0^u \frac{D(u')\,du'}{\xi\Sigma_t(u')}\right] \qquad (6.120)\star$$

6.4 Applications of Fermi Age Theory

a. Diffusion-length Experiment Using Fast Source. The application of the general results developed in the preceding sections of this chapter may be demonstrated by means of two elementary calculations which have some practical importance. The first of these is the estimation of the diffusion length of thermal neutrons in a given material from a knowledge of the thermal-flux distribution in a block of this material. A simplified treatment of this problem has already been given in Sec. 5.5b; however, the calculation presented there was based on the assumption that the neutron source was thermal. We now discard this restriction and consider the more general situation wherein the source yields monoenergetic neutrons of some energy $E_0 > E_{th}$. The present calculation is of interest, therefore, when "thermal-neutron columns" are not immediately available and fast-neutron sources must be employed for the experiment.

The particular problem to be considered is the determination of the thermal-flux distribution along the axis of a long parallelepiped (composed of the material for which the diffusion length is to be measured) which has a plane source of monoenergetic ($E_0 > E_{th}$) neutrons placed at one end. As in the thermal-source experiment of Sec. 5.5b, the thermal-flux distribution is of interest here because the final expression for this function involves the diffusion length of the material. For the region in the block that is far from either end, this expression has a simple approximation of exponential form which is, in fact, similar to the result (5.246). Thus, as in the thermal-source formulation of this problem, the diffusion length can be obtained experimentally from flux measurements along the axis of the block.

In general, the application of the Fermi age model to finite systems which contain localized neutron sources is a complicated problem. We will avoid the mathematical difficulties involved in solving the more general situation described above and consider instead a more simple geometric configuration. In place of the block, which may be the actual geometry used in the experiment, we select an infinite medium (of the material in question) which contains a plane source of neutrons of energy E_0. This particular configuration is proposed because it allows a considerable simplification in the analysis without distorting the final results,

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namely, the expression for the axial distribution of thermal neutrons far from the plane source. Thus, although this elementary model avoids the mathematical complications of the actual geometry, it does yield an accurate description of the physics of interest here.

Consider, then, the problem defined by the following set of assumptions: (1) an infinite medium with scattering and absorption cross sections, and (2) one which contains a plane isotropic source (q_0 neutrons per area per time) of neutrons of energy E_0 placed at the origin $z = 0$. The appropriate set of differential equations for this system is

$$-D(u)\frac{\partial^2}{\partial z^2}\phi(z,u) + \Sigma_a(u)\,\phi(z,u) = -\frac{\partial}{\partial u}q(z,u) \tag{6.121}$$

$$q(z,u) = \xi\Sigma_t(u)\,\phi(z,u) \tag{6.122}$$

$$-D_{\text{th}}\frac{d^2}{dz^2}\phi_{\text{th}}(z) + \Sigma_a^{\text{th}}\phi_{\text{th}}(z) = q(z,u_{\text{th}}) \tag{6.123}$$

The initial condition and neutron-balance requirements are

$$\begin{aligned} &(1)\ \lim_{\substack{\tau\to 0\\ z\neq 0}} \mathcal{Q}(z,u) = 0 \\ &(2)\ \int_0^\infty \mathcal{Q}(z,u)\,dz = \tfrac{1}{2}q_0 \end{aligned} \tag{6.124}$$

where we use, as before, $q(z,u) \equiv \mathcal{Q}(z,u)\,p(u)$. The slowing-down features of this problem were discussed in Sec. 6.2b. The solution for $q(z,u)$ is given in (6.28). This function, evaluated at $u = u_{\text{th}}$, is the source term in (6.123):

$$-D_{\text{th}}\phi_{\text{th}}''(z) + \Sigma_a^{\text{th}}\phi_{\text{th}}(z) = \frac{q_0 p_{\text{th}} e^{-z^2/4\tau_{\text{th}}}}{(4\pi\tau_{\text{th}})^{\frac{1}{2}}} \equiv S(z) \tag{6.125}$$

The general solution to this equation was developed in Sec. 5.2d by use of the kernel method [see Eq. (5.97)]. In the present case the source $S(z)$ from (6.125) may be substituted directly into (5.97), and the integration will yield $\phi_{\text{th}}(z)$. We note, however, that the system in question involves only one space variable z. Thus although the general form (5.97) is applicable here, it is by no means necessary, and, in fact, some labor can be eliminated by using the one-dimensional kernel relation [see Eq. (5.78)], namely,

$$\phi_{\text{th}}(z) = \frac{L}{2D_{\text{th}}}\int_{-\infty}^{\infty} S(\zeta)\,e^{-\varkappa|z-\zeta|}\,d\zeta \tag{6.126}$$

where $L^2 \equiv D_{\text{th}}/\Sigma_a^{\text{th}}$. For this problem S is given by (6.125), and (6.126) takes the form

$$\phi_{\text{th}}(z) = \frac{q_0 p_{\text{th}}}{2\varkappa D_{\text{th}}(4\pi\tau_{\text{th}})^{\frac{1}{2}}}\int_{-\infty}^{\infty}\exp\left(-\frac{\zeta^2}{4\tau_{\text{th}}} - \varkappa|z-\zeta|\right)d\zeta \tag{6.127}$$

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This result may be written in terms of the error function $\text{erf}(x)$, defined by

$$\text{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-s^2} ds \tag{6.128}$$

Thus (6.127) reduces to

$$\phi_{\text{th}}(z) = \frac{q_0 p_{\text{th}} e^{\tau_{\text{th}} \varkappa^2}}{4\varkappa D_{\text{th}}} [2 \cosh \varkappa z + e^{-\varkappa z} \text{erf}(\beta_1) - e^{\varkappa z} \text{erf}(\beta_2)] \tag{6.129}$$

where

$$\beta_1 \equiv \frac{z}{2\sqrt{\tau_{\text{th}}}} - \varkappa \sqrt{\tau_{\text{th}}} \qquad \beta_2 \equiv \frac{z}{2\sqrt{\tau_{\text{th}}}} + \varkappa \sqrt{\tau_{\text{th}}} \tag{6.130}$$

Equation (6.129) gives the thermal-flux distribution, in an infinite medium, that results from a plane source of neutrons at the origin $z = 0$. The resonance-escape probability and the age which appear in this expression are measured from $u = 0$, which corresponds to the energy E_0.

As previously indicated, the only interest in this result is in its form for large values of the variable z. We expect that the asymptotic solution will give the correct form for the thermal flux in the case of the actual geometry for regions far from the source. We obtain the asymptotic form for (6.129) by introducing the expansion for the error function in the case of large x, namely,

$$\text{erf}(x) = 1 - \frac{e^{-x^2}}{x\sqrt{\pi}} \left(1 - \frac{1}{2x^2} + \cdots\right) \tag{6.131}$$

The difference of the two error functions which appear in (6.129) may be written in terms of this expansion, thus

$$e^{-\varkappa z} \text{erf}(\beta_1) - e^{\varkappa z} \text{erf}(\beta_2) = -2 \sinh \varkappa z - \frac{2\varkappa \sqrt{\tau_{\text{th}}} \exp(-z^2/4\tau_{\text{th}} - \varkappa^2 \tau_{\text{th}})}{\sqrt{\pi}\,(z^2/4\tau_{\text{th}} - \varkappa^2 \tau_{\text{th}})} - \cdots$$

The substitution of this expression into (6.129) yields [neglecting terms in $\exp(-z^2/4\tau_{\text{th}})$]

$$\phi_{\text{th}}(z) \sim \left(\frac{q_0 p_{\text{th}} e^{\tau_{\text{th}} \varkappa^2}}{2\varkappa D_{\text{th}}}\right) e^{-\varkappa z} \tag{6.132}$$

This result is to be compared to

$$\phi_{\text{th}}(z) \sim A e^{-k_{300} z} \tag{6.133}$$

which is the solution to the problem of Sec. 5.5d in the case of an infinitely long parallelepiped [cf. Eq. (5.246) for the case $\bar{c} \to \infty$]. For a one-dimensional space, it is clear that [see Eq. (5.248)]

$$k_{300} \equiv \varkappa \tag{6.134}$$

so that Eqs. (6.132) and (6.133) are identical in form.

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Thus, diffusion-length measurements, in which a block of the material in question is exposed to a fast-neutron source, can be carried out by following the procedure used in the case of the thermal source. At large distances from the source, the thermal flux varies exponentially, and the diffusion length can be obtained from simple foil activation measurements along the axis of the specimen. However, as before, these foils should not be placed near the ends. At the far end of the specimen (opposite the irradiated face), the so-called end effects discussed in Sec. 5.5d are encountered. The vicinity of the irradiated face is also to be avoided in positioning the foils inasmuch as the thermal flux in this region takes a Gaussian-like distribution such as described in Sec. 6.2a (see Fig. 6.2). Moreover, if the neutron source is a collimated beam, this distribution will tend to peak well within the interior of the specimen, and the simple exponential behavior will not appear until much farther inside (see Sec. 5.7c and Fig. 5.31).

Finally, it should be noted that the result (6.132) is generally valid, that is, the thermal flux has the same functional dependence at large distances from the source regardless of the energy of the source neutrons.

b. Method of Images: Infinite Slab with Plane Source. The second example that will be considered is the calculation of the slowing-down density, in a slab, which results from a plane source of fast neutrons placed at the center. This example is included so as to demonstrate a technique that can be applied to finite systems which contain localized sources such as planes, points, or lines. The method of Laplace transforms used to obtain the elementary solutions associated with the infinite media (Sec. 6.2) is not useful here because of the conditions which must be satisfied in the finite geometries, namely, that the flux and slowing-down density vanish at the extrapolated boundary and that the initial condition (corresponding to $\tau = 0$) be satisfied at the source plane. In the case of the infinite media, the boundary conditions, of course, did not appear, and the initial condition was easily satisfied by means of a simple integral relation [e.g., (6.20)3]. The basic approach used in these calculations can be applied to the finite system as well; however, in applying this method to the present case, it is necessary to represent the source by a delta function and to expand the results in infinite series of eigenfunctions of the transformed ($\tau \rightarrow s$) differential equation. The solution obtained in this way is not available in closed form and must be expressed by means of an infinite series. No advantage is to be gained in pursuing this course since similar results can be obtained with much less effort by the use of other methods.

We treat the present problem by the method of images. Thus we take the infinite-medium solution for the slowing-down density due to a plane source and, by constructing a system of "alternating sources" outside the slab, reduce the flux at the extrapolated boundaries to zero so

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as to satisfy the required conditions. The differential equation for the slowing-down density is given by (6.19). The solution to this equation must satisfy the boundary conditions:

$$\begin{aligned} &(1)\ \lim_{\tau \to 0} \left[\int_{-a}^{a} Q(x,\tau)\, dx \right] = q_0 \\ &(2)\ Q(x,\tau) = Q(-x,\tau) \\ &(3)\ Q(\pm \tilde{a},\tau) = 0 \end{aligned} \tag{6.135}$$

The last relation follows by virtue of the fact that the slowing-down density is proportional to the fast flux through the coupling equation

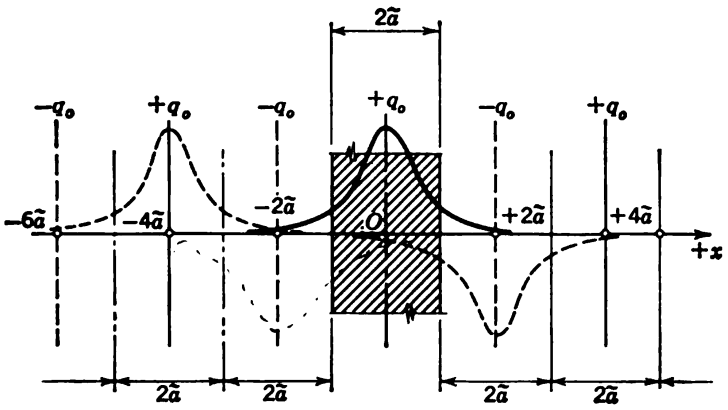

FIG. 6.7 Superposition of images.

(6.6), and, in the usual way, we have required that the flux at all energies satisfy the extrapolated boundary condition. The general solution to (6.19) is, in the case of an infinite medium,

$$Q(x,\tau) = \frac{q_0 e^{-|x-x'|^2/4\tau}}{(4\pi\tau)^{\frac{1}{2}}} \tag{6.136}$$

where $Q(x,\tau) \equiv q(x,\tau)/p(\tau)$, and gives the slowing-down density in an infinite medium (with no resonance absorption) due to a plane source of $\tau = 0$ neutrons at $x = x'$.

The conditions required for the slab geometry may be met by distributing an infinite set of source planes of strength $\pm q_0$ to either side of the slab, as shown in Fig. 6.7. The effect of these alternating "positive" and "negative" sources of neutrons at $\pm 2\tilde{a}$, $\pm 4\tilde{a}$, . . . on the flux is evident from the figure. Note, for example, how the negative source at $+2\tilde{a}$ just cancels the flux at boundary $+\tilde{a}$ but is inadequate to correct for the condition at $-\tilde{a}$. The major correction at $-\tilde{a}$ is provided by the negative source at $-2\tilde{a}$. The small error in the flux at $-\tilde{a}$ due to the source at $+2\tilde{a}$ is now corrected by a positive source at $-4\tilde{a}$, and so on. The slowing-down density at x due to any particular source of $\tau = 0$ neu-

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trons in this alternating system may be written

$$Q_n(x,\tau) = \frac{(-1)^n q_0 e^{-|x-2n\tilde{a}|^2/4\tau}}{(4\pi\tau)^{\frac{1}{2}}} \qquad n = 0, \pm 1, \ldots \tag{6.137}$$

The combined effect of this infinite set is obtained by superimposing the contributions of all the sources; thus,

$$Q(x,\tau) = \sum_{n=-\infty}^{\infty} Q_n(x,\tau) = \frac{q_0}{(4\pi\tau)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-|x-2n\tilde{a}|^2/4\tau} \tag{6.138}$$

In order to demonstrate that this infinite series is indeed the desired solution to the problem, we must show that it satisfies the boundary conditions (6.135). The first condition is seen to be satisfied if we observe that

$$\lim_{\tau\to 0}\left[\frac{1}{(4\pi\tau)^{\frac{1}{2}}} \sum_{n=-\infty}^{-1} \int_{-a}^{a} e^{-(x-2n\tilde{a})^2/4\tau}\, dx\right]$$

$$= \lim_{\tau\to 0}\left[\frac{1}{(4\pi\tau)^{\frac{1}{2}}} \sum_{n=1}^{\infty} \int_{-a}^{a} e^{-(x-2n\tilde{a})^2/4\tau}\, dx\right] = 0$$

and that

$$\lim_{\tau\to 0}\left[\frac{1}{(4\pi\tau)^{\frac{1}{2}}} \int_{-a}^{a} e^{-x^2/4\tau}\, dx\right] = 1$$

Condition (2) is satisfied by virtue of the geometric symmetry of the image system. We can show that condition (3) can also be satisfied by examining the boundary $x = \tilde{a}$. The slowing-down density $Q(\tilde{a},\tau)$ is

$$Q(\tilde{a},\tau) = \frac{q_0}{(4\pi\tau)^{\frac{1}{2}}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-|\tilde{a}-2n\tilde{a}|^2/4\tau}$$

$$= \frac{q_0}{(4\pi\tau)^{\frac{1}{2}}} [e^{-\tilde{a}^2/4\tau} - e^{-\tilde{a}^2/4\tau} + e^{-(3\tilde{a})^2/4\tau} - \cdots] \tag{6.139}$$

Thus it would appear that each term in the series is canceled out by one other, and we obtain

$$Q(\tilde{a},\tau) = 0$$

The result (6.138) satisfies the differential equation (6.19) and the conditions (6.135) and is therefore the solution to the present problem.

In carrying out a calculation it is convenient to approximate the infinite series by taking the predominant terms. A reasonable estimate of $Q(x,\tau)$ is obtained from the expression

$$Q(x,\tau) \simeq \frac{q_0}{(4\pi\tau)^{\frac{1}{2}}} [e^{-x^2/4\tau} - e^{-(x+2\tilde{a})^2/4\tau} - e^{-(x-2\tilde{a})^2/4\tau}] \tag{6.140}$$

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for $x > 0$. Note that the terms in $n \geq 2$ have been neglected since these terms decrease at least as fast as $e^{-n^2/4\tau}$ as n increases.

6.5 Reactor Temperature Coefficients

a. Temperature Changes and Reactor Stability. A problem of considerable importance to the control of nuclear reactors is the evaluation of the temperature coefficient of the reactor system. It is customary to define the temperature coefficient as the change in the multiplication constant (or reactivity) of the system due to a change in the operating temperature of the reactor. Thus the temperature coefficient is a measure of the inherent stability of the reactor. For example, if for some reason there is an increase in the temperature of the reactor (perhaps due to a change in the power demands on the system), then if the temperature coefficient is positive, the multiplication constant will increase, which, in turn, will cause the rate of energy production to increase and further raise the temperature of the system. Such a system would be unstable and would require a positive and continuous influence of a control system to maintain itself in the steady-state condition. On the other hand, if the coefficient is negative, the effect of a temperature rise on the neutron population (and therefore on the power production) would be to decrease the number of neutrons (and power), so that the temperature would then decrease. Thus in these systems temperature "disturbances" would eventually disappear, and the reactor would return to steady state even in the absence of external control orders.

The evaluation of the temperature coefficient for bare homogeneous reactors is a relatively easy task and in many instances amenable to analytical methods. This is not the case, however, for more complex systems such as heterogeneous media and reflected configurations. For these systems, numerical methods are more useful, and changes in multiplication constant are obtained by making repeated calculations for k for successive small changes in the temperature-dependent parameters of the system. Here, attention is confined to the bare homogeneous system, and the emphasis is placed on presenting some of the fundamental concepts involved in the computation of temperature coefficients. The material presented in this section is meant to serve only as a demonstration of a few of the more important ideas and approaches which can be utilized in such calculations. Furthermore, the present methods developed for homogeneous systems may be useful, at times, for obtaining a preliminary understanding of the temperature response of a particular reactor system wherein accurate calculations might be unduly involved and not entirely necessary. In any case, the temperature-sensitive parameters considered here are of general interest to all reactors. The essential difference in approach taken for the treatment of nonhomogene-

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ous systems is due primarily to the fundamentally different forms in which these parameters may be described for these more complex configurations.

Temperature changes in a reactor can affect the neutron population and, therefore, the multiplication constant in two fundamentally different ways, namely, by changing the nuclear (or microscopic) properties of the materials and by changing the density and size (macroscopic properties) of the reactor components. The various properties affected by temperature are: (1) nuclear: (*a*) the neutron slowing-down range, (*b*) thermal group cross sections as influenced by average neutron energy, and (*c*) Doppler effect; (2) physical: (*d*) density of the reactor materials and (*e*) neutron leakage as influenced by reactor size (and migration area).

A change in the neutron slowing-down range is effectively a change in the *thermal base*. Thus temperature changes which directly affect the distribution of the thermal-neutron group do in fact define a new thermal energy (E_{th}). Changes in E_{th} in turn affect such quantities as the age and the resonance-escape probability which are defined by integrals whose limits involve this number.

Temperature changes may also directly affect the neutron cross sections, and this dependence appears in both the thermal-group cross sections and in the fast cross sections. The effect on the thermal group indicated in (*b*) above is sensed through the function $\mathfrak{m}$ [defined by (4.200)] which appears in (4.209). For most of the quantities appearing in the multiplication constant which are functions of the thermal-group cross sections (e.g., f), it will frequently suffice to assume that the cross sections are temperature independent (i.e., are not affected by changes in neutron speed), as with the σ_s, or to assume a simple $1/v$ dependence in the thermal range, as with the σ_a. The temperature dependence of such cross sections is easily determined and will be demonstrated shortly.

There is a second temperature effect on cross sections which appears in the fast range and is especially important in nuclei which possess pronounced resonances, such as the nuclear fuels. Even though many of these materials retain a nearly $1/v$ dependence around thermal energy, the deviation from a simple $1/v$ behavior is not negligible; moreover, in many instances such materials also possess resonances close to the thermal range. These characteristics enter not only into the evaluation of the temperature coefficient of the thermal group properties, but also into the temperature coefficient of such quantities as the resonance-escape probability which involve integrals of cross sections over the fast (resonance) range. Properly, the fast cross sections which are used in such functions must be computed from an integral relation of the type (4.210) which accounts for the thermal motion of the nuclei. The temperature dependence of the fast cross section appears, of course, through the distribution function $\mathfrak{M}$ defined by (4.198) which involves the temperature of the

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material T_N. Thus variations in T_N affect $\mathfrak{M}$, which in turn affects the fast cross-section dependent quantities. This phenomenon is the so-called *Doppler* effect and is discussed in connection with the temperature coefficient of the resonance-escape probability.

The temperature dependence of the material density and reactor size are easily defined in the case of bare homogeneous reactors. A change in the temperature produces, of course, a change in the various nuclear densities which are involved in the macroscopic cross sections. Moreover, the thermal expansions which accompany increases in temperature influence directly the physical size of the system and therefore affect the neutron leakage.

b. The Multiplication Constant. The temperature coefficient of a reactor is obtained by computing the temperature derivative of the multiplication constant. In each of the analytical models so far developed for homogeneous systems, it was found that the multiplication constant could be expressed as a product of several factors. (It will be shown in a later chapter that this is also the case for heterogeneous and multiregional reactors.) On the basis of our most advanced model (Fermi age) we obtained the expression

$$k = \eta \epsilon f p_{\text{th}} g_{\text{th}} p_{\text{NL}} \tag{6.81}$$

The temperature coefficient is computed by taking the temperature derivative of this relation; thus,

$$\frac{1}{k}\frac{\partial k}{\partial T} = \frac{1}{\eta}\frac{\partial \eta}{\partial T} + \frac{1}{p_{\text{th}}}\frac{\partial p_{\text{th}}}{\partial T} + \frac{1}{\epsilon}\frac{\partial \epsilon}{\partial T} + \frac{1}{f}\frac{\partial f}{\partial T} + \frac{1}{g_{\text{th}}}\frac{\partial g_{\text{th}}}{\partial T} + \frac{1}{p_{\text{NL}}}\frac{\partial p_{\text{NL}}}{\partial T} \tag{6.141}\star$$

This equation gives the fractional change in k due to a unit change in temperature T. The total change δk due to a change in temperature δT is clearly

$$\delta k = \left(\frac{\partial k}{\partial T}\right)\delta T \tag{6.142}$$

where the derivative is obtained from (6.141) and evaluated at the original temperature of the system prior to the introduction of the change δT. In some instances it is more convenient to work with the *reactivity* ω, which we define

$$\omega \equiv \frac{k-1}{k} \qquad \text{and} \qquad k = \frac{1}{1-\omega} \tag{6.143}\star$$

It follows that in terms of the reactivity the temperature coefficient takes the form

$$\frac{\partial \omega}{\partial T} = \frac{\partial}{\partial T}\left(1 - \frac{1}{k}\right) = \frac{1}{k^2}\frac{\partial k}{\partial T} \tag{6.144}$$

which gives the change in reactivity due to a unit change in temperature.

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Note that, in the event that the reactor was initially at steady state ($k = 1$),

$$\frac{\partial \omega}{\partial T} = \frac{\partial k}{\partial T} \tag{6.145}$$

c. Temperature Coefficient of the Thermal Utilization. The temperature coefficient of the thermal utilization for a homogeneous system is obtained from the usual relation (3.23); thus we compute

$$\frac{\partial f}{\partial T} = \frac{\partial}{\partial T}\left[\frac{\Sigma_a^{(F)}}{\Sigma_a^{(F)} + \Sigma_a^{(O)}}\right] = f(1 - f)\left[\frac{1}{\Sigma_a^{(F)}}\frac{\partial \Sigma_a^{(F)}}{\partial T} - \frac{1}{\Sigma_a^{(O)}}\frac{\partial \Sigma_a^{(O)}}{\partial T}\right] \tag{6.146}$$

where the cross sections $\Sigma_a^{(F)}$ and $\Sigma_a^{(O)}$ refer to the *thermal group* properties of the fuel and nonfuel components, respectively. The temperature derivative of these cross sections is obtained from the relation

$$\frac{\partial \Sigma}{\partial T} = \frac{\partial}{\partial T}(N\sigma) = N\frac{\partial \sigma}{\partial T} + \sigma\frac{\partial N}{\partial T} \tag{6.147}$$

where N is the nuclear density of the material (nuclei per unit volume) and σ refers to the microscopic thermal-group cross section. If we let $N \equiv \mathfrak{N}/l^3$, then,

$$\frac{1}{N}\frac{\partial N}{\partial T} = -\frac{3}{l}\frac{\partial l}{\partial T} \tag{6.148}$$

where $\mathfrak{N}$ is the total number of nuclei of a particular material in the reactor and l is a characteristic dimension of the system such that the volume of the reactor is the cube of l. It should be noted that $\frac{1}{l}\frac{\partial l}{\partial T}$ is the usual coefficient of thermal expansion of a material. We denote this derivative by a, and observe that it is related to the temperature derivative of the density ρ (mass, or weight, per unit volume) of the material by the equation

$$a \equiv \frac{1}{l}\frac{\partial l}{\partial T} = -\frac{1}{3\rho}\frac{\partial \rho}{\partial T} \tag{6.149}$$

The derivative of the microscopic cross sections σ is obtained from the definition of the thermal group (4.209) and depends, of course, on a detailed knowledge of the energy dependence of σ in the vicinity of kT. Unless otherwise noted, we will assume in this treatment that (1) all absorption cross sections σ_a vary as $1/v$ in the thermal range and (2) scattering and transport cross sections σ_s and σ_{tr} are energy (temperature) independent. In the case of a $1/v$ dependence, it was shown previously that the *thermal-group* cross section is also $1/v$ [see (4.234)]; therefore, in the case of the σ_a,

$$\frac{1}{\sigma_a}\frac{\partial \sigma_a}{\partial T} = -\frac{1}{2T} \tag{6.150}$$

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where T is the "base" temperature and refers to the initial state of the system. If we use the results (6.148) through (6.150) in (6.147), we obtain

$$\frac{1}{\Sigma_a}\frac{\partial \Sigma_a}{\partial T} = -\left(3a + \frac{1}{2T}\right) \quad \text{and} \quad \frac{1}{\Sigma_s}\frac{\partial \Sigma_s}{\partial T} = -3a \tag{6.151}$$

The application of the first of these relations to (6.146) yields

$$\frac{1}{f}\frac{\partial f}{\partial T} = 3(1-f)(a_O - a_F) \tag{6.152}\star$$

where for convenience we have characterized all the nonfuel components by the parameters a_O and l_O. We find, then, that for a homogeneous reactor, wherein we have assumed a $1/v$ dependence for the absorption cross sections, the temperature coefficient of the thermal utilization depends only on the coefficient of thermal expansion of the materials involved. In a system wherein the bulk of the reactor materials are solids, the temperature coefficient of f will be relatively small since the a_i values for solids are in the order of 10^{-6}/°C. However, in the event that an appreciable fraction of the constituents are fluids (for example, the moderator may be in liquid form), the f term in (6.141) may be of comparable magnitude with the others.

d. Temperature Coefficient of the Thermal Nonleakage Probability. The temperature derivative of the thermal nonleakage probability is obtained from (5.217). It is easily shown that

$$\frac{1}{p_{NL}}\frac{\partial p_{NL}}{\partial T} = -p_L\left(\frac{1}{L^2}\frac{\partial L^2}{\partial T} + \frac{1}{B^2}\frac{\partial B^2}{\partial T}\right) \tag{6.153}$$

where p_L is the leakage probability of thermal neutrons according to the definition (5.220). By the usual relation

$$L^2 = \frac{D}{\Sigma_a} = \frac{D}{\Sigma_a^{(O)}}\left[\frac{\Sigma_a^{(O)}}{\Sigma_a^{(F)} + \Sigma_a^{(O)}}\right] \equiv \frac{D}{\Sigma_a^{(O)}}(1-f) \tag{6.154}$$

It is a good approximation to take $D \simeq D_O$, then,

$$L^2 \simeq L_O^2(1-f) \tag{6.155}\star$$

where $L_O^2 \equiv D_O/\Sigma_a^{(O)}$, the diffusion length for the nonfuel materials. It follows from Eq. (6.155) that

$$\frac{\partial L^2}{\partial T} = (1-f)\frac{\partial L_O^2}{\partial T} - L_O^2\frac{\partial f}{\partial T} = L^2\left(\frac{1}{L_O^2}\frac{\partial L_O^2}{\partial T} - \frac{1}{1-f}\frac{\partial f}{\partial T}\right) \tag{6.156}$$

Now,

$$\frac{\partial L_O^2}{\partial T} = L_O^2\left[\frac{1}{D_O}\frac{\partial D_O}{\partial T} - \frac{1}{\Sigma_a^{(O)}}\frac{\partial \Sigma_a^{(O)}}{\partial T}\right] \tag{6.157}$$

Also,

$$\frac{\partial D_O}{\partial T} = \frac{\partial}{\partial T}\left[\frac{1}{3N_O\sigma_{tr}^{(O)}}\right] = -D_O\left[\frac{1}{N_O}\frac{\partial N_O}{\partial T} + \frac{1}{\sigma_{tr}^{(O)}}\frac{\partial \sigma_{tr}^{(O)}}{\partial T}\right] \tag{6.158}$$

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But, by our assumption that $\partial\sigma_{tr}/\partial T \equiv 0$, and the relation (6.148), we have

$$\frac{1}{D_O}\frac{\partial D_O}{\partial T} = 3a_O \tag{6.159}$$

We substitute this result, along with (6.151), into (6.157) and obtain, for the nonfuel components,

$$\frac{\partial L_O^2}{\partial T} = L_O^2\left(6a_O + \frac{1}{2T}\right) \tag{6.160}$$

For the complete system

$$\frac{\partial L^2}{\partial T} = L^2\left(6a_O + \frac{1}{2T} - \frac{1}{1-f}\frac{\partial f}{\partial T}\right) \tag{6.161}$$

The temperature derivative of the buckling B^2 will involve some characteristic dimension l_R of the reactor. If we define

$$B^2 \equiv \left(\frac{\mathfrak{A}}{l_R}\right)^2 \tag{6.162}$$

where $\mathfrak{A}$ will depend on the geometry of the system, then

$$\frac{1}{B^2}\frac{\partial B^2}{\partial T} = -\frac{2}{l_R}\frac{\partial l_R}{\partial T} \tag{6.163}$$

The substitution of (6.163) and (6.161) into (6.153) yields

$$\frac{1}{p_{NL}}\frac{\partial p_{NL}}{\partial T} = p_L\left(\frac{2}{l_R}\frac{\partial l_R}{\partial T} + \frac{1}{1-f}\frac{\partial f}{\partial T} - 6a_O - \frac{1}{2T}\right) \tag{6.164}$$

If we use also (6.152), then,

$$\frac{1}{p_{NL}}\frac{\partial p_{NL}}{\partial T} = p_L\left[\frac{2}{l_R}\frac{\partial l_R}{\partial T} - 3a_O(2-f) - 3fa_F - \frac{1}{2T}\right] \tag{6.165}\star$$

e. Temperature Coefficient of Fast Nonleakage Probability. We compute the temperature coefficient of the fast nonleakage probability to thermal g_{th} from the expression (6.79). Thus

$$\frac{1}{g_{th}}\frac{\partial g_{th}}{\partial T} = -B^2\tau_{th}\left(\frac{1}{\tau_{th}}\frac{\partial \tau_{th}}{\partial T} + \frac{1}{B^2}\frac{\partial B^2}{\partial T}\right) \tag{6.166}$$

As before, the B^2 term is obtained from the relation (6.163). The τ_{th} term we compute from Eq. (6.47). This implies that the Fermi age model is applicable to the system in question and therefore all the requirements for the validity of this model are satisfied. Of immediate concern to the present calculation is the condition that $\Sigma_a(u) \ll \Sigma_s(u)$. Thus, if the integral expression for the age

$$\tau_{th} \equiv \int_0^{u_{th}} \frac{D(u)\,du}{\xi\Sigma_t(u)}$$

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is to yield a good estimate of the slowing-down length for fast neutrons, it is necessary that the concentration of the highly absorbing materials (fuels) in the reactor be low. In these instances, $\Sigma_t \simeq \Sigma_s$, and we may use the form (6.48) for the age τ_{th}. This will be the case in many situations of practical importance. The temperature derivative of τ_{th} may then be written

$$\frac{\partial \tau_{th}}{\partial T} = \frac{\partial}{\partial T} \int_0^{u_{th}} \frac{D(u)\, du}{\xi \Sigma_s(u)} = \frac{D(u_{th})}{\xi \Sigma_s(u_{th})} \frac{\partial u_{th}}{\partial T} + \int_0^{u_{th}} \frac{\partial}{\partial T} \left[\frac{D(u)}{\xi \Sigma_s(u)} \right] du \quad (6.167)$$

We apply again the assumption that σ_s and σ_{tr} are temperature independent; thus with the aid of Eqs. (6.151) and (6.159), (6.167) reduces to the form

$$\frac{\partial \tau_{th}}{\partial T} = -\frac{\alpha_{th}}{T} + 6a_O \tau_{th} \quad (6.168)$$

with α_{th} defined by (6.58). Note that this estimate for the temperature derivative of τ_{th} is based on the assumption that the essential effect of variations in temperature is to modify the density of the reactor materials. We observed above that this is true only if the fuel concentration is low. Otherwise, the relation (6.48) for τ_{th} will not apply, nor for that matter (6.79) for the fast nonleakage probability.

The temperature derivative of g_{th} is obtained by substituting the results (6.163) and (6.168) into (6.166).

$$\frac{1}{g_{th}} \frac{\partial g_{th}}{\partial T} = B^2 \tau_{th} \left(\frac{\alpha_{th}}{T \tau_{th}} - 6a_O + \frac{2}{l_R} \frac{\partial l_R}{\partial T} \right) \quad (6.169)\star$$

f. Temperature Coefficient of η. A first estimate of the temperature coefficient of η may be obtained from the temperature dependence of the fission and radiative-capture cross sections of the fuel. Thus, in the analysis which follows, we neglect entirely the temperature effects on the neutron yield per fission ν.

When working with the thermal-neutron population, the proper definition of η is given in terms of the thermal-group cross sections, and not in terms directly of the measured values as presented in cross-section curves. It is clear then that the present calculation will involve the various f factors derived previously in Sec. 4.7g.

The quantity η is defined by the relation

$$\eta = \frac{\nu \sigma_f}{\sigma_a} \equiv \frac{\nu}{1 + a} \quad (3.21)$$

where for this calculation it is understood that the indicated cross sections refer to the thermal-group properties of the fissionable material. These are related to the measured cross sections through Eq. (4.247). Note

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that for purposes of clarity and for consistency with the preceding analyses we have omitted the superscript "th" in the thermal-group cross-section symbol. In order to differentiate these cross sections from the measured values, we denote the latter by σ'. The prime notation will be extended also to all other quantities referred directly to measured data. Thus

$$\alpha = \frac{\sigma_\gamma}{\sigma_f} \qquad \text{and} \qquad \alpha' = \frac{\sigma'_\gamma}{\sigma'_f}$$

These quantities are related by

$$\alpha(T) = \alpha'(T)\frac{f_\gamma(T)}{f_f(T)} \equiv \alpha'(T) f_\alpha(T) \tag{6.170}$$

Another useful relation is

$$1 + \alpha(T) \equiv [1 + \alpha'(T)] f_{1+\alpha}(T) \tag{6.171}$$

Note that

$$f_{1+\alpha} = \frac{1 + \alpha' f_\alpha}{1 + \alpha'} \tag{6.172}$$

This factor has been computed by J. A. Harvey[1] for the standard fuels, and some values are given in Table 6.2, along with $1 + \alpha'$, as a function of neutron temperature T.

TABLE 6.2 THE QUANTITY $1 + \alpha'$ AND ITS f FACTOR FOR THE STANDARD FUELS

Fuel	Temperature		$1 + \alpha'$	$f_{1+\alpha}$
	°K	ev		
U^{233}	0	0	1.125	1.000
	293.6	0.0253	1.132	0.985
	423	0.0364	1.115	0.993
	575	0.0496	1.095	1.005
	751	0.0648	1.081	1.017
U^{235}	0	.....	1.207	1.000
	293.6	.....	1.183	0.997
	423	.....	1.173	1.001
	575	.....	1.163	1.009
	751	.....	1.157	1.018
Pu^{239}	0	.....	1.407	1.000
	293.6	.....	1.416	1.016
	423	.....	1.418	1.039
	575	.....	1.423	1.075
	751	.....	1.433	1.093

[1] J. A. Harvey and J. E. Sanders, *Progr. in Nuclear Energy*, **1**, 1 (1956), and unpublished work.

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The substitution of the form (6.171) into (3.21) yields for the temperature coefficient of η:

$$\frac{1}{\eta}\frac{\partial \eta}{\partial T} = -\frac{1}{1+\alpha'}\frac{\partial}{\partial T}(1+\alpha') - \frac{1}{f_{1+\alpha}}\frac{\partial f_{1+\alpha}}{\partial T} \qquad (6.173)\star$$

In this formulation we have tacitly assumed that variations in the temperature of the reactor are sensed immediately in the average temperature T_n of the thermal-neutron population. (Note also that we have omitted the subscript n on T_n.) This is a reasonable assumption in the case of a homogeneous system. In these systems all material components of the reactor core are intimately mixed, and a change in the temperature of one implies an equal change in the temperature of all. The neutron temperature T_n, being directly related to the moderator temperature T_N [see Eq. (4.211)], will likewise experience the same variation. This is not the case, however, in heterogeneous systems. Here, variations in the temperature of one component will not in general be immediately sensed in all the other components. This looser thermal coupling is especially significant in the case of the fuel and moderator components; it is due principally to the physical separation of the two as well as to the relatively larger mass (i.e., heat capacity) of the moderator. Thus a change in fuel temperature does not appear instantly as a change in the neutron temperature (the neutron temperature in a heterogeneous reactor is also governed by the moderator temperature). An important consequence, incidentally, of this physical separation of fuel and moderator is the greater temperature sensitivity of the resonance-escape probability.

The result (6.173) may be generally applied for numerical computation. However, no attempt will be made here to obtain a numerical estimate of the temperature coefficient inasmuch as a detailed knowledge of the function $f_{1+\alpha}(T)$ and its derivative is required.

g. Temperature Coefficient of the Resonance-escape Probability. The treatment which follows is intended only to give an idea of some of the factors which enter into the calculation of the temperature coefficient of the resonance-escape probability. There are three such factors: (1) the usual effect of temperature on the physical density of the materials involved, (2) the effect of temperature on the $1/v$ portion of the absorption cross sections, and (3) the effect of temperature on the resonance regions of the cross sections. The first two factors are easily accounted for and can be handled by the methods already developed. The third factor, which is the Doppler effect,[1] is perhaps the most interesting consideration in any calculation of the resonance-escape probability. Unfortunately, its evaluation involves more advanced concepts of nuclear physics than

[1] The importance of Doppler broadening of resonance lines in the calculation of resonance neutron absorption was first pointed out by J. A. Wheeler.

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have been considered so far; thus, the present analysis is limited to some brief remarks about the nature of this effect, and the reader is referred to other sources for more detailed studies.[1] However, for several situations of practical importance to reactor technology, the Doppler effect is a secondary factor in its influence on the temperature coefficient of a reactor. These situations will be discussed shortly; the formulation which follows is best applied to such systems.

An estimate of the temperature coefficient of the resonance-escape probability for a homogeneous system of *low* fuel concentration may be obtained from the general relation (4.113). For the present calculation we write this expression in the form

$$p_{\text{th}} \equiv \exp\left[- \int_{E_{\text{th}}}^{\infty} \frac{\Sigma_a(E)\, dE}{\xi \Sigma_t(E) E} \right] \tag{6.174}$$

If we separate the cross sections into the fuel and nonfuel components by means of the usual notation, then the integral in (6.174), which we denote I, may be written

$$I \equiv \int_{E_{\text{th}}}^{\infty} \frac{[N_F \sigma_a^{(F)}(E) + N_O \sigma_a^{(O)}(E)]\, dE/E}{\xi[N_F \sigma_a^{(F)}(E) + N_O \sigma_a^{(O)}(E) + N_F \sigma_s^{(F)}(E) + N_O \sigma_s^{(O)}(E)]} \tag{6.175}$$

Let us also separate the scattering cross section for the fuel $\sigma_s^{(F)}$ into its two parts, the potential scattering $\sigma_p^{(F)}$ and the resonance scattering $\sigma_{sr}^{(F)}$; thus[2]

$$\sigma_s^{(F)} \equiv \sigma_p^{(F)} + \sigma_{sr}^{(F)} \tag{6.176}$$

In this formulation, the $\sigma_p^{(F)}$ denotes the usual energy-independent portion of the scattering cross section and the $\sigma_{sr}^{(F)}$ the resonance structure. We will assume that the scattering cross section $\sigma_s^{(O)}$ for the nonfuel materials has no resonances and can be well represented by an energy-independent function. Let us define, then, the cross section

$$\sigma_p \equiv \sigma_p^{(F)} + \frac{N_O}{N_F} \sigma_s^{(O)} \tag{6.177}$$

The substitution of this relation into (6.175) yields

$$I = \frac{1}{\xi \sigma_p} \int_{E_{\text{th}}}^{\infty} \frac{[\sigma_a^{(F)} + N_O \sigma_a^{(O)}/N_F]\, dE/E}{1 + \frac{1}{\sigma_p}[\sigma_a^{(F)} + \sigma_{sr}^{(F)} + N_O \sigma_a^{(O)}/N_F]} \tag{6.178}$$

[1] See, e.g., H. Feshbach, G. Goertzel, and H. Yamaguchi, *Nuclear Sci. and Eng.*, **1** (1), 4–20 (March, 1956); see also Secs. 10.3g and 10.3h, especially Table 10.4 and Fig. 10.17.

[2] See, for example, J. M. Blatt and V. F. Weisskopf, "Theoretical Nuclear Physics," pp. 398–411, John Wiley & Sons, Inc., New York, 1952; and D. J. Hughes, "Pile Neutron Research," pp. 258–266, Addison-Wesley Publishing Company, Reading, Mass., 1953.

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where we have assigned ξ some average value by the method outlined in Sec. 4.5c. The above expression is a general form which may be utilized for estimating the *resonance integral* I, and we will consider such calculations later in conjunction with the analysis of heterogeneous reactors. For the present our interest in this result is focused on estimates of the temperature coefficient of the integral proper. In this calculation it will suffice to examine an approximate form for the integral based on the

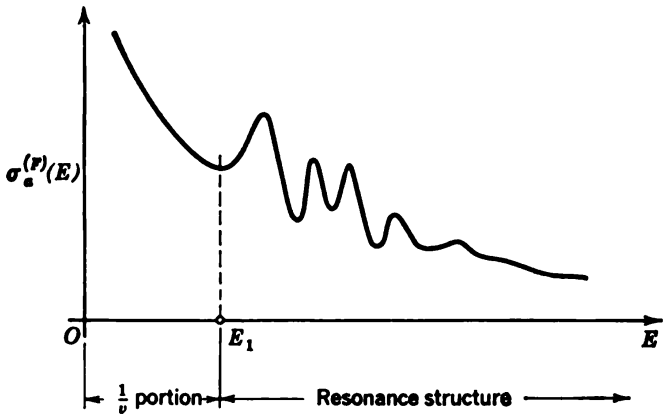

FIG. 6.8 Absorption cross section of a nuclear fuel.

assumption that the nonfuel terms involving the $\sigma_a^{(O)}$ are small in comparison with the fuel terms. Thus,

$$I \simeq \frac{1}{\xi\sigma_p} \int_{E_{th}}^{\infty} \frac{\sigma_a^{(F)}(E)\, dE/E}{1 + \frac{1}{\sigma_p}[\sigma_a^{(F)}(E) + \sigma_{sr}^{(F)}(E)]} \tag{6.179}$$

It should be recognized that the cross sections which appear in this relation are the so-called *effective* cross sections which take into account the thermal motion of the nuclei. We observed in Sec. 4.7c that only in the case of very-high-energy neutrons could one assume that the relative speed (or energy) between neutron and nucleus was well approximated by the absolute speed of the neutron in the laboratory system of coordinates. At the lower end of the energy scale, the thermal motion of the nuclei must be taken into account, and a suitable formula was given in Eq. (4.210) for computing the appropriate reaction rates. In the present case we require the expression for the absorption cross section:

$$v\sigma_a^{(F)}(v) = \int_{\mathbf{V}} \sigma_a^{\infty}(|\mathbf{v} - \mathbf{V}|)|\mathbf{v} - \mathbf{V}|\mathfrak{M}(\mathbf{V};T_N)\, d\mathbf{V} \tag{6.180}$$

where for convenience we have retained the notation in terms of the neutron speed v. The cross section σ_a^{∞} in the integrand takes various forms, depending on the value of v which is involved. For convenience we separate the integral of (6.179) into two parts, the first extending

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from E_{th} to E_1 (some energy just below the first resonance) and a second which begins at E_1 and extends over the entire resonance range (see Fig. 6.8). The energy E_1 is so selected that in the interval (E_{th},E_1) the cross section $\sigma_a^{(F)}(v)$ has approximately a $1/v$ form.[1] In the resonance range, $\sigma_a^{(F)}(v)$ may be regarded as the sum total of contributions from various resonances. We define then

$$\int_{E_{th}}^{\infty} \frac{\sigma_a^{(F)}(E)\, dE/E}{1 + \frac{1}{\sigma_p}[\sigma_a^{(F)}(E) + \sigma_{sr}^{(F)}(E)]} \equiv \mathcal{I}_1 + \mathcal{I}_2 \equiv \mathcal{I} \tag{6.181}$$

where
$$\mathcal{I}_1 \equiv \int_{E_{th}}^{E_1} \frac{(c_0/v)\, dE/E}{1 + c_0/v\sigma_p} \tag{6.182}$$

$$\mathcal{I}_2 \equiv \int_{E_1}^{\infty} \frac{\sigma_a^{(F)}(E)\, dE/E}{1 + \frac{1}{\sigma_p}[\sigma_a^{(F)}(E) + \sigma_{sr}^{(F)}(E)]} \simeq \int_{E_1}^{\infty} \sigma_a^{(F)}(E)\, \frac{dE}{E} \tag{6.183}$$

We have assumed that there are no scattering resonances (i.e., $\sigma_{sr}^{(F)} = 0$) in the interval (E_{th},E_1), and in the interval (E_1,∞) the $1/\sigma_p$ term is much less than unity (i.e., the fuel concentration is very dilute). The c_0 denotes a constant. The cross section which appears in the integral $\mathcal{I}_2$ is obtained from the reaction-rate relation (6.180) by summing the contributions from each resonance, using for the σ_a^{∞} the appropriate single-level Breit-Wigner formula.[2]

Our present interest is not in the accurate evaluation of the resonance integral I but only in obtaining an estimate of its temperature derivative. If we use the form (6.181), we can write the temperature coefficient of p_{th} as

$$\frac{1}{p_{th}}\frac{\partial p_{th}}{\partial T} = -\frac{\partial}{\partial T}\left[\frac{1}{\xi\sigma_p}(\mathcal{I}_1 + \mathcal{I}_2)\right] = \frac{I}{\sigma_p}\frac{\partial \sigma_p}{\partial T} - \frac{1}{\xi\sigma_p}\frac{\partial}{\partial T}(\mathcal{I}_1 + \mathcal{I}_2) \tag{6.184}$$

It follows from our definition of σ_p [see Eq. (6.177)] and the temperature derivatives of the nuclear density N [see (6.148) and (6.149)] that

$$\frac{I}{\sigma_p}\frac{\partial \sigma_p}{\partial T} = \frac{3\Sigma_s^{(0)}}{\sigma_p N_F}(a_0 - a_F) \ln p_{th} \tag{6.185}$$

The temperature derivative of $\mathcal{I}_1$ we compute from (6.182). By changing variables in (6.182) we have

$$\mathcal{I}_1 = 2c_0 \int_{v_{th}}^{v_1} \frac{dv}{v(v + c_0/\sigma_p)} = 2\sigma_p\left[\ln\left(1 + \frac{c_0}{\sigma_p v_{th}}\right) - \ln\left(1 + \frac{c_0}{\sigma_p v_1}\right)\right] \tag{6.186}$$

[1] See Eq. (4.234) which demonstrates that, if $\sigma_a^{\infty} \sim 1/v$, then so also is $\sigma_a^{(F)}(v) \sim 1/v$.
[2] Blatt and Weisskopf, *op. cit.*, pp. 391–394.

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For convenience we let $v_1 \to \infty$ [i.e., we ignore the second term in (6.186) as small compared with the first], then

$$\mathcal{J}_1 \simeq 2\sigma_p \ln\left[1 + \frac{\sigma_a^{(F)}(v_{th})}{\sigma_p}\right] \simeq 2\sigma_a^{(F)}(v_{th}) \tag{6.187}$$

wherein we have used the definition $c_0/v_{th} \equiv \sigma_a^{(F)}(v_{th})$ and assumed that

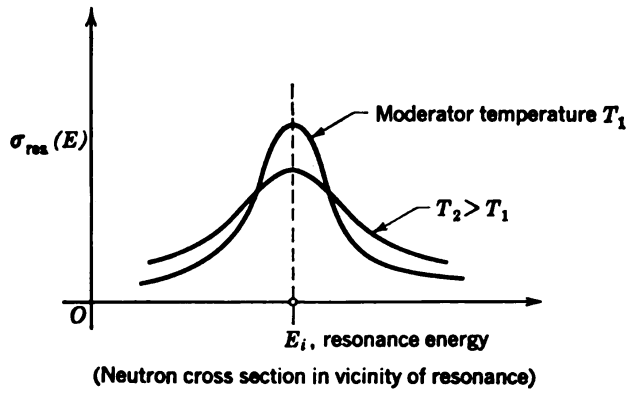

FIG. 6.9 Doppler broadening of resonance.

$\sigma_a^{(F)}(v_{th})/\sigma_p \ll 1$ (a dilute system). Now $\sigma_a^{(F)}(v_{th})$ represents the thermal-group cross section for a $1/v$ dependence; therefore by (4.245),

$$\mathcal{J}_1 = 2\sigma_a^{(F)}(kT_*)\left(\frac{\pi T_*}{4T}\right)^{\frac{1}{2}} \tag{6.188}$$

and the temperature derivative of $\mathcal{J}_1$ is

$$\frac{\partial \mathcal{J}_1}{\partial T} = -\frac{\sigma_a^{(F)}(kT_*)}{T}\left(\frac{\pi T_*}{4T}\right)^{\frac{1}{2}} \tag{6.189}$$

The temperature derivative of the integral $\mathcal{J}_2$ is obtained from (6.183). It has been shown that for many situations of practical interest, in particular for systems wherein the fuel concentration is very low [$\sigma_p \to \infty$ in Eq. (6.177)], $\mathcal{J}_2$ is relatively insensitive to temperature variations.[1] Thus, although a change in the temperature of the medium causes changes in the cross section as given in (6.180) through a change in the distribution of the nuclei represented by $\mathfrak{M}(\mathbf{V},T_N)$, these changes are such as to preserve the area under the individual resonances (see Fig. 6.9). This phenomenon is called *Doppler broadening*. The over-all effect on the resonance integral is small and will be neglected in the present calculation of the temperature coefficient. The principal contribution to the temperature coefficient of p_{th} is derived, then, through the $1/v$ portion of $\sigma_a^{(F)}$. Thus we set $\partial \mathcal{J}_2/\partial T = 0$, and by substituting (6.189) and (6.185)

[1] See, for example, L. Dresner, CF55-12-147, Oak Ridge National Laboratory, Feb. 2, 1956; and *Nuclear Sci. and Eng.*, **1** (1), 68–79 (March, 1956).

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into (6.184) obtain

$$\frac{1}{p_{\text{th}}}\frac{\partial p_{\text{th}}}{\partial T} = \frac{3\Sigma_s^{(O)}}{\sigma_p N_F}(a_O - a_F)\ln p_{\text{th}} + \frac{\sigma_a^{(F)}(kT_*)}{\xi\sigma_p T}\left(\frac{\pi T_*}{4T}\right)^{\frac{1}{2}} \quad (6.190)\star$$

which reveals that the temperature coefficient of the resonance-escape probability for a dilute homogeneous system is positive in so far as its dependence on the fuel cross section is concerned.

6.6 Application to a Heavy-water-moderated Reactor

a. The Argonne Research Reactor: CP-5. We demonstrate the application of the Fermi age model by calculating the critical mass and temperature coefficient for the Argonne National Laboratory research reactor, the CP-5. This reactor has a large heavy-water-moderated core and nearly all the fissions which occur are due to thermal neutrons. It is to be expected, then, that the Fermi age model will yield a good description of the essential features of its neutron physics.

A detailed report on the CP-5 reactor is given in the Atomic Energy Commission document[1] "Research Reactors"; a few of its principal characteristics are summarized here. The reactor core proper is an upright cylinder of D_2O 2 ft high and 2 ft in diameter in which are immersed MTR-type aluminum fuel elements (see Fig. 5.28). The sides and bottom of the core are reflected by 2 ft of D_2O, and this region in turn is surrounded by 2 ft of graphite. The top of the core is reflected by $2\frac{1}{2}$ ft of D_2O alone. The fuel elements, of which there are 16, consist of box-like arrays of fuel plates fabricated of a uranium-aluminum alloy, composed of 17.5 per cent aluminum and 82.5 per cent uranium. For computational purposes we will assume that the volume fraction of aluminum in the core is $v_{\text{Al}} = 0.0688$ and that of D_2O, $v_{D_2O} = 0.914$. The reactor was designed to produce 1,000 kw of heat, and at this power level the temperature of the D_2O is 49°C.

b. Critical-mass Calculation. Our immediate purpose is to compute the mass of U^{235} required to make this reactor critical in the hot-clean condition and to determine the average thermal flux in the core during full-power operation.

If the core of this reactor were entirely bare, we could use Eq. (6.80) directly to determine the critical fuel concentration. Since the present configuration is completely reflected, we require a modified relation which will take into account the effect of the reflector. We observed in Chap. 1 that the purpose of the reflector is to decrease the neutron escapes from the core and thereby reduce the critical fuel concentration of the system. Clearly, if we were to ignore the reflector altogether, our estimate of the

[1] McGraw-Hill Book Company, Inc., New York, 1955.

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critical fuel concentration could be much too large. We should like to attempt a somewhat more realistic calculation. The scheme by which we propose to make such an estimate involves the idea of an *equivalent bare* reactor. Thus we will look for a bare cylindrical reactor which can be made critical with the same fuel concentration as the actual reflected reactor. It is understood that this equivalent bare system will have the same internal configuration of fuel elements and distribution of nonfuel materials as the reflected system. If there were some way to estimate the size of the bare system, then Eqs. (5.204) and (6.80) could be used to compute the critical concentration. A suitable procedure, called the *reflector-savings* method, is developed for this purpose in the general theory of multiregion reactors (Chap. 8). This method allows one to estimate the *increase* in the various dimensions of a bare multiplying medium, over that of a specified reflected configuration, which would be required if both systems were to be critical with the same fuel concentration.

In the present calculation we estimate the proportions of the equivalent bare cylinder by utilizing reflector savings data obtained from more advanced reactor physics models than yet considered in this work.[1] These data, which were computed by H. Garabedian and are based on the two-group theory, indicate that the reflector savings for a D_2O reflector 62 cm thick is about 23 cm. In the CP-5 reactor the 62 cm of D_2O on the sides and bottom of the core is supported by an additional 62 cm of graphite. This thickness of graphite reflector, if used alone, would correspond to a savings[2] of about 50 cm. We will attempt to account for the combined effect of this two-region reflector configuration by selecting an intermediate value of 30 cm for the reflector savings. That is, we are estimating that the presence of the D_2O-C reflectors at the sides and bottom of the core is equivalent to 30 cm of core material in each of those directions. The reflector-savings estimate for the top of the core is 24 cm; this corresponds to a reflector of infinite thickness, according to Garabedian's calculations. The equivalent bare cylinder obtained from these estimates is therefore 60 cm in radius and 115 cm high. Thus we assert that a bare cylinder of this size composed of the same core materials would be critical with the same U^{235} concentration as that required to make the CP-5 critical.

On the basis of the selected bare-core configuration, we estimate the buckling to be

$$B^2 = \left(\frac{\pi}{115}\right)^2 + \left(\frac{2.4}{60}\right)^2 = 0.002353 \text{ cm}^{-2}$$

[1] "Reactor Handbook: Physics," p. 494.
[2] *Ibid.*, p. 484.

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For the proposed calculations we will also use the following data:

$$\eta = 2.08 \qquad \tau_{th} = 130 \text{ cm}^2\ddagger \qquad p_{th} = \epsilon = 1\S$$

The fast nonleakage probability g_{th} based on these data may be computed from Eq. (6.79); it is found to be $g_{th} = 0.7364$.

We compute next the macroscopic cross sections of the core constituents. These calculations are based on the microscopic cross-section data of Table 2.2. In the usual way we estimate first the effective neutron temperature from Eq. (4.211). It can be shown that this temperature is 330°K. We use this figure to compute the macroscopic absorption cross sections for the D_2O and the aluminum based on the assumption of a $1/v$ dependence for σ_a. We obtain from Eqs. (2.54) and (4.245)

$$\Sigma_a^{(D_2O)} = 0.00002311 \text{ cm}^{-1} \qquad \text{and} \qquad \Sigma_a^{(Al)} = 0.0009531 \text{ cm}^{-1}$$

The density of D_2O at 49°C has been taken as 1.08 g/cm³.

The absorption cross section of U^{235} required to support this system is computed from (6.80), which, for the present purpose, is conveniently written in the form

$$\Sigma_a^{(F)} = \frac{DB^2 + \Sigma_a^{(O)}}{\eta\epsilon p_{th} g_{th} - 1} \tag{6.191}$$

The $\Sigma_a^{(O)}$ refers to the nonfuel components D_2O and aluminum and has the value 0.0009762 cm^{-1}. Because of this relatively low nonfuel neutron absorption cross section in the core, it is to be expected that the critical fuel concentration will be low. As a first approximation we take the diffusion coefficient in (6.191) to be that of pure D_2O at 49°C. If we assume that the microscopic transport cross section of D_2O is independent of temperature around thermal energy, then the only effect on Σ_{tr} will be through changes in the density of the material. In that case we may use the relation (see Table 5.1)

$$D(49°\text{C}) = \frac{D(20°\text{C})\rho(20°\text{C})}{\rho(49°\text{C})} = 0.890 \text{ cm} \tag{6.192}$$

It follows that $\Sigma_a^{(F)} = 0.005775$ cm^{-1}. The thermal-group absorption cross section for U^{235} at 330°K is $\sigma_a^{(F)} = 561.4$ barns. For a core volume of 179,000 cm³, this yields a U^{235} critical mass of

$$M_F = \frac{\Sigma_a^{(F)} A_F V_{core}}{\mathcal{A}\sigma_a^{(F)}} = 718.3 \text{ g}$$

‡ Taken from "Reactor Handbook: Physics," p. 770, and corrected for reduced D_2O density at 49°C (see "Reactor Handbook: Engineering," p. 30, for D_2O density versus temperature).

§ Taken from "Reactor Handbook: Physics," p. 770.

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Provided that our guess for the reflector savings is reasonably close, and that the U^{235} is available in the nearly pure form, the above mass estimate should be a reliable figure for the U^{235} required to make the CP-5 critical at the full power condition before the reactor becomes contaminated with fission-product fragments and various other poisons. In order to determine the fuel loading for the contaminated condition, we would require a knowledge of the fission-product buildup during a fuel cycle. Suitable methods for computing these concentrations are developed in Sec. 9.6. In the absence of such information we will estimate the critical loading for the hot dirty condition from the published value of 1.15 kg for the critical mass. The corresponding value of the fission cross section is

$$\Sigma_f = \Sigma_a^{(F)} \left(\frac{\sigma_f}{\sigma_a}\right)_F \left(\frac{1.15}{0.7183}\right) = 0.007681 \text{ cm}^{-1}$$

This result may be used to compute the average thermal flux $\bar{\phi}$ in the CP-5 when fully contaminated and at a full power. If P is the reactor power and ε the energy per fission, then

$$\bar{\phi} = \frac{P}{\varepsilon \Sigma_f V_{\text{core}}} = \frac{10^6}{(3.17 \times 10^{-11})(0.007681)(179{,}000)}$$
$$= 2.29 \times 10^{13} \text{ neutrons/cm}^2\text{/sec}$$

The project data in the "Reactor Handbook" gives 2.3×10^{13} neutrons/cm^2/sec for the average flux.

c. Estimate of Temperature Coefficient. The role of the temperature coefficient in determining the stability of a reactor was discussed in Sec. 6.5a, and some formulas for estimating the value of this coefficient for the various factors which appear in the multiplication constant were developed in subsequent sections. It should be recognized that, in calculating the temperature response of these quantities, the coefficients per se are temperature-dependent functions and may vary in both magnitude and algebraic sign over the entire operating range of the reactor. Thus a complete study of the temperature coefficient for a given system must necessarily consider all possible operational conditions to which the reactor may be subjected. A particular situation which is always of interest in the design of a new system is the initial startup condition; this situation may also be encountered during later stages in the life of the reactor, although under somewhat modified circumstances, whenever the reactor is started up following a prolonged period of shutdown. In each of these situations the reactor is brought to criticality from a *cold condition*. When steady state is achieved at some specified power (flux) level, the temperature stabilizes at some higher steady value. During this entire operation the temperature coefficient, as well as the temperature, is varying in time. Some indication of the thermal sensitivity of the reactor system can be obtained from an estimate of its temperature coefficient.

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We demonstrate the calculation of a temperature coefficient by considering the initial startup condition of the CP-5 and show how this coefficient may be used to predict the excess reactivity (or multiplication) of the reactor at room temperature. Specifically, we compute the temperature coefficient of the hot clean reactor from the temperature derivatives of its thermal utilization, fast nonleakage probability, thermal nonleakage probability, and resonance-escape probability. The change in k for a given change δT in temperature is then easily computed from the relation (6.142).

The data required for this calculation have already been obtained in main part in the preceding study of the CP-5. In addition to these, we will need the coefficients of thermal expansion for the various constituents of the core. These are

$$a_{Al} = 2.38 \times 10^{-5}/°C$$
$$a_F = 1.94 \times 10^{-5}/°C \qquad \text{(for a 30.5\% U-Al alloy)}$$

The linear coefficient for D_2O we obtain from the relation[1] (6.149). Its value is

$$a_{D_2O} = -\frac{1}{3\rho}\frac{\partial \rho}{\partial T} = 9.876 \times 10^{-5}/°C$$

For the calculation which follows we will assume that the reactor parameters listed below are independent of temperature and have the indicated values:

$$B^2 = 0.002353 \text{ cm}^{-2} \qquad \tau_{th} = 130 \text{ cm}^2$$
$$g_{th} = 0.7364 \qquad p_{th} = \epsilon = 1$$

For convenience in reference we list also the following results from the calculations for the CP-5 in Sec. 6.6b:

$$\Sigma_a = 0.006751 \text{ cm}^{-1} \qquad L^2 = 131.8 \text{ cm}^2 \qquad p_{NL} = 0.7633$$
$$p_L = 0.2367 \qquad f = 0.8554 \qquad k = 1$$

These data apply to the critical reactor at 49°C.

We compute first the temperature coefficient of the thermal utilization. By (6.152),

$$\frac{1}{f}\frac{\partial f}{\partial T} = 3(1-f)(a_O - a_F) \simeq 3(1-f)(a_{D_2O} - a_F) = 3.443 \times 10^{-5}/°C$$

The temperature coefficient of the fast nonleakage probability is obtained from (6.169), which requires an estimate of α_{th}. Now

$$\alpha_{th} \equiv \frac{D_{th}}{\xi \Sigma_t}$$

[1] This value is based on the data given in the "Reactor Handbook: Engineering," p. 30.

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For D_{th} we use 0.890 cm, and the denominator we compute from the relation

$$\xi\Sigma_t \equiv \xi_F\Sigma_t^{(F)} + \xi_{D_2O}\Sigma_t^{(D_2O)} + \xi_{Al}\Sigma_t^{(Al)} \simeq \xi_{D_2O}\Sigma_t^{(D_2O)}$$

The value for ξ_{D_2O} we take as 0.527 (see Fig. 4.7) and for $\Sigma_s^{(D_2O)}$, 0.4252 cm^{-1}; these yield $\alpha_{th} = 3.974$. Then, by (6.169),

$$\frac{1}{g_{th}}\frac{\partial g_{th}}{\partial T} = B^2\tau_{th}\left(\frac{\alpha_{th}}{T\tau_{th}} - 6a_O + \frac{2}{l_R}\frac{\partial l_R}{\partial T}\right)$$

If we assume that the reactor does not change size with temperature (i.e., the physical dimensions of the core tank do not change with temperature), then

$$\frac{\partial l_R}{\partial T} \equiv 0$$

and for $a_O = 9.88 \times 10^{-5}/°C$, and $T = 322°K$, we obtain

$$\frac{1}{g_{th}}\frac{\partial g_{th}}{\partial T} = -14.95 \times 10^{-5}/°C$$

The temperature coefficient of the thermal nonleakage probability is computed next. The appropriate relation is given in (6.165)

$$\frac{1}{p_{NL}}\frac{\partial p_{NL}}{\partial T} = p_L\left[\frac{2}{l_R}\frac{\partial l_R}{\partial T} - 3a_O(2 - f) - 3fa_F - \frac{1}{2T}\right]$$

If again we assume that the reactor-core dimensions are temperature independent, the first derivative is zero, and the remaining terms yield

$$\frac{1}{p_{NL}}\frac{\partial p_{NL}}{\partial T} = -45.94 \times 10^{-5}/°C$$

The estimate for the temperature coefficient of the resonance-escape probability is obtained from the relation (6.190) which is based on the assumption that the fuel concentration is very low. More precisely, we require that in the resonance range [see (6.181)]

$$\frac{\sigma_a^{(F)}(E) + \sigma_{sr}^{(F)}(E)}{\sigma_p} \ll 1 \qquad \text{for all } E \tag{6.193}$$

This condition is well satisfied for the concentrations of U^{235} with which we are concerned in the present calculation. Here

$$\sigma_p \equiv \sigma_p^{(F)} + \frac{N_O}{N_F}\sigma_s^{(O)} \simeq \frac{\Sigma_s^{(O)}}{N_F} \tag{6.194}$$

and by (6.190)

$$\frac{1}{p_{th}}\frac{\partial p_{th}}{\partial T} = 3(a_O - a_F)\ln p_{th} + \frac{\sigma_a^{(F)}(kT_*)N_F}{\xi T\Sigma_s^{(O)}}\left(\frac{\pi T_*}{4T}\right)^{\frac{1}{2}} \tag{6.195}$$

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From the CP-5 data we have $p_{th} \equiv 1$; thus the first term is zero. For the second term we compute

$$\sigma_a^{(F)}(kT_*)N_F = (\tfrac{8}{5}\tfrac{8}{6}\tfrac{7}{1})(0.005775) = 0.007072 \text{ cm}^{-1}$$

and for $\Sigma_s^{(O)} \simeq \Sigma_s^{(D_2O)} = 0.4252$ cm^{-1}, $T = 322$°K, $T_* = 293$°K, and $\xi = 0.527$, we obtain

$$\frac{1}{p_{th}} \frac{\partial p_{th}}{\partial T} = 8.20 \times 10^{-5}/\text{°C}$$

These results we now use to estimate the temperature coefficient of the CP-5 at 49°C. Substitution into the general relation (6.141) yields

$$\frac{1}{k} \frac{\partial k}{\partial T} = -49.25 \times 10^{-5}/\text{°C}$$

The excess multiplication δk for the cold clean reactor (at 20°C) is computed from (6.142); thus,

$$\delta k = k \left(\frac{1}{k} \frac{\partial k}{\partial T} \right) \delta T = 0.01428$$

for $\delta T = -29$°C. The multiplication constant at 20°C is therefore $k = 1 + \delta k = 1.0143$, and the reactivity is [see Eq. (6.143)]

$$\omega = \frac{\delta k}{k} = 1.41\%$$

PROBLEMS

6.1 Consider an infinite nonmultiplying medium. Within a spherical region of radius R in this medium, q_0 neutrons per unit time are being emitted uniformly in space at age zero. Show that the slowing-down density $q(r,\tau)$ is given by

$$q(r,\tau) = \frac{2}{r} p(u) \sqrt{\frac{\tau}{\pi}} \left(\frac{q_0}{\frac{4}{3}\pi R^3} \right) \left\{ \frac{r}{4} \sqrt{\frac{\pi}{\tau}} \left[\text{erf} \left(\frac{r+R}{2\sqrt{\tau}} \right) - \text{erf} \left(\frac{r-R}{2\sqrt{\tau}} \right) \right] - \sinh \left(\frac{rR}{2\tau} \right) e^{-(r^2+R^2)/4\tau} \right\}$$

where r is the radial distance from the center of the source sphere, $\tau = \tau(u)$, and

$$\text{erf}(\zeta) \equiv \frac{2}{\sqrt{\pi}} \int_0^\zeta e^{-\lambda^2} \, d\lambda \qquad p(u) \equiv \exp \left\{ - \int_0^u \frac{\Sigma_a(u') \, du'}{\xi \Sigma_t(u')} \right\}$$

6.2 Isotropic sources of zero lethargy neutrons are distributed uniformly over a spherical surface of radius a.

a. What is the slowing-down density due to this source distribution in an infinite medium?

b. Sketch the slowing-down density $q(r,u)$ as a function of r for various values of u.

6.3 A point source in an infinite medium is emitting neutrons of zero lethargy isotropically at a constant rate in time. Assume that the slowing-down process may be described by means of the Fermi age model once the first collision has occurred. Thus at points of first collision, the neutrons enter a continuous slowing-down process.

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a. Find an expression for the mean square distance traveled from the source point to the point at which neutrons acquire lethargy u. Complete the analysis up to the final step of integration over the radial distance variable.

b. If one were to consider the last scattering as a "free ride," what would be the appropriate expression for the slowing-down density $q(r,u)$? Note that neutrons will, in general, reach lethargy u at some point r′ which is different from r, and the probability of a successful last flight to r must be taken into account. (Leave this result in integral form.)

6.4 It is desired to increase the thermal flux in the moderator region of a large reactor in order to accommodate a certain experiment which is to be performed in that region. This increase is to be achieved by introducing a very thin foil (of thickness t) of fissionable material in the vicinity of the experiment. Use the following model to examine this situation: (1) The moderator region is an infinite medium. (2) There are no fission sources anywhere except in the foil, which may be treated as an infinite plane source. (3) In the absence of the foil, there is a uniform source S_0 of thermal neutrons in this medium. (4) The foil is entirely transparent to fast neutrons, but has thermal cross sections $\Sigma_a^{(F)}$ and Σ_f. (5) Neutrons from fissions in the foil appear with zero lethargy and can be assumed to slow down according to Fermi age theory.

a. Derive an expression for the thermal flux as a function of the distance from the foil. For this calculation use the kernel method and recall that if the diffusion equation for thermal neutrons is written in the form

$$-D_{\text{th}}\phi_{\text{th}}''(x) + \Sigma_a^{\text{th}}\phi_{\text{th}}(x) = S(x) \tag{P6.1}$$

then the solution is

$$\phi_{\text{th}}(x) = \int_{-\infty}^{\infty} \varphi(x;x')\, S(x')\, dx' \tag{P6.2}$$

The cross sections in Eq. (P6.1) refer to the diffusion medium, and the source function $S(x)$ may be defined so as to include *all* the remaining neutron *sources* and *sinks* in the medium. The function $\varphi(x;x')$ is the plane-source kernel and gives the flux at x due to a unit source at x'. The final result for $\phi_{\text{th}}(x)$ is to include only the nuclear constants of the system, S_0, and t.

b. Check the solution by showing that it reduces to the proper value when (1) x, the distance from the foil, is large and (2) the foil is removed.

6.5 Consider a bare homogeneous spherical reactor with moderator and fuel cross sections as specified in Prob. 4.8 and Prob. 5.12.

a. Find the critical radius, using $N_{\text{mod}}/N_{\text{fuel}} = 10{,}000$. Include the fast-leakage correction only in so far as it affects the thermal fissions; that is, neglect fast leakage in computing the fast fissions.

b. Suppose that a material which has negligible absorption cross section for neutrons, except at thermal, is spread uniformly throughout a bare critical reactor. If the amount of this material used is such as to cause a change $\delta\Sigma_a^{\text{th}}$ in the total thermal-absorption cross section, find the change $\delta k/k$ due to the change $\delta\Sigma_a^{\text{th}}/\Sigma_a^{\text{th}}$.

c. Suppose, further, that the introduction of the absorber specified in part *b* is to be compensated for by an increase in the size of the reactor; that is, more fuel-moderator mixture is added to the outer dimensions of the system. Find the fractional change $\delta k/k$ in terms of the change $\delta R/R$. Neglect the extrapolation distance and the effect of δR on the fast effect. Show that by neglecting products of L^2B^2 and $B^2\tau_{\text{th}}$ the result may be written

$$\frac{\delta k}{k} \simeq \frac{2B^2(\tau_{\text{th}} + L^2)}{1 + L^2B^2}\frac{\delta R}{R}$$

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d. An absorber of the type described in *b* is added to the reactor in *a* such that $N_{mod}/N_{abs} = 10{,}000$ (where N_{abs} = nuclear density of the added material). The thermal-absorption cross section of this material is 1.0 barn. Find the increase in the size of the reactor necessary to maintain criticality. (Neglect the additional absorber which would be spread over the additional volume of the reactor.) By what percentage does this increase the mass of the fuel?

6.6 *a.* Calculate the critical radius of the reactor of Prob. 6.5, using $N_{mod}/N_{fuel} = 1{,}000$. This time include the fast-leakage factor in all terms of the critical equation. It is suggested that the calculation for criticality be carried out by assuming various values of B^2 and computing the corresponding k. The B^2 which yields $k = 1$ will define the critical size.

b. Calculate $(\delta k/\delta T)$ for a reactor in which fission neutrons appear at one energy. For this calculation assume the following:

(1) The change in temperature affects the fuel density only. This situation could occur in the case of a liquid fuel wherein the coefficient of expansion of the liquid is much greater than that of the solid elements of the system ($\delta l/\delta T = 0$ for all nonfuel components; l = characteristic length).

(2) All microscopic cross sections are temperature independent.

(3) $\dfrac{\partial u_{th}}{\partial T} \equiv 0$

(4) $\Sigma_t(u) \simeq \Sigma_s(u) \simeq \Sigma_s^{(0)}(u)$, where $\Sigma_s^{(0)}(u)$ = scattering cross section of nonfuel components.

c. Compute and plot the leakage, absorptions, and fissions per unit lethargy as a function of lethargy. Graph the thermal contributions by a block at u_{th}.

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CHAPTER 7

TRANSPORT THEORY

7.1 Introduction

A complete description of the neutron population must specify simultaneously the distribution in space, energy, time and direction of motion. The primary objective of this chapter is to develop techniques and computational procedures for determining the angular distribution of neutrons. For this purpose we introduce the following function which gives the most general description of the neutron density:

$$n(\mathbf{r},v,\boldsymbol{\Omega},t)\,d\mathbf{r}\,dv\,d\boldsymbol{\Omega} \equiv \begin{array}{l}\text{number of neutrons at time } t \text{ in volume } d\mathbf{r} \\ \text{about } \mathbf{r}, \text{ whose speeds lie in } dv \text{ about } v \text{ and} \\ \text{whose directions of motion (velocity vectors)} \\ \text{lie in differential solid angle } d\boldsymbol{\Omega} \text{ about} \\ \text{direction } \boldsymbol{\Omega}\end{array} \tag{7.1}$$

In the usual way we define also a neutron flux:

$$\phi(\mathbf{r},v,\boldsymbol{\Omega},t) \equiv v\,n(\mathbf{r},v,\boldsymbol{\Omega},t) \tag{7.2}$$

Thus $\phi(\mathbf{r},v,\boldsymbol{\Omega},t)$ is the total track length traveled per unit time by the neutrons described in (7.1). In most of the analysis presented in this chapter it will be convenient to work with this function rather than with the neutron density.

The general problems with which we will be concerned here deal with homogeneous and isotropic media. Although this is not the most general situation of interest in reactor analysis, the transport model developed for these media has sufficient generality to treat a large variety of practical reactor problems. Thus again we avoid questions of anisotropy and inhomogeneity, arguing that these are highly specialized situations which are not amenable to the general methods of reactor analysis and, therefore lie outside the scope of the present work.

We will lead up to the calculation of the neutron flux defined by (7.2) by introducing first a somewhat more limited description of the neutron population, namely, the one-velocity model. The appropriate function is $\phi(\mathbf{r},\boldsymbol{\Omega},t)$. By omitting, for a time, the problem of describing the distribution of neutrons in energy space, we can focus attention on the directional properties of the neutron motion. Thus our first objective is to construct a picture of the neutron population which gives its dis-

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tribution in both space and direction of motion. This problem is treated in Sec. 7.2. The more general model of the transport theory which accounts also for the distribution of the neutrons in energy space is considered in Sec. 7.3.

The analytical method employed in both of these presentations involves the use of expansions in spherical harmonics. In each case the original integrodifferential equation, which states the neutron-balance condition, is reduced to an infinite set of coupled equations in the various *harmonics* of the function ϕ. By truncating the series expansion for the flux, this set can be further reduced to finite size (depending on the accuracy desired), and it is shown that in first approximation the resulting equations yield the diffusion theory and Fermi age models.

A second general technique for treating the angular distribution of the neutron flux is presented in Sec. 7.4. This is the method of integral equations. Solutions for the *directed flux* $\phi(\mathbf{r},\mathbf{\Omega})$ are derived on the basis of the one-velocity model for various media of infinite extent. The application of these solutions for the infinite medium to systems of finite size is demonstrated in the case of the homogeneous slab and sphere.

Section 7.5 deals with the application of the spherical-harmonics method developed for the one-velocity model of Sec. 7.2 to two situations of interest to reactor analysis. The emphasis is placed on various aspects of the computational procedure involved in such calculations and on the selection of suitable boundary conditions in terms of the directed flux.

The last section is devoted to the application of the transport model of Sec. 7.3, which includes the energy dependence of the neutron flux, to the calculation of the age in hydrogenous mixtures. This is the so-called Fermi-Marshak method. Computations are performed for various metal-water mixtures of practical importance to reactor design, and the results are compared with several other techniques for estimating the neutron age.

7.2 The One-velocity Transport Equation

a. Neutron Balance. We begin our study of the transport model by introducing the one-velocity approximation. For the analysis of this section we use in place of the more general description $\phi(\mathbf{r},v,\mathbf{\Omega},t)$ the following definition of the neutron flux:

$$\phi \equiv \phi(\mathbf{r},\mathbf{\Omega},t) = v\, n(\mathbf{r},\mathbf{\Omega},t) \tag{7.3}$$

$\equiv$ total track length per unit time at time t in unit volume about point $\mathbf{r}$ of v-speed neutrons whose directions of motion lie in the unit solid angle about $\mathbf{\Omega}$

The direction (unit vector) $\mathbf{\Omega}$ is specified by means of the co-latitude

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angle θ and the aximuth angle ψ (see Fig. 7.1). The velocity vector which describes the neutron motion is given in terms of the neutron speed through the relation

$$\mathbf{v} \equiv v\mathbf{\Omega} \tag{7.4}$$

It is important to recognize that the function defined by (7.3) is related to our previous description of the one-velocity flux (5.2) through the equation

$$\phi(\mathbf{r},t) = \int_{\mathbf{\Omega}} \phi(\mathbf{r},\mathbf{\Omega},t)\, d\mathbf{\Omega}$$

Thus our present definition simply selects from the entire population of neutrons of speed v those which are moving in a particular direction.

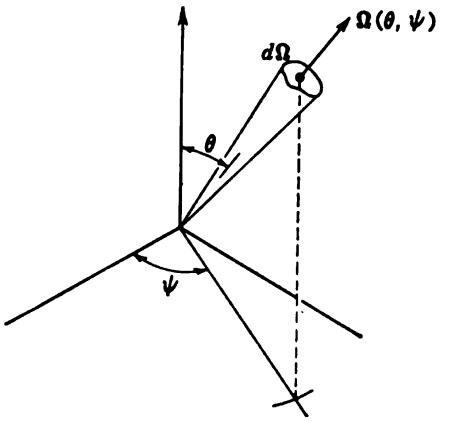

FIG. 7.1 Direction vector in laboratory system of coordinates.

With the foregoing definitions, we may proceed to construct, by considering the neutron balance, an equation governing the neutron density. Consider, then, the set of neutrons of speed v in the volume element $d\mathbf{r}$ about $\mathbf{r}$ at time t which have velocity vectors that lie in $d\mathbf{\Omega}$ about $\mathbf{\Omega}$. From conservation requirements it must be true that

$$\begin{aligned} &\text{I}\left[\begin{array}{l}\text{Number of neutrons}\\ \text{moving in direction } \mathbf{\Omega}\\ \text{which appear per unit}\\ \text{time from sources in } d\mathbf{r}\end{array}\right] + \text{II}\left[\begin{array}{l}\text{number of neutrons of}\\ \text{direction } \mathbf{\Omega} \text{ gained per}\\ \text{unit time in } d\mathbf{r} \text{ from}\\ \text{scattering collisions}\\ \text{which scatter the}\\ \text{neutron from all direc-}\\ \text{tions } \mathbf{\Omega}' \text{ to } \mathbf{\Omega}\end{array}\right] \\ &- \text{III}\left[\begin{array}{l}\text{net number of neutrons}\\ \text{moving in direction } \mathbf{\Omega}\\ \text{that are lost per unit}\\ \text{time by leakage}\\ \text{through the bound-}\\ \text{aries of } d\mathbf{r}\end{array}\right] - \text{IV}\left[\begin{array}{l}\text{number of neutrons of}\\ \text{direction } \mathbf{\Omega} \text{ which are}\\ \text{removed per unit time}\\ \text{from } d\mathbf{r} \text{ by absorption}\\ \text{collisions}\end{array}\right] \\ &- \text{V}\left[\begin{array}{l}\text{number of neutrons of}\\ \text{direction } \mathbf{\Omega} \text{ which are}\\ \text{removed per unit time}\\ \text{from } d\mathbf{r} \text{ by being scat-}\\ \text{tered into a new}\\ \text{direction}\end{array}\right] = \text{VI}\left[\begin{array}{l}\text{change per unit time in}\\ \text{net number of neutrons}\\ \text{of direction } \mathbf{\Omega} \text{ in } d\mathbf{r}\end{array}\right] \end{aligned} \tag{7.5}$$

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Consider now the individual terms in this equation. The first term describes the rate at which the one-velocity neutrons with direction in $d\Omega$ appear from sources in $d\mathbf{r}$. Call this function S; then, the first term of (7.5) is given by

$$S(\mathbf{r},\Omega,t)\, d\mathbf{r}\, d\Omega \tag{7.6}$$

The second term of (7.5) gives the rate at which neutrons arrive in the differential element $d\mathbf{r}\, d\Omega$ by being scattered into the direction about Ω from *all* other directions. To determine this quantity we compute first the number of neutrons of direction Ω' that are scattered into the direction Ω by collisions which occur in $d\mathbf{r}$ at time t. The integral of this quantity over all directions Ω' yields the required term. Now,

$$\Sigma_s\phi(\mathbf{r},\Omega',t)\, d\mathbf{r}\, d\Omega' \tag{7.7}$$

gives the total number of scattering collisions in $d\mathbf{r}$ per unit time experienced by neutrons with directions in $d\Omega'$ about Ω'. Of this number a certain fraction $f(\Omega;\Omega')\, d\Omega$ will emerge from these collisions moving in directions that lie in $d\Omega$ about Ω. Formally, we define the function $f(\Omega;\Omega')$ as the frequency function for the change of direction from Ω' to Ω. Note that [cf. Eq. (2.11)]

$$f(\Omega;\Omega')\, d\Omega \equiv \begin{array}{l}\text{probability that a neutron with an initial velocity}\\ \text{directed along } \Omega', \text{ when scattered, emerges from}\\ \text{the collision with a velocity vector which lies in the}\\ \text{solid angle } d\Omega \text{ about } \Omega\end{array} \tag{7.8}\star$$

Thus the product of (7.7) and (7.8) gives the number of neutrons initially directed along Ω' which are scattered into the direction about Ω while occupying the volume $d\mathbf{r}$. The integral of this product over all Ω' gives the total number of such contributions to $d\mathbf{r}\, d\Omega$ from all directions. For the second term of (7.5) we have, then,

$$\int_{\Omega'} \Sigma_s\phi(\mathbf{r},\Omega',t)\, f(\Omega;\Omega')\, d\Omega'\, d\Omega\, d\mathbf{r} \tag{7.9}$$

The third term of (7.5), which represents the net losses from $d\mathbf{r}\, d\Omega$ due to the leakage of neutrons through the boundaries of $d\mathbf{r}$, may be derived from a detailed count of the neutron passages through the surfaces of the volume element $d\mathbf{r} \equiv dx\, dy\, dz$ shown in Fig. 7.2. This element is selected for computational convenience such that its surfaces are oriented parallel to a Cartesian coordinate system erected in the space in question. Let us consider first the face $dx\, dy$ at station z. Now the number of neutrons with velocity vectors directed along Ω (in $d\Omega$) which will pass through this face in unit time is the number of such neutrons included in the volume which would be swept out by translating the face $dx\, dy$ a distance equal to v in the negative Ω direction. (This swept region of

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space is shown in Fig. 7.2.) Thus the number of neutrons entering $dx\,dy\,dz$ through the face $dx\,dy$ at z per unit time with directions in $d\Omega$ about Ω is

$$n(x,y,z,\Omega,t)\,d\Omega(\mathbf{v}\cdot\mathbf{k})\,dx\,dy \tag{7.10}$$

where $\mathbf{k}$ is the unit vector in the z direction. Similarly, the number

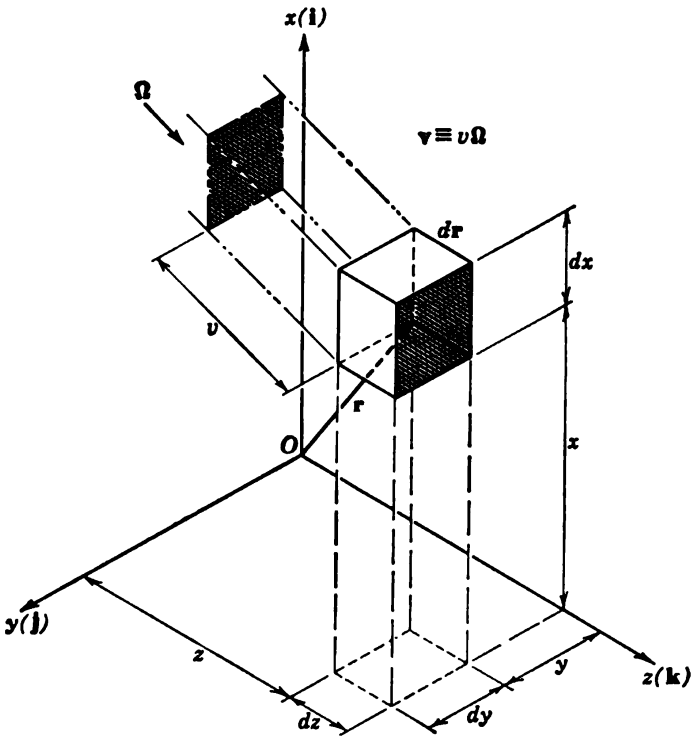

FIG. 7.2 Volume element $dx\,dy\,dz$.

of neutrons with direction Ω leaving the volume element through the $dx\,dy$ face at $z + dz$ per unit time is given by

$$n(x,y,z+dz,\Omega,t)\,d\Omega(\mathbf{v}\cdot\mathbf{k})\,dx\,dy$$

If dz is taken sufficiently small, a good approximation to $n(x,y,z+dz,\Omega,t)$ is

$$n(x,y,z+dz,\Omega,t) \simeq n(x,y,z,\Omega,t) + \frac{\partial}{\partial z}\,n(x,y,z,\Omega,t)\,dz$$

Therefore the net loss from $d\mathbf{r}$ per unit time of neutrons of direction Ω through the $dx\,dy$ faces is

$$\frac{\partial}{\partial z}\,n(x,y,z,\Omega,t)\,dx\,dy\,dz\,d\Omega\,(\mathbf{v}\cdot\mathbf{k}) \equiv \frac{\partial}{\partial z}\,\phi(\mathbf{r},\Omega,t)(\Omega\cdot\mathbf{k})\,d\mathbf{r}\,d\Omega \tag{7.11}$$

where we have used (7.4) and the usual definition for the flux. Analogous considerations for the other two faces lead to the following result: The

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net loss per unit time from $d\mathbf{r}$ of neutrons of direction $\boldsymbol{\Omega}$ in $d\boldsymbol{\Omega}$ by leakage (transport) across all faces is equal to

$$\left[\frac{\partial}{\partial x}\phi(\mathbf{r},\boldsymbol{\Omega},t)(\boldsymbol{\Omega}\cdot\mathbf{i}) + \frac{\partial}{\partial y}\phi(\mathbf{r},\boldsymbol{\Omega},t)(\boldsymbol{\Omega}\cdot\mathbf{j}) + \frac{\partial}{\partial z}\phi(\mathbf{r},\boldsymbol{\Omega},t)(\boldsymbol{\Omega}\cdot\mathbf{k})\right] d\mathbf{r}\, d\boldsymbol{\Omega}$$
$$= \boldsymbol{\Omega}\cdot\left(\mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\right) d\mathbf{r}\, d\boldsymbol{\Omega} = \boldsymbol{\Omega}\cdot\nabla\phi\, d\mathbf{r}\, d\boldsymbol{\Omega} \quad (7.12)$$

where $\mathbf{i}$ and $\mathbf{j}$ are the unit vectors in the x and y directions, respectively, and the definition for the gradient operator ∇ has been introduced.

Term IV in (7.5) is the number of neutrons lost from $d\mathbf{r}\, d\boldsymbol{\Omega}$ per unit time by absorption; this is given by

$$\Sigma_a\phi(\mathbf{r},\boldsymbol{\Omega},t)\, d\boldsymbol{\Omega}\, d\mathbf{r} \quad (7.13)$$

Term V gives the rate at which neutrons are scattered out of $d\mathbf{r}\, d\boldsymbol{\Omega}$ into new directions. This term may be represented by the form (7.7) with $\boldsymbol{\Omega}'$ replaced by $\boldsymbol{\Omega}$.

The last term VI is the time-rate of change of the neutron density in $d\mathbf{r}\, d\boldsymbol{\Omega}$. This may be written

$$\frac{\partial}{\partial t} n(\mathbf{r},\boldsymbol{\Omega},t)\, d\boldsymbol{\Omega}\, d\mathbf{r} = \frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},\boldsymbol{\Omega},t)\, d\boldsymbol{\Omega}\, d\mathbf{r} \quad (7.14)$$

The substitution of the results (7.6) through (7.14) into the verbal statement (7.5) yields the equation

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},\boldsymbol{\Omega},t) + \boldsymbol{\Omega}\cdot\nabla\phi(\mathbf{r},\boldsymbol{\Omega},t) + \Sigma_t\phi(\mathbf{r},\boldsymbol{\Omega},t)$$
$$= S(\mathbf{r},\boldsymbol{\Omega},t) + \int_{\boldsymbol{\Omega}'}\Sigma_s\phi(\mathbf{r},\boldsymbol{\Omega}',t)\, f(\boldsymbol{\Omega};\boldsymbol{\Omega}')\, d\boldsymbol{\Omega}' \quad (7.15)\star$$

where we have combined the losses from scattering and absorption into a single term, using the total cross section Σ_t, and have canceled out the common factor $d\mathbf{r}\, d\boldsymbol{\Omega}$. This integrodifferential equation for the neutron flux is usually called the *one-velocity transport equation*. In the general case it is very difficult to work with Eq. (7.15); however, in many cases the angular dependence of the quantities involved is sufficiently weak that certain approximations may be made which reduce (7.15) to a more tractable set of equations. In the work which follows we demonstrate one procedure for doing this which involves the use of expansions in spherical harmonics.

b. Frequency Function for Scattering Collision. Before proceeding with the analysis it will be convenient to exhibit some of the properties of the scattering function defined in (7.8). A useful geometric device for describing this function consists of defining it as the probability that the velocity vector of a scattered neutron pierce a particular area on the sur-

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face of a sphere drawn about the collision point (see Fig. 7.3). Thus if the collision occurs at the center of the sphere, then $f(\mathbf{\Omega};\mathbf{\Omega}')\,d\mathbf{\Omega}$ is the probability that the vector $\mathbf{v}$ passes through the area $d\mathbf{\Omega}$ on its surface.

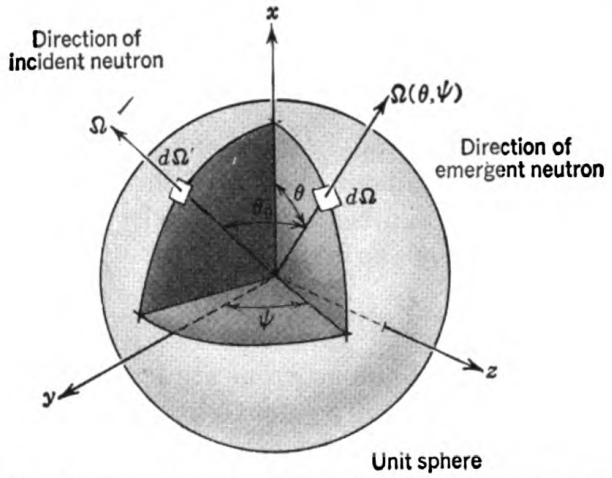

FIG. 7.3 Scattering angle in laboratory system of coordinates.

From the definition of a frequency function, it follows that

$$\int_{\mathbf{\Omega}} f(\mathbf{\Omega};\mathbf{\Omega}')\,d\mathbf{\Omega} = 1 \tag{7.16}$$

One of the fundamental assumptions in this analysis is that the medium in question is homogeneous and isotropic. An immediate consequence of this assumption is that f can be a function only of the angle between the directions $\mathbf{\Omega}$ and $\mathbf{\Omega}'$. We call this angle θ_0 and define

$$\cos\theta_0 \equiv \mu_0 = \mathbf{\Omega}\cdot\mathbf{\Omega}'$$

If we introduce also the function

$$\mathfrak{y}(\mu_0;\mathbf{\Omega}')\,d\mu_0 \equiv \begin{pmatrix}\text{probability that a neutron moving in direction}\\ \mathbf{\Omega}', \text{ when scattered, emerges with a new direction}\\ \text{whose cosine lies in } d\mu_0 \text{ about } \mu_0\end{pmatrix} \tag{7.17}$$

then it follows that

$$f(\mathbf{\Omega};\mathbf{\Omega}') = \frac{1}{2\pi}\,\mathfrak{y}(\mu_0;\mathbf{\Omega}') \tag{7.18}\star$$

Thus the general function f may be replaced in this treatment by the function $\mathfrak{y}$ which applies specifically to the isotropic medium. We use this formulation throughout the remainder of this analysis.

c. Expansion in Spherical Harmonics. The transport equation (7.15) may be reduced to a set of differential equations by introducing a suitable series representation for each of the functions which are involved. In this formulation we make the expansions in terms of an orthonormal set of spherical harmonics. We begin with the frequency function $\mathfrak{y}$

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defined in (7.17). Since this function is defined by means of the single variable μ_0, a suitable series representation can be given in terms of the Legendre polynomials[1] $P_\eta(\mu_0)$. Thus

$$\mathfrak{y}(\mu_0;\mathbf{\Omega}') \equiv \sum_{\eta=0}^{\infty} \left(\frac{2\eta+1}{2}\right) \mathfrak{y}_\eta P_\eta(\mu_0) \tag{7.19}$$

The coefficients $\mathfrak{y}_\eta$ are obtained in the usual way by applying the orthogonality properties of the Legendre polynomials. It follows that

$$\mathfrak{y}_\eta = \int_{-1}^{1} \mathfrak{y}(\mu_0;\mathbf{\Omega}')\, P_\eta(\mu_0)\, d\mu_0 \tag{7.20}$$

Let θ' and ψ' denote the arguments of the direction $\mathbf{\Omega}'$, and let $\mu \equiv \cos\theta$. Clearly, μ_0 is completely defined by the quantities μ', μ, ψ', and ψ. The Legendre polynomial $P_\eta(\mu_0)$ may be written in terms of these variables by the application of the addition theorem[2] which involves the associated functions P_η^β.

$$P_\eta(\mu_0) = \sum_{\beta=-\eta}^{\eta} \frac{(\eta-\beta)!}{(\eta+\beta)!} P_\eta^\beta(\mu)\, P_\eta^\beta(\mu')\, e^{i\beta(\psi-\psi')} \tag{7.21}$$

The definition of the associated functions used in this treatment follows that given by Sommerfeld:[3]

$$P_\eta^\beta(\mu) \equiv \sin^\beta\theta \frac{d^\beta}{d\mu^\beta} P_\eta(\mu) \tag{7.22}$$

This definition applies for $\eta \geq 0$ and $-\eta \leq \beta \leq \eta$. The associated functions with negative index β are related to those with positive index through the equation

$$P_\eta^{-\beta} = (-1)^\beta \frac{(\eta-\beta)!}{(\eta+\beta)!} P_\eta^\beta \tag{7.23}$$

which applies for the β and η intervals given above. Note that

$$P_\eta^0(\mu) = P_\eta(\mu) \tag{7.24}$$

$$P_\eta^\beta(\mu) = 0 \qquad \text{for } \beta > \eta \tag{7.25}$$

Some of the associated functions of lower index are given below:

$$\begin{aligned} P_2^2(\mu) &= 3(1-\mu^2) & P_2^{-1}(\mu) &= -\tfrac{1}{2}\mu(1-\mu^2)^{\frac{1}{2}} \\ P_2^1(\mu) &= 3\mu(1-\mu^2)^{\frac{1}{2}} & P_2^{-2}(\mu) &= \tfrac{1}{8}(1-\mu^2) \\ P_2^0(\mu) &= \tfrac{1}{2}(3\mu^2-1) & P_1^1(\mu) &= -2P_1^{-1}(\mu) = (1-\mu^2)^{\frac{1}{2}} \end{aligned} \tag{7.26}$$

[1] Refer to Eq. (4.21) and the discussion for definition of the lower-degree polynomials.

[2] A. Sommerfeld, "Partial Differential Equations in Physics," p. 134, Academic Press, Inc., New York, 1949.

[3] *Ibid.*, pp. 126–129.

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The substitution of Eq. (7.21) into (7.19), followed by the application of Eq. (7.18), yields a series representation for the scattering frequency function f.

$$f(\Omega;\Omega') = \frac{1}{2\pi}\sum_{\eta=0}^{\infty}\left(\frac{2\eta+1}{2}\right)\mathfrak{y}_\eta \sum_{\beta=-\eta}^{\eta}\frac{(\eta-\beta)!}{(\eta+\beta)!}P_\eta^\beta(\mu)\,P_\eta^\beta(\mu')\,e^{i\beta(\psi-\psi')} \tag{7.27}$$

We follow this general procedure to obtain suitable expansions for the neutron flux and the source function S. For these functions, however, it will be necessary to develop the expansions in the complete set of spherical harmonics. In this work we define these to be the orthonormal set:

$$Y_n^m(\Omega) \equiv Y_n^m(\mu,\psi) \equiv H_n^m P_n^m(\mu)e^{im\psi} \tag{7.28}$$

with $n = 0, 1, \ldots$ and $-n \leq m \leq n$. The complex conjugate of this function is

$$Y_n^{m*}(\Omega) \equiv H_n^m P_n^m(\mu)e^{-im\psi} = (-1)^m Y_n^{-m}(\Omega) \tag{7.29}$$

which applies for both positive and negative m. The coefficients H_n^m are defined by the relation

$$H_n^m \equiv \left[\frac{(2n+1)(n-m)!}{4\pi(n+m)!}\right]^{\frac{1}{2}} \tag{7.30}$$

Note that

$$H_n^{-m} = \frac{(n+m)!}{(n-m)!}H_n^m \tag{7.31}$$

The orthogonality property of the spherical harmonics is stated

$$\int_\Omega Y_\alpha^{\beta*}(\Omega)\,Y_n^m(\Omega)\,d\Omega \equiv \int_{-1}^{1}d\mu\int_0^{2\pi}Y_\alpha^{\beta*}(\mu,\psi)\,Y_n^m(\mu,\psi)\,d\psi = \delta_m^\beta\delta_n^\alpha \tag{7.32}$$

The symbol δ_n^α denotes the Kronecker delta and is zero if $n \neq \alpha$ and unity if $n = \alpha$.

The flux and the source may be expanded in terms of the functions (7.28). Thus

$$\phi(\mathbf{r},\Omega,t) \equiv \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\phi_n^m(\mathbf{r},t)\,Y_n^m(\Omega) \tag{7.33}$$

$$S(\mathbf{r},\Omega,t) \equiv \sum_{n=0}^{\infty}\sum_{m=-n}^{n}S_n^m(\mathbf{r},t)\,Y_n^m(\Omega) \tag{7.34}$$

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with the coefficients to be computed from the integrals

$$\phi_n^m(\mathbf{r},t) \equiv \int_{\mathbf{\Omega}} \phi(\mathbf{r},\mathbf{\Omega},t)\ Y_n^m(\mathbf{\Omega})\ d\mathbf{\Omega} \tag{7.35}$$

$$S_n^m(\mathbf{r},t) \equiv \int_{\mathbf{\Omega}} S(\mathbf{r},\mathbf{\Omega},t)\ Y_n^m(\mathbf{\Omega})\ d\mathbf{\Omega} \tag{7.36}$$

An inspection of the series for $f(\mathbf{\Omega};\mathbf{\Omega}')$ given in (7.27) reveals that by regrouping the various factors behind the summation signs one can express this series in terms of the spherical harmonics (7.28). It is easy to show that (7.27) may then be written in the form

$$f(\mathbf{\Omega};\mathbf{\Omega}') = \sum_{\eta=0}^{\infty} \sum_{\beta=-\eta}^{\eta} \mathfrak{y}_\eta Y_\eta^\beta(\mathbf{\Omega})\ Y_\eta^{\beta*}(\mathbf{\Omega}') \tag{7.37}$$

The transport equation (7.15) may now be expressed in terms of these expansions. The substitution of (7.33), (7.34), and (7.37) into (7.15) yields

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \mathbf{\Omega}\cdot\nabla + \Sigma_t\right) \sum_{n,m} \phi_n^m(\mathbf{r},t)\ Y_n^m(\mathbf{\Omega}) - \sum_{n,m} S_n^m(\mathbf{r},t)\ Y_n^m(\mathbf{\Omega})$$

$$= \int_{\mathbf{\Omega}'} \Sigma_s \sum_{n,m} \phi_n^m(\mathbf{r},t)\ Y_n^m(\mathbf{\Omega}') \sum_{\eta=0}^{\infty} \mathfrak{y}_\eta \sum_{\beta=-\eta}^{\eta} Y_\eta^\beta(\mathbf{\Omega})\ Y_\eta^{\beta*}(\mathbf{\Omega}')\ d\mathbf{\Omega}' \tag{7.38}$$

The integral in this equation may be reduced to a simpler form; thus

$$\Sigma_s \sum_{n,m} \phi_n^m(\mathbf{r},t) \sum_{\eta=0}^{\infty} \mathfrak{y}_\eta \sum_{\beta=-\eta}^{\eta} Y_\eta^\beta(\mathbf{\Omega}) \int_{\mathbf{\Omega}'} Y_n^m(\mathbf{\Omega}')\ Y_\eta^{\beta*}(\mathbf{\Omega}')\ d\mathbf{\Omega}'$$

$$= \Sigma_s \sum_{n,m} \phi_n^m(\mathbf{r},t)\ \mathfrak{y}_n Y_n^m(\mathbf{\Omega})$$

where we have applied the orthogonality property (7.32). The substitution of this result into (7.38) yields, upon rearranging,

$$\sum_{n,m} Y_n^m(\mathbf{\Omega}) \left[\left(\frac{1}{v}\frac{\partial}{\partial t} + \mathbf{\Omega}\cdot\nabla + \Sigma_t - \Sigma_s\mathfrak{y}_n\right)\phi_n^m(\mathbf{r},t) - S_n^m(\mathbf{r},t)\right] = 0 \tag{7.39}$$

This infinite set may be separated into coupled equations in the harmonics ϕ_n^m and S_n^m by multiplying through by $Y_\alpha^{\beta*}$ and integrating over all $\mathbf{\Omega}$ with the aid of the orthogonality property (7.32). For the first, third, fourth, and fifth terms we obtain immediately

$$\frac{1}{v}\frac{\partial}{\partial t}\phi_\alpha^\beta(\mathbf{r},t) + \Sigma_t\phi_\alpha^\beta(\mathbf{r},t) - \Sigma_s\mathfrak{y}_\alpha\phi_\alpha^\beta(\mathbf{r},t) - S_\alpha^\beta(\mathbf{r},t) \tag{7.40}$$

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The leakage term may be evaluated by expanding the operator $\mathbf{\Omega} \cdot \nabla$ into the form

$$\mathbf{\Omega} \cdot \nabla = \mu \frac{\partial}{\partial x} + \sin\theta \cos\psi \frac{\partial}{\partial y} + \sin\theta \sin\psi \frac{\partial}{\partial z} \tag{7.41}$$

and by using the exponential forms for the trigonometric functions of ψ. Then, the second term in (7.39) yields

$$\begin{aligned}
&\sum_{n,m} Y_n^m(\mathbf{\Omega})\, \mathbf{\Omega} \cdot \nabla \phi_n^m(\mathbf{r},t) \\
&\quad = \sum_{n,m} H_n^m \left\{ \left[\frac{(n+m)P_{n-1}^m(\mu) + (n+1-m)P_{n+1}^m(\mu)}{2n+1} \right] e^{im\psi} \frac{\partial}{\partial x} \right. \\
&\qquad + \frac{1}{2} P_n^m(\mu) \left(\frac{\partial}{\partial y} - i \frac{\partial}{\partial z} \right) e^{i\psi(m+1)} \sin\theta \\
&\qquad \left. + \frac{1}{2} P_n^m(\mu) \left(\frac{\partial}{\partial y} + i \frac{\partial}{\partial z} \right) e^{i\psi(m-1)} \sin\theta \right\} \phi_n^m(\mathbf{r},t)
\end{aligned} \tag{7.42}$$

where we have used the recursion relation[1]

$$\mu P_n^m(\mu) = \frac{(n+m)P_{n-1}^m(\mu) + (n+1-m)P_{n+1}^m(\mu)}{2n+1} \tag{7.43}$$

If the first term of (7.42) is multiplied by $Y_\alpha^{\beta *}(\mathbf{\Omega})$ and integrated over $\mathbf{\Omega}$, it can be shown that the result is

$$\begin{aligned}
&\sum_{n,m} \frac{(n+m)H_n^m}{2n+1} \int_{\mathbf{\Omega}} P_{n-1}^m(\mu) e^{im\psi} Y_\alpha^{\beta *}(\mathbf{\Omega})\, d\mathbf{\Omega} \left(\frac{\partial}{\partial x} \phi_n^m \right) \\
&\qquad = \left[\frac{(\alpha+1-\beta)(\alpha+1+\beta)}{(2\alpha+3)(2\alpha+1)} \right]^{\frac{1}{2}} \frac{\partial}{\partial x} \phi_{\alpha+1}^{\beta}
\end{aligned}$$

The remaining terms in (7.42) may be reduced in the same way. For these calculations the recursion relations

$$P_n^m(\mu) = \frac{P_{n+1}^{m+1}(\mu) - P_{n-1}^{m+1}(\mu)}{(2n+1)\sin\theta} \tag{7.44}$$

$$\begin{aligned}
P_n^m(\mu) = \frac{1}{(2n+1)\sin\theta} &[(n+2-m)(m-1-n)P_{n+1}^{m-1}(\mu) \\
&+ (n+m-1)(m+n)P_{n-1}^{m-1}(\mu)]
\end{aligned} \tag{7.45}$$

are needed. By this procedure the contributions of all the terms in (7.42) may be evaluated, and if these results are combined with (7.40), it can be shown that the set of equations (7.39) reduces to the form

[1] *Ibid.*, p. 132.

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$$
\begin{aligned}
-\frac{1}{v}\frac{\partial}{\partial t}\phi_\alpha^\beta(\mathbf{r},t) &- \Sigma_t\phi_\alpha^\beta(\mathbf{r},t) + S_\alpha^\beta(\mathbf{r},t) + \Sigma_s\eta_\alpha\phi_\alpha^\beta(\mathbf{r},t) \\
&= \left[\frac{(\alpha+1-\beta)(\alpha+1+\beta)}{(2\alpha+3)(2\alpha+1)}\right]^{\frac{1}{2}}\frac{\partial}{\partial x}\phi_{\alpha+1}^\beta(\mathbf{r},t) \\
&\quad + \left[\frac{(\alpha-\beta)(\alpha+\beta)}{(2\alpha-1)(2\alpha+1)}\right]^{\frac{1}{2}}\frac{\partial}{\partial x}\phi_{\alpha-1}^\beta(\mathbf{r},t) \\
&+ \frac{1}{2}\left(\frac{\partial}{\partial y} - i\frac{\partial}{\partial z}\right)\left\{\left[\frac{(\alpha+\beta)(\alpha+\beta-1)}{(2\alpha+1)(2\alpha-1)}\right]^{\frac{1}{2}}\phi_{\alpha-1}^{\beta-1}(\mathbf{r},t)\right. \qquad (7.46)\star \\
&\quad \left. - \left[\frac{(\alpha-\beta+2)(\alpha-\beta+1)}{(2\alpha+3)(2\alpha+1)}\right]^{\frac{1}{2}}\phi_{\alpha+1}^{\beta-1}(\mathbf{r},t)\right\} \\
&+ \frac{1}{2}\left(\frac{\partial}{\partial y} + i\frac{\partial}{\partial z}\right)\left\{-\left[\frac{(\alpha-\beta)(\alpha-\beta-1)}{(2\alpha+1)(2\alpha-1)}\right]^{\frac{1}{2}}\phi_{\alpha-1}^{\beta+1}(\mathbf{r},t)\right. \\
&\quad \left. + \left[\frac{(\alpha+\beta+1)(\alpha+\beta+2)}{(2\alpha+3)(2\alpha+1)}\right]^{\frac{1}{2}}\phi_{\alpha+1}^{\beta+1}(\mathbf{r},t)\right\}
\end{aligned}
$$

The index α in this set takes on all integer values $\alpha = 0, 1, 2, \ldots$, and β takes all values from $-\alpha$ to $+\alpha$. For cases of practical importance Eqs. (7.46) may be reduced to a finite set of simultaneous equations by taking only the first few values of α, say, $\alpha = 1, 2, \ldots 5$. Several applications of Eqs. (7.46) to the calculation of the angular distribution of neutrons in situations of interest to reactor analysis are presented in Sec. 7.5.

d. Diffusion Approximation. A first-order approximation of Eqs. (7.46) effectively removes the angular dependence of the neutron flux. However, this approximation is extremely useful and, in fact, forms the basis of the diffusion-theory model developed in Chap. 5. For this reason it will be instructive to examine the details of the approximation and to demonstrate the rigorous derivation of the diffusion equation (5.21).

It will be recalled that the diffusion approximation is based upon the assumption that the neutron-flux distribution is nearly isotropic (see Secs. 5.1b and 5.1d). This assumption is equivalent, analytically, to the requirement that the series expansion for the flux (7.33) be well represented by the first two terms. Consider, therefore, the set of Eqs. (7.46) for values of $\alpha = 0,1$. This means that in the series expansion (7.33) we retain only the terms involving ϕ_0^0, ϕ_1^{-1}, ϕ_1^0, and ϕ_1^1, all coefficients of higher indices being taken as identically zero. In order that our analysis be consistent, we will also require that the source function S be isotropic so that all $S_\alpha^\beta \equiv 0$, except S_0^0. Note that to assume otherwise is to imply that the sources have strong directional properties, and it would follow that the angular distribution of the neutron population in their immediate vicinity would also be nonisotropic.

With these conditions imposed on Eqs. (7.46), it is clear that this infinite

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coupled set reduces to the four equations

$$-\left(\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t\right)\phi_0^0+S_0^0+\Sigma_s\mathfrak{y}_0\phi_0^0=\frac{1}{\sqrt{3}}\frac{\partial}{\partial x}\phi_1^0$$
$$-\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)\phi_1^{-1}+\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\phi_1^1 \quad (7.47)$$

$$-\left(\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t\right)\phi_1^{-1}+\Sigma_s\mathfrak{y}_1\phi_1^{-1}=-\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\phi_0^0 \quad (7.48)$$

$$-\left(\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t\right)\phi_1^0+\Sigma_s\mathfrak{y}_1\phi_1^0=\frac{1}{\sqrt{3}}\frac{\partial}{\partial x}\phi_0^0 \quad (7.49)$$

$$-\left(\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t\right)\phi_1^1+\Sigma_s\mathfrak{y}_1\phi_1^1=\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)\phi_0^0 \quad (7.50)$$

The first two coefficients of the scattering function $\mathfrak{y}$ are obtained from the relation (7.20). These are

$$\mathfrak{y}_0=\int_{-1}^{1}\mathfrak{y}(\mu_0;\Omega')\,P_0(\mu_0)\,d\mu_0=\int_{-1}^{1}\mathfrak{y}(\mu_0;\Omega')\,d\mu_0\equiv 1 \quad (7.51)$$

$$\mathfrak{y}_1=\int_{-1}^{1}\mathfrak{y}(\mu_0;\Omega')\,P_1(\mu_0)\,d\mu_0=\int_{-1}^{1}\mathfrak{y}(\mu_0;\Omega')\mu_0\,d\mu_0\equiv\overline{\mu_0} \quad (7.52)$$

where $\overline{\mu_0}$ represents the average value of the cosine of the scattering angle [cf. Eq. (4.30)]. We demonstrate first the reduction of Eqs. (7.47) through (7.50) to the familiar diffusion equation for the case of a system at *steady state*. Thus the derivatives $\partial/\partial t\to 0$, and it follows immediately that Eqs. (7.48), (7.49), and (7.50) for the functions ϕ_1^{-1}, ϕ_1^0, and ϕ_1^1 yield the solutions

$$\phi_1^{-1}=\frac{1}{\sqrt{6}\,(\Sigma_t-\Sigma_s\overline{\mu_0})}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\phi_0^0 \quad (7.53)$$

$$\phi_1^0=-\frac{1}{\sqrt{3}\,(\Sigma_t-\Sigma_s\overline{\mu_0})}\frac{\partial}{\partial x}\phi_0^0 \quad (7.54)$$

$$\phi_1^1=-\frac{1}{\sqrt{6}\,(\Sigma_t-\Sigma_s\overline{\mu_0})}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)\phi_0^0 \quad (7.55)$$

where we have used (7.52). As a notational convenience we introduce the symbol D to define the ratio

$$D\equiv\frac{1}{3(\Sigma_t-\Sigma_s\overline{\mu_0})} \quad (7.56)\star$$

We call this quantity the *diffusion coefficient* and recognize that it is the function previously given by Eq. (5.44). The substitution of the results (7.53) through (7.55) into (7.47) yields a single *second-order* partial differential equation for ϕ_0^0. If we apply also the definition (7.56), then for

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the steady-state condition the appropriate equation is found to be

$$-\Sigma_a\phi_0^0 + S_0^0 = -\frac{\partial}{\partial x} D \frac{\partial}{\partial x} \phi_0^0 - \frac{\partial}{\partial y} D \frac{\partial}{\partial y} \phi_0^0 - \frac{\partial}{\partial z} D \frac{\partial}{\partial z} \phi_0^0 = -\nabla \cdot D\nabla\phi_0^0 \tag{7.57}$$

The functions ϕ_0^0 and S_0^0 may be replaced by the "total" quantities $\phi(\mathbf{r})$ and $S(\mathbf{r})$, inasmuch as each represents the isotropic component of their respective functions. We compute, therefore, the total functions $\phi(\mathbf{r})$ and $S(\mathbf{r})$ from relations of the form (7.5). For this computation we require appropriate expansions for the steady-state functions; thus, the relations (7.33) and (7.34) are replaced by the set

$$\begin{aligned} \phi(\mathbf{r},\boldsymbol{\Omega}) &= \sum_{n,m} \phi_n^m(\mathbf{r})\, Y_n^m(\boldsymbol{\Omega}) \\ S(\mathbf{r},\boldsymbol{\Omega}) &= \sum_{n,m} S_n^m(\mathbf{r})\, Y_n^m(\boldsymbol{\Omega}) \end{aligned} \tag{7.58}$$

Then, the total track length $\phi(\mathbf{r})$ of all neutrons of speed v, regardless of their direction of motion, is given by

$$\phi(\mathbf{r}) \equiv \int_{\boldsymbol{\Omega}} \phi(\mathbf{r},\boldsymbol{\Omega})\, d\boldsymbol{\Omega} = \sum_{n,m} \phi_n^m(\mathbf{r}) \int_{\boldsymbol{\Omega}} Y_n^m(\boldsymbol{\Omega})\, d\boldsymbol{\Omega} \equiv \frac{\phi_0^0}{H_0^0} \tag{7.59}$$

and the total number $S(\mathbf{r})$ of neutrons of speed v produced by sources is

$$S(\mathbf{r}) \equiv \int_{\boldsymbol{\Omega}} S(\mathbf{r},\boldsymbol{\Omega})\, d\boldsymbol{\Omega} = \sum_{n,m} S_n^m(\mathbf{r}) \int_{\boldsymbol{\Omega}} Y_n^m(\boldsymbol{\Omega})\, d\boldsymbol{\Omega} \equiv \frac{S_0^0}{H_0^0} \tag{7.60}$$

where we have used the definition of the spherical harmonics (7.28) and the orthogonality property (7.32).

The substitution of these results into (7.57) yields

$$-\nabla \cdot D\nabla\phi(\mathbf{r}) + \Sigma_a\phi(\mathbf{r}) = S(\mathbf{r}) \tag{7.61}\star$$

which in the case of a uniform medium reduces to the usual form (5.22).

e. Net Current. The net current relation (5.19) introduced in the development of the diffusion theory may be derived from the first-order solution to the transport equation. For this purpose we introduce the following definition:

$$J_s(\mathbf{r}) \equiv \begin{array}{l}\text{net number of neutrons which pass per unit time through} \\ \text{a unit area at } \mathbf{r} \text{ which is so oriented that the normal to its} \\ \text{surface is in the direction } \mathbf{s}\end{array} \tag{7.62}$$

This number can be computed from the neutron density $n(\mathbf{r},\boldsymbol{\Omega})\, d\boldsymbol{\Omega}$, which gives the number of neutrons per unit volume around $\mathbf{r}$ with velocities which lie in $d\boldsymbol{\Omega}$ about $\boldsymbol{\Omega}$. The number of these neutrons which

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pass through the area dA shown in Fig. 7.4 per unit time is given by

$$[v\, n(\mathbf{r},\Omega)\, d\Omega](\Omega \cdot \mathbf{s})\, dA = \phi(\mathbf{r},\Omega)(\Omega \cdot \mathbf{s})\, dA\, d\Omega \tag{7.63}$$

where $\mathbf{s}$ is the direction normal to dA. This relation is the contribution to the net current through dA (with normal $\mathbf{s}$) of neutrons with velocities in $d\Omega$ about Ω. The net current through dA of neutrons moving in all directions $J_s(\mathbf{r})\, dA$ is obtained by integrating (7.63) over all Ω.

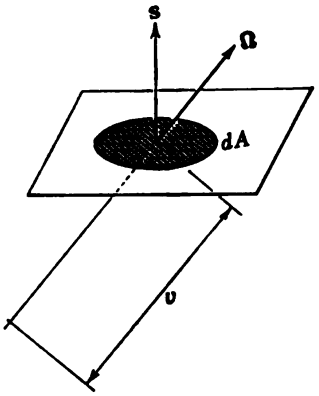

FIG. 7.4 Neutron velocity and the surface dA.

$$\begin{aligned} J_s(\mathbf{r})\, dA &= dA \int_\Omega \phi(\mathbf{r},\Omega)(\Omega \cdot \mathbf{s})\, d\Omega \\ &= dA \left\{ \mathbf{s} \cdot \left[\int_\Omega \phi(\mathbf{r},\Omega)\Omega\, d\Omega \right] \right\} \end{aligned} \tag{7.64}$$

The expression at the extreme right is of the form

$$J_s(\mathbf{r}) = \mathbf{s} \cdot \mathbf{J}(\mathbf{r}) \tag{7.65}$$

Thus the net current through dA of normal $\mathbf{s}$ may be regarded as the component in the $\mathbf{s}$ direction of a vector $\mathbf{J}(\mathbf{r})$ which we shall call the *vector net current* and define as [cf. Eqs. (7.64) and (7.65)]

$$\mathbf{J}(\mathbf{r}) \equiv \int_\Omega \phi(\mathbf{r},\Omega)\Omega\, d\Omega \tag{7.66}\star$$

The integral is evaluated by noting that Ω may be written in the form

$$\Omega \equiv \mathbf{i} \cos\theta + \mathbf{j} \sin\theta \cos\psi + \mathbf{k} \sin\theta \sin\psi \tag{7.67}$$

(see Fig. 7.3) and using the spherical-harmonics expansion for $\phi(\mathbf{r},\Omega)$ given in (7.58):

$$\mathbf{J}(\mathbf{r}) = \sum_{n,m} \phi_n^m(\mathbf{r}) \int_\Omega Y_n^m(\Omega)\Omega\, d\Omega \tag{7.68}$$

We note that

$$\cos\theta = \frac{Y_1^0(\Omega)}{H_1^0} \tag{7.69}$$

$$\sin\theta \cos\psi = \frac{1}{2H_1^1}[Y_1^1(\Omega) - Y_1^{-1}(\Omega)] \tag{7.70}$$

$$\sin\theta \sin\psi = \frac{1}{i2H_1^1}[Y_1^1(\Omega) + Y_1^{-1}(\Omega)] \tag{7.71}$$

If these relations are used in (7.68) and the integrations performed with the aid of the orthogonality properties of the Y_n^m, it may be shown that

$$\mathbf{J}(\mathbf{r}) = \frac{\phi_1^0(\mathbf{r})}{H_1^0}\mathbf{i} + \frac{1}{2H_1^1}[\phi_1^1(\mathbf{r}) - \phi_1^{-1}(\mathbf{r})]\mathbf{j} - \frac{1}{2iH_1^1}[\phi_1^1(\mathbf{r}) + \phi_1^{-1}(\mathbf{r})]\mathbf{k} \tag{7.72}$$

A first-order estimate of this function may be obtained by using

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expressions of the form[1] (7.53) through (7.55) for ϕ_1^0, ϕ_1^{-1}, and ϕ_1^1. The result is [with the application of (7.30) for H_n^m]

$$\mathbf{J}(\mathbf{r}) = -D\sqrt{4\pi}\left[\mathbf{i}\frac{\partial}{\partial x}\phi_0^0(\mathbf{r}) + \mathbf{j}\frac{\partial}{\partial y}\phi_0^0(\mathbf{r}) + \mathbf{k}\frac{\partial}{\partial z}\phi_0^0(\mathbf{r})\right]$$

As in the case of the diffusion equation (7.61), this expression may be written in terms of the total flux $\phi(\mathbf{r})$. If we use, then, (7.59) and note that $H_0^0 = (4\pi)^{-\frac{1}{2}}$, it follows that

$$\mathbf{J}(\mathbf{r}) = -D\nabla\phi(\mathbf{r})$$

which is the relation (5.19).

f. One-dimensional Problems. The general results obtained in Sec. 7.2d reduce to much simpler form in the event that the flux and the source are functions of a single space variable x. In these situations the system must have azimuthal symmetry, and the angular distribution of the flux and the source may be completely specified by a single angle, namely, the angle θ which the neutron direction of motion makes with the x axis. If $\cos\theta \equiv \mu$, then it is convenient to define the flux in these systems in the following manner:

$$\phi(x,\mu,t)\,d\mu \equiv \begin{array}{l}\text{total track length per unit volume per unit time}\\ \text{around } x \text{ of all neutrons whose directions of motion}\\ \text{with respect to the } x \text{ axis are such that the cosines}\\ \text{of these directions lie in the conical shell } d\mu \text{ about } \mu\end{array} \tag{7.73}$$

In our selection of a suitable series expansion for the scattering function $\mathfrak{y}$ we found it convenient to use the Legendre functions, since both $\mathfrak{y}$ and the P_n are defined by a single variable μ_0 over the same interval $(-1,1)$. It is clear that the flux defined by (7.73) has these same properties in so far as the variable μ is concerned. Thus we select as a suitable series representation for (7.73) the form [cf. Eq. (7.19)]

$$\phi(x,\mu,t) \equiv \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)\phi_n(x,t)\,P_n(\mu) \tag{7.74}$$

The coefficients $\phi_n(x,t)$ are obtained by the application of the orthogonality properties of the Legendre polynomials. In the usual way we compute

$$\phi_n(x,t) = \int_{-1}^{1}\phi(x,\mu,t)\,P_n(\mu)\,d\mu \tag{7.75}$$

These coefficients are related to the quantities $\phi_n^{0|}(x,t)$ used in the general spherical-harmonics expansion of the function $\phi(\mathbf{r},\boldsymbol{\Omega},t)$ [see Eq. (7.35)].

[1] Note that the analysis of the steady-state problem is identical to the time-dependent case considered in Sec. 7.2d, except that all coefficients used in the expansions are functions only of $\mathbf{r}$.

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This relationship is easily demonstrated. First, we observe that $\phi(x,\mu,t)$ is obtained from $\phi(\mathbf{r},\mathbf{\Omega},t)$ by integrating the latter function over all azimuthal directions ψ, since $\phi(x,\mu,t)$ gives the track length of all neutrons whose directions of motion are at an angle θ to the x axis regardless of their azimuthal orientation. We have, then,

$$\phi(x,\mu,t) \equiv \int_0^{2\pi} \phi(x,\mathbf{\Omega},t)\, d\psi$$

$$= \sum_{n,m} \phi_n^m(x,t)\, H_n^m P_n^m(\mu) \int_0^{2\pi} e^{im\psi}\, d\psi = 2\pi \sum_{n=0}^{\infty} \phi_n^0(x,t)\, H_n^0 P_n(\mu) \quad (7.76)$$

where we have used the expansion (7.33) for $\phi(x,\mathbf{\Omega},t)$, the definition (7.28) for the Y_n^m, and recognized that $P_n^0(\mu) \equiv P_n(\mu)$, and that the integral over ψ is zero except when $m \equiv 0$. If we compare the result (7.76) with the form (7.74), it follows that

$$\phi_n = \frac{\phi_n^0}{H_n^0} \quad (7.77)$$

An appropriate set of coupled differential equations analogous to the relations (7.46) may be developed for the one-dimensional formulation of the transport equation by performing a series of operations on the general result (7.39). First we integrate this equation over all ψ so as to collect the contributions to the neutron population of all processes which yield neutrons whose direction cosines are μ irrespective of their azimuthal orientation. This step is analogous to the integration performed in (7.76). If we use the fact that for a one-dimensional system

$$\mathbf{\Omega} \cdot \nabla = \mu \frac{\partial}{\partial x} \quad (7.78)$$

then it may be shown that the ψ integration of (7.39) yields

$$\sum_{n=0}^{\infty} H_n^0 P_n(\mu) \left[\left(\frac{1}{v} \frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x} + \Sigma_t - \Sigma_s \eta_n \right) \phi_n^0(x,t) - S_n^0(x,t) \right] = 0 \quad (7.79)$$

where we have introduced a description of the source S which is consistent with the requirements of the present system, namely [see Eq. (7.34)],

$$S(x,\mu,t) \equiv \int_0^{2\pi} S(x,\mathbf{\Omega},t)\, d\psi = 2\pi \sum_{n=0}^{\infty} S_n^0(x,t)\, H_n^0 P_n(\mu)$$

$$\equiv \sum_{n=0}^{\infty} \left(\frac{2n+1}{2} \right) S_n(x,t)\, P_n(\mu) \quad (7.80)$$

and the coefficients S_n^0 and S_n are related exactly as ϕ_n^0 and ϕ_n.

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The second step in the development is to multiply (7.79) through by $P_\alpha(\mu)$ and integrate over the interval $-1 \leq \mu \leq 1$. We obtain:

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \Sigma_t - \Sigma_s\mathfrak{y}_\alpha\right) H_\alpha^0\phi_\alpha^0\left(\frac{2}{2\alpha+1}\right) - H_\alpha^0 S_\alpha^0\left(\frac{2}{2\alpha+1}\right) + \sum_{n=0}^{\infty} H_n^0 \frac{\partial}{\partial x}\phi_n^0 \int_{-1}^{1} \mu P_n(\mu)\, P_\alpha(\mu)\, d\mu = 0 \quad (7.81)$$

where we have used again the orthogonality properties of the P_n functions and have suppressed, for the moment, the argument of ϕ_α^0. The integral is evaluated by using the recursion relation (7.43); for the present case $m \equiv 0$, and this equation reduces to

$$\mu P_n(\mu) = \frac{nP_{n-1}(\mu) + (n+1)P_{n+1}(\mu)}{2n+1} \quad (7.82)$$

The substitution of (7.82) into (7.81) yields, upon integration,

$$\left(\frac{1}{v}\frac{\partial}{\partial t} + \Sigma_t - \Sigma_s\mathfrak{y}_\alpha\right) H_\alpha^0\phi_\alpha^0\left(\frac{2}{2\alpha+1}\right) - H_\alpha^0 S_\alpha^0\left(\frac{2}{2\alpha+1}\right) + \frac{2(\alpha+1)H_{\alpha+1}^0}{(2\alpha+3)(2\alpha+1)}\frac{\partial}{\partial x}\phi_{\alpha+1}^0 + \frac{2\alpha H_{\alpha-1}^0}{(2\alpha-1)(2\alpha+1)}\frac{\partial}{\partial x}\phi_{\alpha-1}^0 = 0 \quad (7.83)$$

This result may be written in terms of the coefficients ϕ_n and S_n by applying the relation (7.77), and the corresponding one for S. Thus

$$-\frac{1}{v}\frac{\partial}{\partial t}\phi_\alpha(x,t) - \Sigma_t\phi_\alpha(x,t) + S_\alpha(x,t) + \Sigma_s\mathfrak{y}_\alpha\phi_\alpha(x,t) = \left(\frac{\alpha+1}{2\alpha+1}\right)\frac{\partial}{\partial x}\phi_{\alpha+1}(x,t) + \left(\frac{\alpha}{2\alpha+1}\right)\frac{\partial}{\partial x}\phi_{\alpha-1}(x,t) \quad (7.84)\star$$

where $\alpha = 0, 1, \ldots$. This set of coupled equations corresponds to the relations (7.46), and the complete description of the flux and source is to be obtained from the expansions (7.74) and (7.80). As in the more general formulation, estimates of the spatial and angular distribution of a one-velocity neutron population can be obtained to various degrees of accuracy depending upon the value of α at which the set (7.84) is truncated. The sample applications of the spherical-harmonics method mentioned previously will be performed, using the above set of equations.

We conclude the present discussion of the one-dimensional problems in transport theory with some remarks about the diffusion approximation and the relation of the partial currents introduced in Chap. 5 to the directed, or angular, flux. For this purpose we require the steady-state first-order approximations to the Eqs. (7.84). These are obtained by retaining only terms for $\alpha = 0, 1$; thus, the diffusion theory is based on the requirement that terms in $\alpha \geq 2$ are negligibly small in comparison

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to those for $\alpha < 2$. The appropriate set of equations are [using $S_\alpha = 0$ for $\alpha > 0$ and Eqs. (7.51) and (7.52)]

$$\frac{\partial}{\partial x}\phi_1(x) + \Sigma_a\phi_0(x) = S_0(x) \tag{7.85}$$

$$\frac{1}{3}\frac{\partial}{\partial x}\phi_0(x) + (\Sigma_t - \Sigma_s\overline{\mu_0})\phi_1(x) = 0 \tag{7.86}$$

which when combined yield

$$-D\frac{\partial^2}{\partial x^2}\phi_0(x) + \Sigma_a\phi_0(x) = S_0(x) \tag{7.87}$$

$$\phi_1(x) = -D\frac{\partial}{\partial x}\phi_0(x) \tag{7.88}$$

where we have used (7.56) and assumed that the medium is homogeneous. Now, from (7.75),

$$\phi_0(x) = \int_{-1}^{1}\phi(x,\mu)\,d\mu \equiv \phi(x) \equiv \text{total track length of all neutrons} \tag{7.89}$$

$$\phi_1(x) = \int_{-1}^{1}\phi(x,\mu)\mu\,d\mu \tag{7.90}$$

We can identify ϕ_1 from the results of our neutron-leakage calculation for Eq. (7.5). If we note that $\phi(x,\mu) \equiv n(x,\mu)(\mathbf{v}\cdot\mathbf{i})$ and bear in mind the definition of (7.73), it is clear that

$$\phi(x,\mu)\mu\,d\mu \equiv \text{number of neutrons crossing unit area per unit time (with area oriented so that its normal is in } x \text{ direction) in a direction whose angle with } x \text{ direction has cosine in } d\mu \text{ about } \mu \tag{7.91}$$

Thus the integral (7.90) is the net neutron current in the x direction at x. Moreover, if we write

$$J(x) \equiv \phi_1(x) = \int_{-1}^{0}\phi(x,\mu)\mu\,d\mu + \int_{0}^{1}\phi(x,\mu)\mu\,d\mu \tag{7.92}$$

and use the first-order (diffusion) approximation

$$\phi(x,\mu) = \tfrac{1}{2}\phi_0(x) + \tfrac{3}{2}\mu\phi_1(x) \tag{7.93}\star$$

from (7.74) in (7.92), we obtain

$$J(x) = [\tfrac{1}{4}\phi_0(x) + \tfrac{1}{2}\phi_1(x)] - [\tfrac{1}{4}\phi_0(x) - \tfrac{1}{2}\phi_1(x)] \equiv j_+(x) - j_-(x) \tag{7.94}$$

where we have defined

$$j_+(x) \equiv \int_{0}^{1}\phi(x,\mu)\mu\,d\mu = \tfrac{1}{4}\phi_0(x) + \tfrac{1}{2}\phi_1(x) \tag{7.95}$$

$$j_-(x) \equiv \int_{0}^{-1}\phi(x,\mu)\mu\,d\mu = \tfrac{1}{4}\phi_0(x) - \tfrac{1}{2}\phi_1(x) \tag{7.96}$$

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If we note that $\phi_1 = -D\phi_0' = -D\phi'$, then it is seen that these results for the partial currents are identical to the relations (5.49).

g. Boundary Conditions. A general set of boundary conditions were described in Sec. 5.1e, and it was pointed out that all these conditions could not be satisfied in detail when used in conjunction with the diffusion theory. In addition, it was mentioned that these conditions could be met only when applied with the more precise transport-theory model. As a matter of fact, not even this is possible, from a practical viewpoint. To illustrate this point, consider the vacuum-interface condition stated in (5.54). An analytical statement of this condition for the one-dimensional model of Sec. 7.2f would be

$$\phi(0,\mu) \equiv 0 \qquad \mu \leq 0 \tag{7.97}\star$$

where for convenience we have specified the vacuum boundary to be at $x = 0$. According to (7.97) no neutrons may return from the directions $-1 \leq \mu \leq 0$, i.e., from the vacuum. Now the variable μ takes on all values in the indicated interval; but, in actual computation, the expression for the directed flux must be obtained from a truncated set of equations based on (7.84), and this solution will necessarily involve a finite number of arbitrary constants. Thus it is not possible to satisfy (7.97) rigorously. In actual practice this difficulty is circumvented by introducing an alternate set of conditions,[1] which although not rigorous are at least consistent with the statement (7.97). One such set was suggested by Marshak, and these require that the following integral of the flux vanish at a vacuum interface:

$$\int_{-1}^{0} \phi(0,\mu)P_m(\mu)\, d\mu = 0 \qquad m = 1, 3, 5, \ldots \tag{7.98}$$

It is interesting to note that the integral for $m = 1$ yields the relation

$$\int_{-1}^{0} \phi(0,\mu)\mu\, d\mu = 0 \tag{7.99}$$

We note immediately from (7.96) that this integral is in fact the partial current in the direction of negative x, i.e., the return current from the vacuum. Thus, condition (7.99) requires that there be a zero net current from the vacuum; however, it should be recognized that inward currents from certain directions are still possible, provided of course that the gains from these be exactly balanced by flows outward from other directions. This condition on the inward current is precisely the statement of (5.57a) used in the diffusion theory. The vacuum-interface condition based on the requirement that the integral (7.98) for odd m vanish at the surface is a convenient extension, then, of the diffusion-theory relation.

[1] A. M. Weinberg and L. C. Noderer, "Theory of Neutron Chain Reaction," vol. I, pp. 11–18, Oak Ridge National Laboratory, CF 51-5-98, May 15, 1951.

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The rigorous statement for the interface condition between two dissimilar media is that the directed flux for each specified angle be continuous through the interface boundary. If we denote these media I and II, then we require that

$$\phi_{\mathrm{I}}(\mathbf{R},\mathbf{\Omega}) \equiv \phi_{\mathrm{II}}(\mathbf{R},\mathbf{\Omega}) \qquad \text{for all } \mathbf{\Omega} \tag{7.100}\star$$

where the vector $\mathbf{R}$ refers to the coordinates of a specific point on the interface boundary, and $\mathbf{\Omega}$ is any general direction. Thus (7.100) states that there be the same number of neutrons per unit volume around point $\mathbf{R}$ in medium I that are moving in direction $\mathbf{\Omega}$ as there are per unit volume around $\mathbf{R}$ in medium II. In the event that the flux is a function of a single space variable x, (7.100) takes the form

$$\phi_{\mathrm{I}}(x_0,\mu) = \phi_{\mathrm{II}}(x_0,\mu) \qquad -1 \leq \mu \leq 1 \tag{7.101}$$

where x_0 denotes the position of the interface. A numerical example of the application of this boundary condition for the P_3 approximation[1] is given in Sec. 7.5.

h. Time-dependent Equation in First-order Approximation. We conclude the present section with a discussion of the approximations involved in deriving a first-order solution to the transport equation which includes the time dependence. As in the case of the steady-state results obtained in Sec. 7.2d, time-dependent relations (7.47) through (7.50) may also be reduced to a single second-order partial differential equation involving the total track length $\phi(\mathbf{r},t)$. However, the essential features of this equation may be determined from a study of the one-dimensional system developed in Sec. 7.2f. Consider, therefore, a system wherein the spatial distribution of both the flux and the source can be specified by a single variable x and the angular distribution by the variable μ. The appropriate relations for the first-order approximation are obtained from (7.84) by taking only the terms in $\alpha = 0, 1$. Again we use $S_\alpha \equiv 0$ for $\alpha > 0$ and for convenience assume that S is not a function of time; then,

$$-\frac{1}{v}\frac{\partial}{\partial t}\phi_0(x,t) - \Sigma_a\phi_0(x,t) + S_0(x) = \frac{\partial}{\partial x}\phi_1(x,t) \tag{7.102}$$

$$-\frac{1}{v}\frac{\partial}{\partial t}\phi_1(x,t) - \Sigma_t\phi_1(x,t) + \Sigma_s\overline{\mu}_0\phi_1(x,t) = \frac{1}{3}\frac{\partial}{\partial x}\phi_0(x,t) \tag{7.103}$$

This pair of equations may be reduced to a single second-order equation in ϕ_0 by differentiating (7.102) with respect to t and (7.103) with respect to x and eliminating the mixed (t and x) derivative term in ϕ_1. The

[1] It has become customary to refer to the order of an approximation in the spherical-harmonics technique by indicating the highest-order harmonic retained from the infinite set of coupled equations.

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result is easily shown to be

$$\left(\frac{1 + 3\Sigma_a D}{v}\right)\frac{\partial}{\partial t}\phi(x,t) + \frac{3D}{v^2}\frac{\partial^2}{\partial t^2}\phi(x,t) = S(x) - \Sigma_a\phi(x,t) + D\frac{\partial^2}{\partial x^2}\phi(x,t) \quad (7.104)$$

where we have used (7.56) for D, along with the fact that $\phi_0(x,t) = \phi(x,t)$ and $S_0(x) \equiv S(x)$ [see, for example, (7.75)]. In treating specific problems with this equation, it may be necessary to use Eqs. (7.102) and (7.103) to evaluate the initial and/or boundary conditions.

It is of interest to compare Eq. (7.104) with the general form (5.21) which had been assumed in the development of the diffusion-theory model. Two distinct differences are revealed: (1) an additional term in the second derivative of t has appeared, and (2) the coefficient of the first derivative term in t has been altered. Inasmuch as the above result was obtained directly from the transport equation, one might expect Eq. (7.104) to be a more accurate statement of the nonstationary problem within the limitations of first-order theory. Evidently, the diffusion-theory model (5.21) implies certain additional assumptions or limitations which have resulted in the omission of the two features noted above. Some of these are considered in the discussion which follows. It should be recognized, however, that the adequacy of the first-order formulation (7.104) can be determined in the final analysis only by comparison with experiment since the error involved in the omission of the various higher-order terms is not evident, a priori.

Before discussing the discrepancies between Eqs. (7.104) and (5.21) it would be in order to comment further on the nature of Eq. (7.104). This equation in its general form is the so-called *telegraphist's* equation.[1] The unique feature of this equation is that it describes physical phenomena which exhibit both wavelike characteristics and residual disturbance effects. The wavelike behavior is given by the second-order term in t, and the residual disturbance effect, by the first-order term in t. Wave effects propagate with a finite velocity, and time-dependent neutron phenomena should properly include such a term since time-wise variations in neutron population cannot be sensed at a given distance from the source in less time than it takes a neutron of speed v to reach that point. However, after the passage of this "wave," the disturbance persists and this effect is described by the first-order term.

The essential feature of diffusion phenomena, on the other hand, is that all disturbances propagate with "infinite" velocity, i.e., any disturbance at any point in the medium is sensed everywhere instantly. It is of interest, then, to determine the limiting form of Eq. (7.104) for very large

[1] See, for example, P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," Part I, p. 865, McGraw-Hill Book Company, Inc., New York, 1953.

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neutron speed. For this calculation let us impose the condition that both vD and $\Sigma_a v$ remain finite as $v \to \infty$. Thus in the limit we require that the diffusion medium have vanishingly small absorption cross section and a very large transport cross section. In taking the limit, it is convenient to write (7.104) in the form

$$\left[1 + \frac{1}{v} 3\Sigma_a(vD)\right] \frac{\partial n}{\partial t} + \frac{3}{v^2}(vD) \frac{\partial^2 n}{\partial t^2} = S - (\Sigma_a v)n - (vD) \frac{\partial^2 n}{\partial x^2} \quad (7.105)$$

as $v \to \infty$, this equation reduces to

$$\frac{\partial n}{\partial t} = S - \Sigma_a v n + Dv \frac{\partial^2 n}{\partial x^2}$$

which is in the form (5.21). Thus the differential equation previously selected to describe time variations in the neutron population is a good approximation only if the absorption cross section is very small and the transport cross section very large. For those situations in which these conditions are not well satisfied, Eq. (7.104) may yield a better one-velocity first-order relation.

7.3 The Boltzmann Equation with Energy Dependence

a. Assumptions and Limitations. The details of the neutron slowing-down process and the role which it plays in determining the principal nuclear properties of a reactor have been discussed at some length in the introductory sections of Chaps. 4 and 6. In order to retain a greater degree of generality in this development, we select as the basis for the present model only the most elementary physical features of the processes involved. In this way we derive a general statement of the Boltzmann (transport) equation which includes the distribution of the neutrons in energy space. The only limitations which are imposed on the general results of this analysis are the two simplifying assumptions listed below:

(1) The medium is isotropic and homogeneous
(2) Neutron slowing down is due only to elastic-scattering collisions with nuclei

(7.106)

The first assumption is customarily made in dealing with problems of reactor physics. Problems that involve inhomogeneity and anisotropy on a microscopic scale are highly specialized situations and fall outside the scope of this work. Gross inhomogeneities, such as multiregion media, on the other hand, are of considerable interest here. The transport model is extremely useful for treating heterogeneous reactor configurations and boundary conditions; some remarks have already been made in Sec. 7.2g on the latter application.

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The purpose of assumption (2) is to simplify the calculation. As previously noted, the slowing-down process is due to the joint effects of both elastic- and inelastic-scattering collisions, and the relative importance of the two processes is determined by the neutron energy and composition of the media. By ignoring the inelastic process in the present analysis we imply that the system in question is relatively free of heavy nuclei. These systems will therefore tend to be thermal. Another reason for omitting the inelastic phenomenon is that a useful and accurate model is not yet available for describing the energy spectrum of neutrons emerging from an inelastic collision. In the absence of such information one can always utilize experimental data, but this approach is most effective when working with the multigroup formulation.

The assumptions (7.106) are the only limitations to be imposed in the derivation of the Boltzmann equation, and it is important to note that no specific relation has been selected to describe the neutron-scattering function. Thus the results of this analysis will be generally applicable to many systems of practical importance. In order to demonstrate the application of these results a particular scattering law is later selected, namely, isotropic scattering in the center-of-mass system of coordinates. However, this specific calculation will not be introduced until the general relations have been developed.

The purpose of the analysis which follows is to establish a suitable neutron-balance relation which takes into account the dependence of the relevant functions upon the neutron energy, direction of motion, time and spatial coordinates. The balance condition, which is an integro-differential equation, is then reduced to an infinite, but simpler, set of coupled integro-differential equations by expanding the flux, the neutron sources, and the scattering function in spherical harmonics. The derivation of the Fermi age model from these results is then demonstrated.

b. Neutron Balance. A neutron-balance condition may be written in terms of the flux $\phi(\mathbf{r},v,\mathbf{\Omega},t)\, d\mathbf{r}\, d\mathbf{\Omega}\, dv$ [see Eq. (7.2)], and this relation will give the net change in the number of neutrons which occupy the volume $d\mathbf{r}$, whose speeds lie between v and $v + dv$, and whose velocity vectors lie in the solid angle $d\mathbf{\Omega}$ about $\mathbf{\Omega}$. A statement analogous to (7.5) which accounts for all the physical processes that can affect this population may be written for the present case. In this formulation, however, we must include in addition to the specifications of (7.5) the requirement that the neutrons involved in the counting process be limited to those which occupy the speed interval dv about v. Thus Eq. (7.5) may be immediately extended to the present application if we replace the quantity $\phi(\mathbf{r},\mathbf{\Omega},t)\, d\mathbf{r}\, d\mathbf{\Omega}$ by $\phi(\mathbf{r},v,\mathbf{\Omega},t)\, d\mathbf{r}\, dv\, d\mathbf{\Omega}$. There is one important modification which must be made, however; this occurs in the term which involves the scattering integral. Before discussing this, let us first

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consider the remaining terms. For these we can write

$$\left[\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},v,\boldsymbol{\Omega},t) + \boldsymbol{\Omega}\cdot\nabla\phi(\mathbf{r},v,\boldsymbol{\Omega},t) + \Sigma_t(v)\,\phi(\mathbf{r},v,\boldsymbol{\Omega},t) - S(\mathbf{r},v,\boldsymbol{\Omega},t)\right] d\mathbf{r}\,dv\,d\boldsymbol{\Omega} \tag{7.107}$$

The first term gives the time rate of change of the neutrons which occupy the space $d\mathbf{r}\,dv\,d\boldsymbol{\Omega}$, the second, the losses due to the net transport of neutrons through the boundaries of $d\mathbf{r}$, the third, the losses from this space due to neutrons being absorbed or scattered out or "slowed out" (of the dv interval), and the fourth, the contributions from sources.

Now for the scattering-integral term. The total number of neutrons scattered into $d\mathbf{r}\,dv\,d\boldsymbol{\Omega}$ per unit time from all possible speeds v' and directions of motion $\boldsymbol{\Omega}'$ is obtained by taking the number of neutrons in $d\mathbf{r}\,dv'd\boldsymbol{\Omega}'$ which experience scattering collisions, namely,

$$\Sigma_s(v')\,\phi(\mathbf{r},v',\boldsymbol{\Omega}',t)\,d\mathbf{r}\,dv'\,d\boldsymbol{\Omega}'$$

and multiplying it by the probability that a scattering collision in $d\mathbf{r}\,dv'\,d\boldsymbol{\Omega}'$ throws the neutron into speed in dv about v with direction in $d\boldsymbol{\Omega}$ about $\boldsymbol{\Omega}$. We denote this probability $f(v,\boldsymbol{\Omega};v',\boldsymbol{\Omega}')\,dv\,d\boldsymbol{\Omega}$ and define [cf Eq. (7.8)].

$$f(v,\boldsymbol{\Omega};v',\boldsymbol{\Omega}')\,dv\,d\boldsymbol{\Omega} \equiv \begin{array}{l}\text{probability that a neutron with an initial}\\ \text{speed } v' \text{ and direction of motion } \boldsymbol{\Omega}', \text{ when}\\ \text{scattered, emerges from the collision with}\\ \text{a speed in } dv \text{ about } v \text{ and direction of}\\ \text{motion in } d\boldsymbol{\Omega} \text{ about } \boldsymbol{\Omega}\end{array} \tag{7.108}$$

The product of the total scatterings and this probability integrated over all accessible v' and $\boldsymbol{\Omega}'$ gives the total gain in the neutron population of the differential element $d\mathbf{r}\,dv\,d\boldsymbol{\Omega}$ due to the neutrons scattered in from higher speeds and different directions of motion:

$$\int_{v',\boldsymbol{\Omega}'} \phi(\mathbf{r},v',\boldsymbol{\Omega}',t)\,\Sigma_s(v')\,f(v,\boldsymbol{\Omega};v',\boldsymbol{\Omega}')\,d\mathbf{r}\,dv'\,d\boldsymbol{\Omega}'\,dv\,d\boldsymbol{\Omega} \tag{7.109}$$

The total neutron balance is obtained by adding (7.109) to (7.107) according to the sign convention of (7.5). If the differential $d\mathbf{r}\,dv\,d\boldsymbol{\Omega}$ is divided out, the result is found to be

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},v,\boldsymbol{\Omega},t) + \Sigma_t(v)\,\phi(\mathbf{r},v,\boldsymbol{\Omega},t) + \boldsymbol{\Omega}\cdot\nabla\phi(\mathbf{r},v,\boldsymbol{\Omega},t) = S(\mathbf{r},v,\boldsymbol{\Omega},t) + \int_{\mathbf{v}'} \phi(\mathbf{r},v',\boldsymbol{\Omega}',t)\,\Sigma_s(v')\,f(v,\boldsymbol{\Omega};v',\boldsymbol{\Omega}')\,dv'\,d\boldsymbol{\Omega}' \tag{7.110}\star$$

where $\mathbf{v}' \equiv v'\boldsymbol{\Omega}'$ and denotes the velocity *vector* of the incident neutron. Equation (7.110) is the Boltzmann (transport) relation which describes the distribution of neutrons in space, velocity, and time. This relation

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is completely general within the limitations of the conditions (7.106). Specific situations can be described by defining the source function S and the scattering function $f(v,\mathbf{\Omega};v',\mathbf{\Omega}')$.

c. Azimuthally Symmetric Scattering from Stationary Nuclei. In line with the concluding remarks above, further calculation is not possible until additional information on the properties of the system in question is supplied. This brings us to the third general assumption which will be imposed in the derivation, namely,

$$\text{(3) All nuclei of the medium are stationary} \qquad (7.111)$$

As we well recognize, this assumption implies that the neutron speeds involved in (7.110) are much greater than the speeds due to the thermal motion of the nuclei. Thus, all results obtained throughout the remainder of this analysis apply only to the neutron population well above the thermal range.

The application of this assumption, along with (1) of (7.106), to the Boltzmann equation reduces the scattering function $f(v,\mathbf{\Omega};v',\mathbf{\Omega}')$ to a much simpler form. If the medium is isotropic and homogeneous, then the scattering process must be azimuthally symmetric about the initial direction of motion of the neutron (see Sec. 7.2b). Or, more specifically, the probability that a neutron be scattered into any particular direction must depend only on its initial and final energies and on the angle between its original direction $\mathbf{\Omega}'$ and its direction after collision $\mathbf{\Omega}$. As before, we denote this angle θ_0 (see Fig. 7.3). If we define the directions $\mathbf{\Omega}' \equiv \mathbf{\Omega}'(\theta',\psi')$ and $\mathbf{\Omega} \equiv \mathbf{\Omega}(\theta,\psi)$, then the scattering angle θ_0 in the laboratory system of coordinates may be written in terms of the components of these vectors by means of the relation

$$\cos\theta_0 \equiv \mu_0 \equiv \mathbf{\Omega}\cdot\mathbf{\Omega}' \equiv \mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos(\psi-\psi') \qquad (7.112)$$

where $\mu \equiv \cos\theta$ and $d\mathbf{\Omega} \equiv \sin\theta\, d\theta\, d\psi \equiv |d\mu\, d\psi|$. The frequency function for azimuthally symmetric scattering can be obtained from the general function $f(v,\mathbf{\Omega};v',\mathbf{\Omega}')$. If we include a statement of the condition in the definition of this function, then f may be written [cf. Eq. (2.9) as another example of a conditional frequency]

$$f(v,\mathbf{\Omega};v',\mathbf{\Omega}')\,dv\,d\mu\,d\psi \equiv f(v,\mu,\psi;\mathbf{v}')\,dv\,d\mu\,d\psi \qquad (7.113)$$

$\equiv$ probability that, given a neutron with velocity vector $\mathbf{v}'$ before a scattering collision, the neutron speed after collision is in dv about v and its direction, in $d\mu\,d\psi$ about $\mathbf{\Omega}(\mu,\psi)$

The velocity vector $\mathbf{v}'$ is used here to represent the condition on the probability, namely, that the initial coordinates of the neutron be v', μ', and ψ'. For the present calculation, scattering collisions are assumed to be azimuthally symmetric about $\mathbf{\Omega}'$. That is, it is equally probable that the final vector $\mathbf{\Omega}$ will lie in any position on the conical surface which has

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Ω' as its axis and a vertex angle θ_0 (see Fig. 7.5). Thus the final direction of motion will depend only on the scattering angle between the initial and final directions. We define, therefore, the function [cf. Eq. (7.17)]

$$\mathfrak{q}(v,\mu_0;\mathbf{v}')\,dv\,d\mu_0 \equiv \begin{array}{l}\text{probability that a neutron with velocity } \mathbf{v}' \\ \text{is scattered into the differential element} \\ dv\,d\mu_0 \text{ about } v \text{ and } \mu_0\end{array} \tag{7.114}$$

to describe azimuthally symmetric collisions. This function is related to f by [cf. Eq. (7.18)]

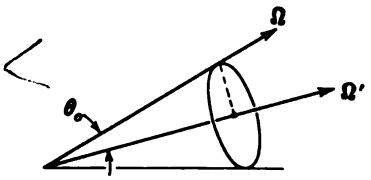

FIG. 7.5 Scattering angle θ_0 in laboratory system of coordinates.

$$f(v,\mu,\psi;\mathbf{v}') = \frac{1}{2\pi}\,\mathfrak{q}(v,\mu_0;\mathbf{v}') \tag{7.115}$$

We will use the function $\mathfrak{q}$ henceforth in the integral of (7.110).

d. Expansion in Spherical Harmonics. The neutron-balance relation based on the assumptions (7.106) and (7.111) may be written in the form

$$\frac{1}{v}\frac{\partial}{\partial t}\,\phi(\mathbf{r},v,\boldsymbol{\Omega},t) + \Sigma_t(v)\,\phi(\mathbf{r},v,\boldsymbol{\Omega},t) + \boldsymbol{\Omega}\cdot\nabla\phi(\mathbf{r},v,\boldsymbol{\Omega},t)$$
$$= S(\mathbf{r},v,\boldsymbol{\Omega},t) + \frac{1}{2\pi}\int_{\mathbf{v}'}\phi(\mathbf{r},v',\boldsymbol{\Omega}',t)\,\Sigma_s(v')\,\mathfrak{q}(v,\mu_0;\mathbf{v}')\,dv'\,d\mu'\,d\psi' \tag{7.116}$$

with the aid of (7.115). As this equation now stands, further calculation is not possible until some additional information is supplied on the functions ϕ, S, and $\mathfrak{q}$. For convenience, we assume that all these functions can be expanded in infinite series of spherical harmonics. On this basis we may write for the frequency function $\mathfrak{q}$

$$\mathfrak{q}(v,\mu_0;\mathbf{v}') \equiv \sum_{\eta=0}^{\infty}\left(\frac{2\eta+1}{2}\right)\mathfrak{q}_\eta(v;\mathbf{v}')\,P_\eta(\mu_0) \tag{7.117}$$

where in the usual way we compute

$$\mathfrak{q}_\eta(v;\mathbf{v}') \equiv \int_{-1}^{1}\mathfrak{q}(v,\mu_0;\mathbf{v}')\,P_\eta(\mu_0)\,d\mu_0 \tag{7.118}$$

Expansion (7.117) may now be substituted into the scattering integral of (7.116); however, it is necessary first to express the functions $P_\eta(\mu_0)$ in terms of the variables θ, ψ, θ', and ψ'. The addition theorem for Legendre polynomials (7.21) may be used for this purpose, and the procedure involved is identical to that used in Sec. 7.2c for the series representation of the function $\mathfrak{y}(\mu_0;\boldsymbol{\Omega}')$. In carrying out this calculation it is convenient to introduce also the spherical harmonics $Y_n^m(\boldsymbol{\Omega})$ defined by

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Eqs. (7.28) through (7.30). After some algebra, it can be shown that

$$\mathfrak{q}(v,\mu_0;\mathbf{v}') = 2\pi \sum_{\eta=0}^{\infty} \sum_{\beta=-\eta}^{\eta} \mathfrak{q}_\eta(v;\mathbf{v}')\ Y_\eta^\beta(\mathbf{\Omega})\ Y_\eta^{\beta*}(\mathbf{\Omega}') \tag{7.119}$$

Note that this result is entirely analogous to (7.37).

The flux and source terms are expanded in the complete set $Y_n^m(\mathbf{\Omega})$; take

$$\phi(\mathbf{r},v,\mathbf{\Omega},t) \equiv \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \phi_n^m(\mathbf{r},v,t)\ Y_n^m(\mathbf{\Omega}) \tag{7.120}$$

$$S(\mathbf{r},v,\mathbf{\Omega},t) \equiv \sum_{n=0}^{\infty} \sum_{m=-n}^{n} S_n^m(\mathbf{r},v,t)\ Y_n^m(\mathbf{\Omega}) \tag{7.121}$$

with

$$\phi_n^m(\mathbf{r},v,t) \equiv \int_{\mathbf{\Omega}} Y_n^m(\mathbf{\Omega})\ \phi(\mathbf{r},v,\mathbf{\Omega},t)\ d\mathbf{\Omega} \tag{7.122}$$

$$S_n^m(\mathbf{r},v,t) \equiv \int_{\mathbf{\Omega}} Y_n^m(\mathbf{\Omega})\ S(\mathbf{r},v,\mathbf{\Omega},t)\ d\mathbf{\Omega} \tag{7.123}$$

The substitution of expansions (7.119) through (7.121) into (7.116) yields

$$\left[\frac{1}{v}\frac{\partial}{\partial t} + \mathbf{\Omega}\cdot\nabla + \Sigma_t(v)\right] \sum_{n,m} \phi_n^m(\mathbf{r},v,t)\ Y_n^m(\mathbf{\Omega}) - \sum_{n,m} S_n^m(\mathbf{r},v,t)\ Y_n^m(\mathbf{\Omega})$$
$$= \int_{v'} \Sigma_s(v') \sum_{n,m} \phi_n^m(\mathbf{r},v',t) \sum_{\eta=0}^{\infty} \mathfrak{q}_\eta(v;\mathbf{v}')\ dv' \sum_{\beta=-\eta}^{\eta} Y_\eta^\beta(\mathbf{\Omega})$$
$$\times \int_{\mathbf{\Omega}'} Y_n^m(\mathbf{\Omega}')\ Y_\eta^{\beta*}(\mathbf{\Omega}')\ d\mathbf{\Omega}' \tag{7.124}$$

The application of the orthogonality property (7.32) to the integral over $\mathbf{\Omega}'$ reduces the right-hand side of Eq. (7.124) to

$$\sum_{n,m} Y_n^m(\mathbf{\Omega}) \int_{v'} \Sigma_s(v')\ \phi_n^m(\mathbf{r},v',t)\ \mathfrak{q}_n(v;\mathbf{v}')\ dv'$$

and (7.124) may then be written

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{n} Y_n^m(\mu,\psi) \left\{\left[\frac{1}{v}\frac{\partial}{\partial t} + \mathbf{\Omega}\cdot\nabla + \Sigma_t(v)\right] \phi_n^m(\mathbf{r},v,t) - S_n^m(\mathbf{r},v,t) \right.$$
$$\left. - \int_{v'} \phi_n^m(\mathbf{r},v',t)\ \Sigma_s(v')\ \mathfrak{q}_n(v;\mathbf{v}')\ dv' \right\} = 0 \tag{7.125}$$

The next step in the analysis is the separation of this equation into a set of coupled differential equations in the harmonics for the flux and the source ϕ_n^m and S_n^m. The procedure is straightforward and requires the multiplication of (7.125) by some harmonic $Y_\alpha^{\beta*}(\mathbf{\Omega})$ and integration over μ and ψ. The details of this calculation are identical to those encountered

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in the reduction of the one-velocity transport equation to a set of coupled relations in the various harmonics [see Eqs. (7.39) through (7.46)]. Thus it is easy to show that [cf. Eqs. (7.39) and (7.125)]

$$
\begin{aligned}
-\left[\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t(v)\right]\phi_\alpha^\beta(\mathbf{r},v,t)+S_\alpha^\beta(\mathbf{r},v,t)+\int_{v'}\phi_\alpha^\beta(\mathbf{r},v',t)\,\Sigma_s(v')\,\mathfrak{q}_\alpha(v;\mathbf{v}')\,dv' \\
=\left[\frac{(\alpha+1-\beta)(\alpha+1+\beta)}{(2\alpha+3)(2\alpha+1)}\right]^{\frac{1}{2}}\frac{\partial}{\partial x}\phi_{\alpha+1}^\beta(\mathbf{r},v,t) \\
+\left[\frac{(\alpha-\beta)(\alpha+\beta)}{(2\alpha+1)(2\alpha-1)}\right]^{\frac{1}{2}}\frac{\partial}{\partial x}\phi_{\alpha-1}^\beta(\mathbf{r},v,t) \\
+\frac{1}{2}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)\left\{\left[\frac{(\alpha+\beta)(\alpha+\beta-1)}{(2\alpha+1)(2\alpha-1)}\right]^{\frac{1}{2}}\phi_{\alpha-1}^{\beta-1}(\mathbf{r},v,t)\right. \\
\left.-\left[\frac{(\alpha-\beta+2)(\alpha-\beta+1)}{(2\alpha+3)(2\alpha+1)}\right]^{\frac{1}{2}}\phi_{\alpha+1}^{\beta-1}(\mathbf{r},v,t)\right\} \\
+\frac{1}{2}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\left\{-\left[\frac{(\alpha-\beta)(\alpha-\beta-1)}{(2\alpha+1)(2\alpha-1)}\right]^{\frac{1}{2}}\phi_{\alpha-1}^{\beta+1}(\mathbf{r},v,t)\right. \\
\left.+\left[\frac{(\alpha+\beta+2)(\alpha+\beta+1)}{(2\alpha+3)(2\alpha+1)}\right]^{\frac{1}{2}}\phi_{\alpha+1}^{\beta+1}(\mathbf{r},v,t)\right\}
\end{aligned}
\tag{7.126}\star
$$

where $\alpha = 0, 1, \ldots$ and $-\alpha \leq \beta \leq \alpha$.

e. Isotropic Scattering in Center-of-mass System. Again, in order to make further progress in this development, we require additional information about the system which is to be described by Eqs. (7.126). The next step we take is to specify further details about the scattering function $\mathfrak{q}$. Before introducing this information, however, it is convenient to transform the relevant functions from the speed variable v to the lethargy variable u. We make this transformation in the case of the function $\mathfrak{q}$ by defining a new function $\mathfrak{s}$ such that

$$
\mathfrak{s}(u,\mu_0;\mathbf{v}')J\,du\,d\mu_0 = \mathfrak{q}(v,\mu_0;\mathbf{v}')\,dv\,d\mu_0 \tag{7.127}
$$

where the symbol J denotes the Jacobian

$$
J \equiv \left|\frac{du}{dv}\right| \tag{7.128}
$$

and the lethargy u corresponds to the speed v. Thus (7.127) states that, if $\mathfrak{q}\,dv\,d\mu_0$ is the probability that a scattered neutron acquires coordinates v and μ_0, then this must be identically the probability that it have also the coordinates u and μ_0. Note that the Jacobian relates the differential volume in v, μ_0 space to the corresponding volume in u, μ_0 space.

In order to use the new scattering function $\mathfrak{s}$ in the general results (7.126) we require a series expansion of the form (7.119). Take, then

$$
\mathfrak{s}(u,\mu_0;\mathbf{v}') \equiv \mathfrak{s}(u,\mu_0;u',\boldsymbol{\Omega}') \equiv \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)\mathfrak{s}_n(u;u',\boldsymbol{\Omega}')\,P_n(\mu_0) \tag{7.129}
$$

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where we have replaced the statement of the initial velocity $\mathbf{v}'$ by the initial direction of motion $\mathbf{\Omega}'$ and the initial lethargy u'. The coefficients $\mathfrak{s}_n$ are computed from the relation

$$\mathfrak{s}_n(u;u',\mathbf{\Omega}') = \int_{-1}^{1} \mathfrak{s}(u,\mu_0;u',\mathbf{\Omega}')\, P_n(\mu_0)\, d\mu_0 \tag{7.130}$$

The expansion (7.129) is not yet in its most useful form. Further simplification may be achieved by observing that the frequency function $\mathfrak{s}(u,\mu_0;u',\mathbf{\Omega}')$ may be written in terms of the conditional frequencies $\mathfrak{y}(\mu_0;u,u',\mathbf{\Omega}')$ and $\mathfrak{g}(u;u',\mathbf{\Omega}')$ [see Eq. (2.11)]; thus

$$\mathfrak{s}(u,\mu_0;u',\mathbf{\Omega}') \equiv \mathfrak{y}(\mu_0;u,u',\mathbf{\Omega}')\, \mathfrak{g}(u;u',\mathbf{\Omega}') \tag{7.131}$$

where

$\mathfrak{y}(\mu_0;u,u',\mathbf{\Omega}')\, d\mu_0 \equiv$ probability that, given a neutron with coordinates u' and $\mathbf{\Omega}'$ before collision and lethargy u after collision, the neutron has been scattered through angle $\cos^{-1}\mu_0$ in (L) (7.132)

$\mathfrak{g}(u;u',\mathbf{\Omega}')\, du \equiv$ probability that, given a neutron with coordinates u' and $\mathbf{\Omega}'$, the neutron is scattered into lethargy interval du about u (7.133)

It is evident from the definition of the frequency $\mathfrak{y}$ and the energy relation (4.18) that $\mathfrak{y}(\mu_0;u,u',\mathbf{\Omega}')$ is necessarily a delta function. This follows from the fact that, given the initial and final lethargies of a scattered neutron, there exists only one angle through which the scattering can occur. That angle (or the corresponding μ_0 which we denote by ζ) is obtained from the energy relation

$$E = \frac{E'}{(1+A)^2}\,(A^2 + 2\eta A + 1) \tag{4.18}$$

where E' is the neutron energy before collision and η is the cosine of the scattering angle in the center-of-mass (C) system. The cosine η is related to the corresponding cosine μ_0 for the (L) system through the equation

$$\mu_0 \equiv \mu_0(\eta) = \frac{1 + A\eta}{(1 + 2A\eta + A^2)^{\frac{1}{2}}} \tag{4.27}$$

If we transform from the variable E to u and eliminate η from (4.27) by means of (4.18), we obtain

$$\mu_0(u,u') \equiv \zeta = \tfrac{1}{2}[(A+1)e^{\frac{1}{2}(u'-u)} - (A-1)e^{-\frac{1}{2}(u'-u)}] \tag{7.134}$$

The angle $\cos^{-1}\mu_0$ is related, then, to the difference of the initial and final lethargies of the neutron through the above expression.

The integral (7.130) for computing the coefficients $\mathfrak{s}_n$ may be written in terms of the functions $\mathfrak{y}$ and $\mathfrak{g}$. If we recognize that $\mathfrak{y}$ is a delta

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function, and write

$$\mathfrak{y}(\mu_0;u,u',\mathbf{\Omega}') \equiv \delta(\mu_0 - \zeta) \tag{7.135}$$

then the substitution of this result, along with the application of (7.131), into (7.130) yields

$$\mathfrak{s}_n(u;u',\mathbf{\Omega}') = \mathfrak{g}(u;u',\mathbf{\Omega}') \int_{-1}^{1} \delta(\mu_0 - \zeta)\, P_n(\mu_0)\, d\mu_0 = \mathfrak{g}(u;u',\mathbf{\Omega}')\, P_n(\zeta) \tag{7.136}$$

This result is a general relation from which we can compute the coefficients $\mathfrak{s}_n$ to describe any arbitrary scattering process which satisfies the four fundamental assumptions already imposed. All we require in any case is the appropriate form of the function $\mathfrak{g}$ which relates the initial and final lethargies of the scattered neutron. Thus at this point we must introduce additional details about the physics.

As we have previously observed there is one scattering law which, although simple in form, describes reasonably well many of the physical situations of interest to reactor physics. We introduce this feature as a fourth assumption.

(4) Elastic scattering is isotropic in the center-of-mass system of coordinates (7.137)

The coefficients $\mathfrak{s}_n$ which correspond to this particular scattering process are easily computed if we recall that the frequency function for the change of lethargy in the case of isotropic scattering in (C) was given by [cf. Eq. (4.50)]

$$\begin{aligned} \mathfrak{g}(u;u',\mathbf{\Omega}') &\equiv \frac{e^{u'-u}}{1-\alpha} \qquad u' \leq u \leq u' + \ln\frac{1}{\alpha} \\ &\equiv 0 \qquad\quad \text{elsewhere} \end{aligned} \tag{7.138}$$

The substitution of this expression into (7.136) yields

$$\mathfrak{s}_n(u;u',\mathbf{\Omega}') = \frac{P_n(\zeta)e^{u'-u}}{1-\alpha} \tag{7.139}$$

Thus these functions relate the initial and final lethargies of the scattered neutron with the cosine of the scattering angle in the (L) system. We will use these results in the analysis which follows.

f. Boltzmann Equation in First-order Harmonics and Isotropic Scattering. The first-order approximation to the set (7.126) is obtained by truncating the series expansion for the flux at $\alpha = 1$. This reduces the infinite set of equations to four simultaneous equations in the harmonics ϕ_0^0, ϕ_1^{-1}, ϕ_1^0, and ϕ_1^1. All higher-order terms are assumed to be small as compared with these and are therefore neglected. In setting up these equations it will be convenient to use the lethargy in place of the speed and

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redefine the flux by means of the relation

$$\phi(\mathbf{r},u,t)\,|du| \equiv \phi(\mathbf{r},v,t)\,|dv| \tag{7.140}$$

We have already transformed the scattering function $\mathfrak{q}$ to the corresponding function $\mathfrak{s}$ in lethargy, and the transformation of the remaining quantities in (7.126) is straightforward; therefore we write immediately

$$-\left[\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t(u)\right]\phi_0^0(\mathbf{r},u,t)+S_0^0(\mathbf{r},u,t)+\int_{u'}\phi_0^0(\mathbf{r},u't)\,\Sigma_s(u')\,\mathfrak{s}_0(u;u',\boldsymbol{\Omega}')\,du'$$
$$=\frac{1}{\sqrt{3}}\frac{\partial}{\partial x}\phi_1^0(\mathbf{r},u,t)-\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)\phi_1^{-1}(\mathbf{r},u,t)$$
$$+\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\phi_1^1(\mathbf{r},u,t) \tag{7.141}$$

$$-\left[\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t(u)\right]\phi_1^{-1}(\mathbf{r},u,t)+\int_{u'}\phi_1^{-1}(\mathbf{r},u',t)\,\Sigma_s(u')\,\mathfrak{s}_1(u;u',\boldsymbol{\Omega}')\,du'$$
$$=-\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\phi_0^0(\mathbf{r},u,t) \tag{7.142}$$

$$-\left[\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t(u)\right]\phi_1^0(\mathbf{r},u,t)+\int_{u'}\phi_1^0(\mathbf{r},u',t)\,\Sigma_s(u')\,\mathfrak{s}_1(u;u',\boldsymbol{\Omega}')\,du'$$
$$=\frac{1}{\sqrt{3}}\frac{\partial}{\partial x}\phi_0^0(\mathbf{r},u,t) \tag{7.143}$$

$$-\left[\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t(u)\right]\phi_1^1(\mathbf{r},u,t)+\int_{u'}\phi_1^1(\mathbf{r},u',t)\,\Sigma_s(u')\,\mathfrak{s}_1(u;u',\boldsymbol{\Omega}')\,du'$$
$$=\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)\phi_0^0(\mathbf{r},u,t) \tag{7.144}$$

where for simplicity we have taken the source to be isotropic. Note that this set of equations corresponds to Eqs. (7.47) through (7.50).

For the special case of isotropic scattering in (C), we can compute the coefficients of the scattering function from (7.139). We require here only the first two:

$$\mathfrak{s}_0=\frac{e^{u'-u}}{1-\alpha}\qquad \mathfrak{s}_1=\frac{\zeta e^{u'-u}}{1-\alpha} \tag{7.145}$$

where ζ is given by (7.134). The substitution of these expressions into the Eqs. (7.141) through (7.144) yields the first-order approximation to the Boltzmann equation for the case of isotropic scattering. It is important to recognize that even in this form the flux distribution is given by a set of integrodifferential equations; thus, although the problem has been considerably simplified, the final analysis is by no means trivial.

g. Combined Slowing-down Diffusion Equation. An elementary solution to Eqs. (7.141) through (7.144), which is the basis for the Fermi age theory, can be obtained if we impose an additional limitation on the

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properties of the system. Let us assume, therefore, that

(5) Collision densities $\Sigma_s(u')\phi_\alpha^\beta(\mathbf{r},u',t)$ are slowly varying functions over the lethargy range of the scattering integral (7.146)

In this case we may expand $\Sigma_s(u')\phi_\alpha^\beta(\mathbf{r},u',t)$ in a Taylor series about the point u. From these expansions we select the approximations

$$\Sigma_s(u')\ \phi_0^0(\mathbf{r},u',t) \simeq \Sigma_s(u)\ \phi_0^0(\mathbf{r},u,t) + (u' - u)\ \frac{\partial}{\partial u}\left[\Sigma_s(u)\ \phi_0^0(\mathbf{r},u,t)\right] \quad (7.147)$$

$$\Sigma_s(u')\ \phi_1^\beta(\mathbf{r},u',t) \simeq \Sigma_s(u)\ \phi_1^\beta(\mathbf{r},u,t) \quad (7.148)$$

If these approximations are used in the scattering integrals of Eqs. (7.141) through (7.144), the resulting expressions may be evaluated in terms of elementary functions and the integral equations thereby reduced to a set of partial differential equations. The details of the calculation follow. We observe first that the range of integration in the scattering integrals extends from $u + \ln \alpha$ to u, where α is defined by (4.20). (These limits were discussed in Sec. 4.2d.) Then, if we substitute the expression for s_0 from (7.145) and the approximation (7.147) into the integral of (7.141), we obtain

$$\begin{aligned}\frac{1}{1-\alpha}\int_{u+\ln\alpha}^{u} \phi_0^0(\mathbf{r},u',t)\ \Sigma_s(u')e^{u'-u}\,du' \\ \simeq \frac{\Sigma_s(u)\ \phi_0^0(\mathbf{r},u,t)}{1-\alpha}\int_{u+\ln\alpha}^{u} e^{u'-u}\,du' \\ + \frac{1}{1-\alpha}\frac{\partial}{\partial u}\left[\Sigma_s(u)\ \phi_0^0(\mathbf{r},u,t)\right]\int_{u+\ln\alpha}^{u}(u'-u)e^{u'-u}\,du'\end{aligned} \quad (7.149)$$

Also, the substitution of (7.148) into the scattering integrals of Eqs. (7.142) through (7.144), along with the expression for s_1 from (7.145), yields the results

$$\begin{aligned}\frac{1}{1-\alpha}\int_{u+\ln\alpha}^{u} \phi_1^\beta(\mathbf{r},u',t)\ \Sigma_s(u')\zeta e^{u'-u}\,du' \\ \simeq \frac{\Sigma_s(u)\ \phi_1^\beta(\mathbf{r},u,t)}{1-\alpha}\int_{u+\ln\alpha}^{u}\zeta e^{u'-u}\,du'\end{aligned} \quad (7.150)$$

The three integrals which appear in these expressions are easily computed, and it may be shown that

$$\frac{1}{1-\alpha}\int_{u+\ln\alpha}^{u} e^{u'-u}\,du' = 1 \quad (7.151)$$

$$\frac{1}{1-\alpha}\int_{u+\ln\alpha}^{u}(u'-u)e^{u'-u}\,du' \equiv -\xi \quad (7.152)$$

$$\frac{1}{1-\alpha}\int_{u+\ln\alpha}^{u}\zeta e^{u'-u}\,du' \equiv \frac{2}{3A} = \overline{\mu_0} \quad (7.153)$$

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where we have used the definition (4.42) for ξ and (4.30) for $\overline{\mu_0}$. The function ζ which appears in (7.153) is the cosine of the scattering angle in (C) which corresponds to the lethargy change from u' to u.

We substitute the results (7.151) through (7.153) into the expressions (7.149) and (7.150), and these in turn are used to approximate the set of equations (7.141) through (7.144). Thus,

$$-\left(\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_a\right)\phi_0^0-\xi\frac{\partial}{\partial u}(\Sigma_s\phi_0^0)+S_0^0=\frac{1}{\sqrt{3}}\frac{\partial}{\partial x}\phi_1^0 - \frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)\phi_1^{-1}+\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\phi_1^1 \tag{7.154}$$

$$-\left(\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t\right)\phi_1^{-1}+\overline{\mu_0}\Sigma_s\phi_1^{-1}=-\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\phi_0^0 \tag{7.155}$$

$$-\left(\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t\right)\phi_1^0+\overline{\mu_0}\Sigma_s\phi_1^0=\frac{1}{\sqrt{3}}\frac{\partial}{\partial x}\phi_0^0 \tag{7.156}$$

$$-\left(\frac{1}{v}\frac{\partial}{\partial t}+\Sigma_t\right)\phi_1^1+\overline{\mu_0}\Sigma_s\phi_1^1=\frac{1}{\sqrt{6}}\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)\phi_0^0 \tag{7.157}$$

This set of four partial differential equations, then, are the first-order approximations to the Boltzmann equation which satisfy the requirements (7.106), (7.111), (7.137), and (7.146).

The steady-state solutions to these equations are easily obtained by setting $\partial/\partial t \to 0$ and defining a new function $\phi(\mathbf{r},u,t) \to \phi(\mathbf{r},u)$. The solutions for the higher harmonics are

$$\phi_1^{-1}=\sqrt{\frac{3}{2}}\,D\left(\frac{\partial}{\partial y}+i\frac{\partial}{\partial z}\right)\phi_0^0 \tag{7.158}$$

$$\phi_1^0=-\sqrt{3}\,D\frac{\partial}{\partial x}\phi_0^0 \tag{7.159}$$

$$\phi_1^1=-\sqrt{\frac{3}{2}}\,D\left(\frac{\partial}{\partial y}-i\frac{\partial}{\partial z}\right)\phi_0^0 \tag{7.160}$$

where we have used the definition

$$D(u)\equiv\frac{1}{3[\Sigma_t(u)-\overline{\mu_0}\Sigma_s(u)]} \tag{7.161}$$

for the diffusion coefficient. The substitution of Eqs. (7.158) through (7.160) into the steady-state form of (7.154) yields a single second-order equation in ϕ_0^0 [cf. Eq. (7.57)].

$$-\Sigma_a\phi_0^0+S_0^0-\xi\frac{\partial}{\partial u}(\Sigma_s\phi_0^0)=-\nabla\cdot D\nabla\phi_0^0 \tag{7.162}$$

This result may be written in terms of the total flux $\phi(\mathbf{r},u)$, which is

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related to ϕ_0^0 by the equation [cf. Eqs. (7.59) and (7.60)]

$$\phi(\mathbf{r},u) = \frac{\phi_0^0(\mathbf{r},u)}{H_0^0} \qquad \text{and also} \qquad S(\mathbf{r},u) = \frac{S_0^0(\mathbf{r},u)}{H_0^0} \tag{7.163}$$

The substitution of these functions into (7.162) yields, for systems wherein Σ is independent of $\mathbf{r}$,

$$-D(u)\nabla^2\phi(\mathbf{r},u) + \Sigma_a(u)\,\phi(\mathbf{r},u) = S(\mathbf{r},u) - \xi\frac{\partial}{\partial u}[\Sigma_s(u)\,\phi(\mathbf{r},u)] \tag{7.164}\star$$

As will be demonstrated shortly, this is essentially the form assumed for the combined slowing-down diffusion equation of Chap. 6.

h. Coupling Equation. It was shown in Chap. 6 that the Fermi age model when combined with a suitable coupling equation and a diffusion equation for the thermal flux gave a reasonably accurate picture of the space and energy distributions of the neutron population in a reactor. A suitable expression for the thermal-flux distribution has already been developed, and it was shown that the source term in this relation was in fact the slowing-down density at thermal energy. In order to combine the description of the fast-flux distribution given by (7.164) with an equation for the thermal flux, we require, therefore, an expression which couples the fast flux with the slowing-down density.

We derive here an expression for the slowing-down density which is based on the assumption that the lethargy values in question are far removed from the source value. Let

$$q(\mathbf{r},u) \equiv \begin{array}{l}\text{number of neutrons slowing past lethargy}\\ u \text{ per unit time in unit volume around}\\ \text{space point } \mathbf{r}\end{array} \tag{7.165}$$

This number is given by the combined contributions of all the scatterings which occur at lower lethargies and which can throw neutrons into the lethargy range $\geq u$. The appropriate expression is

$$q(\mathbf{r},u) = \int_{u+\ln\alpha}^{u} \Sigma_s(u')\,\phi(\mathbf{r},u')\left(\frac{e^{u'-u}-\alpha}{1-\alpha}\right)du' \tag{7.166}$$

As in the case of the scattering integrals which appear in the Boltzmann equation, we assume that $\Sigma_s(u')\phi(\mathbf{r},u')$ is a slowly varying function over the interval of integration and therefore may be approximated by the first two terms of a Taylor series,

$$\Sigma_s(u')\,\phi(\mathbf{r},u') \simeq \Sigma_s(u)\,\phi(\mathbf{r},u) + (u'-u)\frac{\partial}{\partial u}[\Sigma_s(u)\,\phi(\mathbf{r},u)]$$

If this expansion is substituted into the integral of (7.166) and the inte-

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gration carried out, the resulting expression for the slowing-down density is

$$q(\mathbf{r},u) = \xi\Sigma_s(u)\,\phi(\mathbf{r},u) + a\,\frac{\partial}{\partial u}[\Sigma_s(u)\,\phi(\mathbf{r},u)] \qquad (7.167)\star$$

where

$$a \equiv \frac{\alpha \ln^2 \alpha}{2(1-\alpha)} - \xi \qquad (7.168)$$

Equation (7.167) is the accurate expression for the coupling relation between the thermal-flux equation and the combined slowing-down diffusion equation.

We summarize the above results:

$$-D(u)\,\nabla^2\phi(\mathbf{r},u) + \Sigma_a(u)\,\phi(\mathbf{r},u) = S(\mathbf{r},u) - \xi\,\frac{\partial}{\partial u}[\Sigma_s(u)\,\phi(\mathbf{r},u)] \qquad (7.164)$$

$$q(\mathbf{r},u) = \xi\Sigma_s(u)\,\phi(\mathbf{r},u) + a\,\frac{\partial}{\partial u}[\Sigma_s(u)\,\phi(\mathbf{r},u)] \qquad (7.167)$$

$$-D_{th}\nabla^2\phi_{th}(\mathbf{r}) + \Sigma_a^{th}\phi_{th}(\mathbf{r}) = q(\mathbf{r},u_{th}) \qquad (7.169)$$

i. Fermi Age Model. We develop the relations used in the Fermi age model by considering successive approximations to the three equations (7.164), (7.167), and (7.169). We begin by solving Eqs. (7.164) and (7.167) for $q(\mathbf{r},u)$; thus,

$$q(\mathbf{r},u) = \xi\Sigma_s(u)\,\phi(\mathbf{r},u) + \gamma[\Sigma_a(u)\,\phi(\mathbf{r},u) - D(u)\nabla^2\phi(\mathbf{r},u) - S(\mathbf{r},u)] \qquad (7.170)$$

where

$$\gamma \equiv -\frac{a}{\xi} \qquad (7.171)\star$$

The first term on the right-hand side of (7.170) is the predominant one for reactors of reasonable size. For very small reactors, the second term can be of comparable magnitude because of the large leakage (represented here by the Laplacian term). The number γ is of the order of ξ and ranges from zero to unity (see Fig. 4.18).

In the case of large reactors, wherein the absorption is known to be small, a good approximation to (7.170) can be obtained by neglecting the γ terms entirely. This approximation is acceptable, however, only if the source term is also small in the lethargy range of interest. In a multiplying medium such as a reactor, the neutron sources are the fissions. In the derivation of the slowing-down density, it was assumed that the lethargies in question were far removed from the representative value for the fission spectrum. But, at these lethargies the contribution of the fission spectrum $\chi(u)$ to the neutron population is very small. Thus the omission of the source term does not introduce large errors. A first-order approximation to (7.170) may be taken, then, as

$$q(\mathbf{r},u) \simeq \xi\Sigma_s(u)\,\phi(\mathbf{r},u) \qquad (7.172)$$

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and the corresponding expression for (7.164) is

$$-D(u)\nabla^2\phi(\mathbf{r},u) + \Sigma_a(u)\,\phi(\mathbf{r},u) = S(\mathbf{r},u) - \frac{\partial}{\partial u}\,q(\mathbf{r},u)$$

which is Eq. (6.5). The source is included in this relation since it may very well be of comparable magnitude to the other terms. It could be argued that its omission from (7.170) was permissible only because it represents a correction on a correction (the γ term).

A second-order approximation to (7.170) for large reactors is obtained by neglecting the leakage term only:

$$q(\mathbf{r},u) \simeq \xi\Sigma_s(u)\,\phi(\mathbf{r},u) + \gamma\Sigma_a(u)\,\phi(\mathbf{r},u) - \gamma S(\mathbf{r},u) \qquad (7.173)$$

This expression may be simplified if we use the approximation $\gamma \simeq \xi$; then

$$q(\mathbf{r},u) \simeq \xi\Sigma_t(u)\,\phi(\mathbf{r},u) - \xi S(\mathbf{r},u) \qquad (7.174)$$

which is to be used in conjunction with (7.164).

A set of equations which has proved to be very useful in reactor calculations, wherein the Fermi age model is applicable, is

$$q(\mathbf{r},u) = \xi\Sigma_t(u)\,\phi(\mathbf{r},u) \qquad (6.6)$$

$$-D(u)\nabla^2\phi(\mathbf{r},u) + \Sigma_a(u)\,\phi(\mathbf{r},u) = S(\mathbf{r},u) - \frac{\partial}{\partial u}\,q(\mathbf{r},u) \qquad (6.5)$$

which are the equations used in Chap. 6. These relations hold if $\Sigma_a(u) \ll \Sigma_s(u)$, and u is much larger than the values corresponding to the bulk of the fission neutrons.

We conclude the present section by summarizing the basic assumptions that have been introduced in arriving at the Fermi age model as represented by (6.5) and (6.6). (1) The medium is isotropic and homogeneous. (2) Neutron slowing down is due to elastic-scattering collisions alone. (3) All nuclei of the medium are stationary. (4) Elastic scattering is isotropic in the center-of-mass system. (5) The angular distribution of the flux is nearly isotropic. (6) The collision density is a slowly varying function of lethargy over the maximum allowable jump per collision. (7) The neutron constants (Σ,D) are not functions of the spatial coordinates. (8) The reactor is large. (9) $\xi \simeq \gamma$ and $\Sigma_a \ll \Sigma_s$.

7.4 Methods of Integral Equations: One-velocity Model

A second general approach to the problem of determining the angular distribution of the neutron flux may be developed by utilizing the methods of integral equations. These methods are generally quite powerful, and formal solutions are readily derived even for nonhomogeneous systems with complex source distributions. Unfortunately, the mathematical techniques required for these applications are of a somewhat sophisticated

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nature; moreover, actual computations, even for systems of relatively simple geometry, can become quite involved and laborious. However, there are specific reactor-physics situations in which the integral-equation methods are indeed easily applied and can yield accurate results with reasonable effort. A good example of this is the Serber-Wilson method presented in Chap. 8. The methods and general results developed in this section serve as a basis for these applications.

In the presentation which follows, we derive the formal solution to the one-velocity transport equation (7.15) for several media of practical importance. The general treatment follows that of Case, de Hoffmann, and Placzek.[1] The reader is referred to the work of these authors for a much more complete development of the general methods and results.

a. Neutron Distribution in Media with Zero Cross Section. As an introductory calculation we derive the neutron-flux distribution in a system which has zero cross section; media with more complex neutron properties are examined subsequently. If $\phi(\mathbf{r},\mathbf{\Omega})$ is the *directed* (angular) flux in the usual definition, then for the present system to be at steady state, we must require that the neutron losses from unit volume around $\mathbf{r}$ and from unit solid angle about $\mathbf{\Omega}$, which can be due to leakage alone, be balanced by the neutrons supplied from the sources $S(\mathbf{r},\mathbf{\Omega})$. It follows, then, that [cf. Eqs. (7.5) and (7.15)]

$$\mathbf{\Omega} \cdot \nabla\phi(\mathbf{r},\mathbf{\Omega}) = S(\mathbf{r},\mathbf{\Omega}) \tag{7.175}$$

It is convenient for the purpose of integration to write the leakage term $\mathbf{\Omega} \cdot \nabla\phi$ as a partial derivative along the direction $\mathbf{\Omega}$. Thus, if s is a variable which increases along the direction $\mathbf{\Omega}$, then

$$\mathbf{\Omega} \cdot \nabla = \frac{\partial}{\partial s} \tag{7.176}$$

and (7.175) takes the form

$$\frac{\partial}{\partial s}\phi(\mathbf{r},\mathbf{\Omega}) = S(\mathbf{r},\mathbf{\Omega}) \tag{7.177}$$

Integration over the variable s yields

$$\phi(\mathbf{r},\mathbf{\Omega}) = \int_s S(\mathbf{r},\mathbf{\Omega})\, ds + \phi_0(\mathbf{r},\mathbf{\Omega}) \tag{7.178}$$

The function $\phi_0(\mathbf{r},\mathbf{\Omega})$ denotes the solution to the homogeneous equation

$$\mathbf{\Omega} \cdot \nabla\phi(\mathbf{r},\mathbf{\Omega}) = 0 \tag{7.179}$$

and it will be arbitrarily taken as zero for all subsequent analyses. A

[1] K. M. Case, F. de Hoffmann, and G. Placzek, "Introduction to the Theory of Neutron Diffusion," Los Alamos Scientific Laboratory, 1953, available through the Government Printing Office, Washington, D.C.

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more convenient way of writing the integral of (7.178) is (see Fig. 7.6)

$$\phi(\mathbf{r},\mathbf{\Omega}) = \int_0^\infty S(\mathbf{r} - R\mathbf{\Omega},\mathbf{\Omega})\, dR \qquad (7.180)\star$$

The solution (7.180) states that the total directed flux $\phi(\mathbf{r},\mathbf{\Omega})$ is the sum total of contributions from all sources which lie on the segment of a line through $\mathbf{r}$ parallel to $\mathbf{\Omega}$ for which $0 \leq R \leq \infty$. Clearly, sources off this line cannot contribute to the flux at $\mathbf{r}$ (in direction $\mathbf{\Omega}$), nor can sources *on* the line at stations $R < 0$.

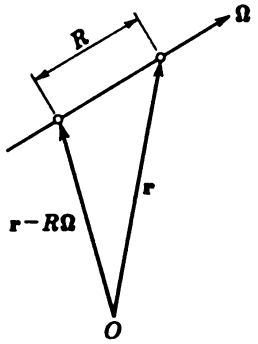

FIG. 7.6 Integration variable R and field point $\mathbf{r}$.

The integral equation (7.180) is the solution to the differential equation (7.175), and this general result will be very useful in the work which follows. As will be shown shortly, the neutron-balance equations for other, more complex systems may be reduced to the form (7.175); consequently, their solutions can be written down immediately from (7.180). Before proceeding with these calculations, however, we will develop an alternate expression for the solution (7.180) which is obtained by transforming the line integral to a volume integral. This form will be especially convenient for describing the more complex systems.

The transformation is carried out by introducing the *angular* delta function $\delta_2(\mathbf{\Omega} \cdot \mathbf{\Omega}')$, which is defined

$$\delta_2(\mathbf{\Omega} \cdot \mathbf{\Omega}') = 0 \qquad \mathbf{\Omega}' \neq \mathbf{\Omega} \qquad (7.181)$$

It has the property that

$$\int_{\mathbf{\Omega}'} g(\mathbf{\Omega}')\, \delta_2(\mathbf{\Omega} \cdot \mathbf{\Omega}')\, d\mathbf{\Omega}' = g(\mathbf{\Omega}) \qquad (7.182)$$

Note that the function δ_2 differs from the usual delta function for direction $\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}' - 1)$, where [cf. Eq. (7.135)]

$$\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}' - 1) \equiv \delta(\mu_0 - 1) \qquad (7.183)$$

We have used here our earlier notation $\mathbf{\Omega} \cdot \mathbf{\Omega}' \equiv \cos\theta_0 \equiv \mu_0$. If (7.183) is integrated over all directions (i.e., over all possible orientations of $\mathbf{\Omega}'$ to $\mathbf{\Omega}$, see Fig. 7.1), we obtain

$$\int_{\mathbf{\Omega}} \delta(\mathbf{\Omega} \cdot \mathbf{\Omega}' - 1)\, d\mathbf{\Omega} = \int_{-1}^{1} \delta(\mu_0 - 1)\, d\mu_0 \int_0^{2\pi} d\psi_0 = 2\pi \qquad (7.184)$$

Thus the functions $\delta_2(\mathbf{\Omega} \cdot \mathbf{\Omega}')$ and $\delta(\mathbf{\Omega} \cdot \mathbf{\Omega}' - 1)$ are related through the equation

$$\delta_2(\mathbf{\Omega} \cdot \mathbf{\Omega}') = \frac{1}{2\pi}\, \delta(\mathbf{\Omega} \cdot \mathbf{\Omega}' - 1) \qquad (7.185)$$

The angular delta function δ_2 may be used to define the source $S(\mathbf{r},\mathbf{\Omega})$ which appears in the integral of (7.180). For this purpose we introduce

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the identity

$$S(\mathbf{r},\boldsymbol{\Omega}) \equiv \int_{\boldsymbol{\Omega}'} S(\mathbf{r},\boldsymbol{\Omega}')\, \delta_2(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')\, d\boldsymbol{\Omega}' \tag{7.186}$$

Then, (7.180) may be written

$$\begin{aligned}\phi(\mathbf{r},\boldsymbol{\Omega}) &= \int_0^\infty dR \int_{\boldsymbol{\Omega}'} S(\mathbf{r} - R\boldsymbol{\Omega}',\boldsymbol{\Omega}')\, \delta_2(\boldsymbol{\Omega}\cdot\boldsymbol{\Omega}')\, d\boldsymbol{\Omega}' \\ &= \int_0^\infty \int_{\boldsymbol{\Omega}'} S\left(\mathbf{r} - \boldsymbol{\rho}, \frac{\boldsymbol{\rho}}{R}\right) \delta_2\left(\boldsymbol{\Omega}\cdot\frac{\boldsymbol{\rho}}{R}\right) dR\, d\boldsymbol{\Omega}'\end{aligned} \tag{7.187}$$

where we have defined the vector $\boldsymbol{\rho}$ by the relations

$$\boldsymbol{\rho} \equiv R\boldsymbol{\Omega}' \qquad d\boldsymbol{\rho} \equiv R^2\, dR\, d\boldsymbol{\Omega}' \tag{7.188}$$

The above expression may be further simplified by using the integration variable $\mathbf{r}'$ which is related to $\boldsymbol{\rho}$ through the equation

$$\mathbf{r}' \equiv \mathbf{r} - \boldsymbol{\rho} \qquad \text{and} \qquad R \equiv |\mathbf{r} - \mathbf{r}'| \equiv |\boldsymbol{\rho}| \tag{7.189}$$

We use this variable in (7.187) and note that the integration extends over all space V.

$$\begin{aligned}\phi(\mathbf{r},\boldsymbol{\Omega}) &= \int_V S\left(\mathbf{r} - \boldsymbol{\rho}, \frac{\boldsymbol{\rho}}{R}\right) \delta_2\left(\boldsymbol{\Omega}\cdot\frac{\boldsymbol{\rho}}{R}\right) \frac{d\boldsymbol{\rho}}{R^2} \\ &= \int_V \frac{S\left(\mathbf{r}', \dfrac{\mathbf{r} - \mathbf{r}'}{R}\right) \delta_2\left(\boldsymbol{\Omega}\cdot\dfrac{\mathbf{r} - \mathbf{r}'}{R}\right) d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2}\end{aligned} \tag{7.190}$$

This result may be used to compute the *total* track length $\phi(\mathbf{r})$, which is defined by Eq. (7.59). The substitution of (7.190) into (7.59) yields

$$\begin{aligned}\phi(\mathbf{r}) &= \int_{\boldsymbol{\Omega}} \int_V \frac{S\left(\mathbf{r}', \dfrac{\mathbf{r} - \mathbf{r}'}{R}\right) \delta_2\left(\boldsymbol{\Omega}\cdot\dfrac{\mathbf{r} - \mathbf{r}'}{R}\right) d\mathbf{r}'\, d\boldsymbol{\Omega}}{|\mathbf{r} - \mathbf{r}'|^2} \\ &= \int_V \frac{S\left(\mathbf{r}', \dfrac{\mathbf{r} - \mathbf{r}'}{R}\right) d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2}\end{aligned} \tag{7.191}$$

We demonstrate the use of these results by evaluating the integrals (7.190) and (7.191) for the case of a point source. Let

$$S(\mathbf{r}',\boldsymbol{\Omega}) \equiv S_0 A(\boldsymbol{\Omega})\, \delta(\mathbf{r}' - \mathbf{r}_0) \tag{7.192}$$

which states that the source distribution S is a point source of strength S_0 located at $\mathbf{r}_0$ and emitting neutrons with an angular distribution $A(\boldsymbol{\Omega})$. The substitution of this expression for S into (7.190) gives for the directed flux

$$\phi(\mathbf{r},\boldsymbol{\Omega}) = \frac{S_0\, A\left(\dfrac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}\right) \delta_2\left(\boldsymbol{\Omega}\cdot\dfrac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}\right)}{|\mathbf{r} - \mathbf{r}_0|^2} \tag{7.193}$$

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which states that there is a flux at $\mathbf{r}$ arising from neutrons traveling in direction $\boldsymbol{\Omega}$ *only if* $\boldsymbol{\Omega}$ lies along the vector $\mathbf{r} - \mathbf{r}_0$; its magnitude is given by $S_0 A/|\mathbf{r} - \mathbf{r}_0|^2$. In the same way we compute for the total flux

$$\phi(\mathbf{r}) = \frac{S_0 A\left(\dfrac{\mathbf{r} - \mathbf{r}_0}{|\mathbf{r} - \mathbf{r}_0|}\right)}{|\mathbf{r} - \mathbf{r}_0|^2} \tag{7.194}$$

This result tells us that the total flux of neutrons at $\mathbf{r}$ due to a point source at $\mathbf{r}_0$ is precisely the number of neutrons which leave $\mathbf{r}_0$ that are headed directly toward $\mathbf{r}$, reduced, of course, by the usual $1/r^2$ factor.

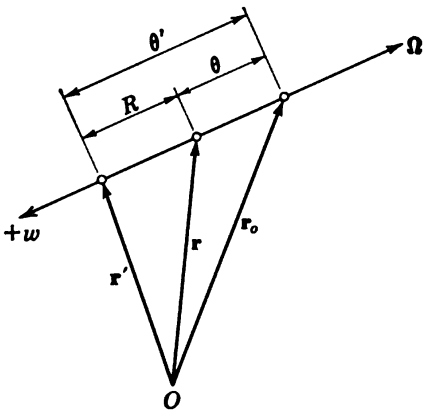

FIG. 7.7 The stations $\mathbf{r}$ and $\mathbf{r}'$.

b. Neutron Distribution in Purely Absorbing Media. The neutron distribution in a *purely absorbing* medium may be derived from the general results for the medium of zero cross section presented in Sec. 7.4a. We begin with a statement of the neutron-balance condition. The appropriate differential equation is obtained from (7.175) by adding an absorption term to the left-hand side; thus,

$$\boldsymbol{\Omega} \cdot \nabla\phi(\mathbf{r},\boldsymbol{\Omega}) + \Sigma_a(\mathbf{r})\, \phi(\mathbf{r},\boldsymbol{\Omega}) = S(\mathbf{r},\boldsymbol{\Omega}) \tag{7.195}$$

which states that the only loss mechanism for neutrons from unit volume around $\mathbf{r}$ and unit solid angle about $\boldsymbol{\Omega}$ is by transport $(\boldsymbol{\Omega} \cdot \nabla\phi)$ and absorption $(\Sigma_a\phi)$. Gains are due only to the source S. Note that this formulation does not yet include the neutron-scattering process.

Equation (7.195) may be solved by reducing it to the form (7.175), for which the general solution (7.180) has already been obtained. This transformation is accomplished by introducing the integration factor e^{β}, where β, the *optical thickness*, is defined by the equation

$$\beta(\mathbf{r};\mathbf{r}_0) \equiv \int_0^{|\mathbf{r}_0 - \mathbf{r}|} \Sigma_a(\mathbf{r}_0 - w\boldsymbol{\Omega})\, dw \tag{7.196}$$

The variable w denotes various stations on the line parallel to $\boldsymbol{\Omega}$ which passes through the field point $\mathbf{r}$ (see Fig. 7.7). Thus we define $\beta(\mathbf{r};\mathbf{r}_0)$ as the "optical thickness" of the straight-line path joining the field point $\mathbf{r}$ with some reference point $\mathbf{r}_0$. Call the length of this path θ, i.e., let

$$\theta \equiv |\mathbf{r}_0 - \mathbf{r}| \tag{7.197}$$

This number, then, corresponds to the variable s in Eq. (7.176). The derivative $\boldsymbol{\Omega} \cdot \nabla$ may therefore be written

$$\boldsymbol{\Omega} \cdot \nabla = \frac{\partial}{\partial\theta} = \frac{\partial\beta}{\partial\theta}\frac{\partial}{\partial\beta} \tag{7.198}$$

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But by (7.196) and (7.197)

$$\frac{\partial \beta}{\partial \theta} = \Sigma_a(\mathbf{r}_0 - |\mathbf{r}_0 - \mathbf{r}|\boldsymbol{\Omega}) \equiv \Sigma_a(\mathbf{r}) \tag{7.199}$$

We use this expression, along with (7.198), in (7.195) and obtain

$$\Sigma_a \frac{\partial \phi}{\partial \beta} + \Sigma_a \phi = S \tag{7.200}$$

The integration factor e^{β} may now be introduced. If we note that

$$\frac{\partial}{\partial \beta}(\phi e^{\beta}) = \phi e^{\beta} + e^{\beta}\frac{\partial \phi}{\partial \beta}$$

then (7.200) may be written

$$\frac{\partial}{\partial \beta}(\phi e^{\beta}) = \frac{S e^{\beta}}{\Sigma_a}$$

which is in the form (7.177). Integration over β gives

$$\phi = \int \frac{S e^{\beta' - \beta}\, d\beta'}{\Sigma_a} \tag{7.201}$$

where

$$\beta' \equiv \beta(\mathbf{r}';\mathbf{r}_0) \equiv \int_0^{|\mathbf{r}_0 - \mathbf{r}'|} \Sigma_a(\mathbf{r}_0 - w\boldsymbol{\Omega})\, dw \tag{7.202}$$

We use the definition for β from (7.196) and obtain (see Fig. 7.7)

$$\beta' - \beta \equiv \int_{|\mathbf{r}_0 - \mathbf{r}|}^{|\mathbf{r}_0 - \mathbf{r}'|} \Sigma_a(\mathbf{r}_0 - w\boldsymbol{\Omega})\, dw \equiv \int_{\theta}^{\theta'} \Sigma_a(\mathbf{r}_0 - w\boldsymbol{\Omega})\, dw \equiv \int_0^R \Sigma_a(\mathbf{r} - w\boldsymbol{\Omega})\, dw \tag{7.203}$$

In the notation of (7.196),

$$\int_0^R \Sigma_a(\mathbf{r} - w\boldsymbol{\Omega})\, dw \equiv \beta(\mathbf{r} - \mathbf{R};\mathbf{r}) \equiv \beta(\mathbf{r}';\mathbf{r}) \tag{7.204}$$

with $\mathbf{r}' \equiv \mathbf{r} - \mathbf{R}$. If we use this result in (7.201) and note that

$$d\beta' = \Sigma_a\, d\theta' = \Sigma_a\, dR$$

we obtain, finally [cf. Eq. (7.180)]

$$\phi(\mathbf{r},\boldsymbol{\Omega}) = \int_0^{\infty} S(\mathbf{r} - \mathbf{R},\boldsymbol{\Omega}) e^{-\beta(\mathbf{r} - \mathbf{R};\mathbf{r})}\, dR \tag{7.205}\star$$

Thus the expression for the directed flux in a purely absorbing medium may be obtained from the result for the medium with zero cross section by merely introducing the attenuation factor $e^{-\beta}$ into the integrand of (7.180).

The expression for the total flux $\phi(\mathbf{r})$ is computed from (7.59), using the result (7.205). We first transform (7.205) to the volume integral form corresponding to (7.190). This is

$$\phi(\mathbf{r},\boldsymbol{\Omega}) = \int_V \frac{S(\mathbf{r}',\boldsymbol{\Omega}')\ \delta_2(\boldsymbol{\Omega} \cdot \boldsymbol{\Omega}') e^{-\beta(\mathbf{r}';\mathbf{r})}\, d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} \tag{7.206}$$

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Substitution into (7.59) yields

$$\phi(\mathbf{r}) = \int_V \frac{S(\mathbf{r}',\mathbf{\Omega}')e^{-\beta(\mathbf{r}';\mathbf{r})}\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \tag{7.207}$$

It must be remembered that the $\mathbf{\Omega}'$ which appears in (7.206) and (7.207) is in fact the direction given by $\mathbf{R}/R$.

We demonstrate the use of these results by computing $\phi(\mathbf{r},\mathbf{\Omega})$ and $\phi(\mathbf{r})$ for the case of a point source. As an example let us take an isotropic source of strength S_0 at the origin; then,

$$S(\mathbf{r},\mathbf{\Omega}) \equiv \frac{1}{4\pi} S_0\delta(\mathbf{r}) \tag{7.208}$$

Substitution into (7.206) yields

$$\phi(\mathbf{r},\mathbf{\Omega}) = \frac{S_0}{4\pi r^2}\,\delta_2\left(\mathbf{\Omega}\cdot\frac{\mathbf{r}}{r}\right)e^{-\beta(0;\mathbf{r})} \tag{7.209}$$

which states that the flux of neutrons with direction $\mathbf{\Omega}$ at $\mathbf{r}$ is zero except when $\mathbf{\Omega}$ coincides with the line connecting $\mathbf{r}$ with the position of the source. And, in this case, the flux is the total number of neutrons emitted per unit solid angle ($S_0/4\pi r^2$) reduced by the attenuation losses according to $e^{-\beta}$. The total flux may be computed in the same way. Substitution of (7.208) into (7.207) gives

$$\phi(\mathbf{r}) = \frac{S_0 e^{-\beta(0;\mathbf{r})}}{4\pi r^2} \tag{7.210}$$

as is to be expected.

The special case of a homogeneous medium is of great practical importance. In such media the optical thickness reduces to the simple form

$$\beta(\mathbf{r}';\mathbf{r}) \equiv \Sigma_a|\mathbf{r}'-\mathbf{r}| \tag{7.211}\star$$

and the results (7.206) and (7.207) may be written

$$\phi(\mathbf{r},\mathbf{\Omega}) = \int_0^\infty S(\mathbf{r}-\mathbf{R},\mathbf{\Omega})e^{-\Sigma_a R}\,dR \equiv \int_V \frac{S(\mathbf{r}',\mathbf{\Omega}')\,\delta_2(\mathbf{\Omega}\cdot\mathbf{\Omega}')e^{-\Sigma_a|\mathbf{r}'-\mathbf{r}|}\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \tag{7.212}$$

and

$$\phi(\mathbf{r}) = \int_V \frac{S(\mathbf{r}',\mathbf{\Omega}')e^{-\Sigma_a|\mathbf{r}'-\mathbf{r}|}\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \tag{7.213}$$

For the case of the isotropic point source (7.208), these solutions yield

$$\phi(\mathbf{r},\mathbf{\Omega}) = \frac{S_0\delta_2(\mathbf{\Omega}\cdot\mathbf{r}/r)e^{-\Sigma_a r}}{4\pi r^2} \tag{7.214}$$

$$\phi(\mathbf{r}) = \frac{S_0 e^{-\Sigma_a r}}{4\pi r^2} \tag{7.215}$$

An interesting and useful application of the concepts developed for the purely absorbing medium is in the calculation of *escape probabilities* from arbitrarily shaped volumes. If $P(\mathbf{r})$ is the probability that a

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neutron born at $\mathbf{r}$ within the volume V escape, then the *average escape probability* P_{esc} for a distribution $S(\mathbf{r})$ of isotropic neutron sources in V is defined by

$$P_{esc} \equiv \frac{\int_V S(\mathbf{r})\, P(\mathbf{r})\, d\mathbf{r}}{\int_V S(\mathbf{r})\, d\mathbf{r}} \tag{7.216}\star$$

In general for convex shapes, $P(\mathbf{r})$ is the total neutron current emerging from the surface of volume V arising from a unit isotropic source located at $\mathbf{r}$ inside V. To compute this quantity we begin with the probability for escape through the element of area dA on the surface A of volume V (see Fig. 7.8). If $d\Omega$ denotes the element of solid angle

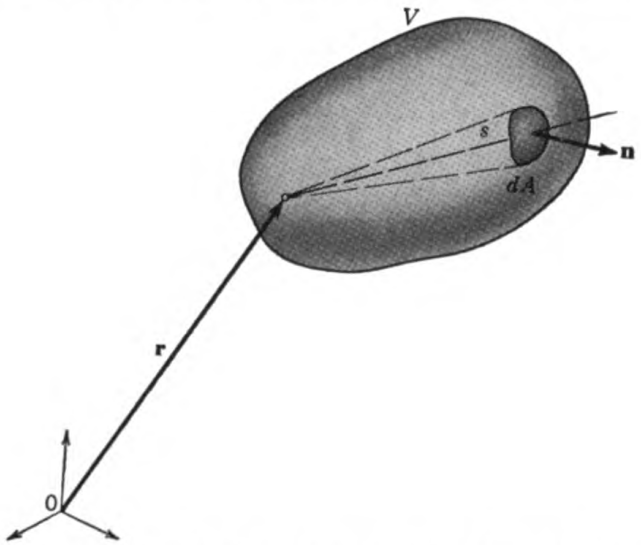

FIG. 7.8 Volume V and orientation of neutron path.

subtended by the area dA from the point $\mathbf{r}$, then $d\Omega/4\pi$ is the fraction of neutrons emitted by the source which is headed in the direction of dA. Now suppose V contains a homogeneous material of absorption cross section Σ_a, and let the distance of dA from $\mathbf{r}$ be s, then $\exp(-\Sigma_a s)$ is the probability that a neutron will travel from $\mathbf{r}$ to dA without experiencing an absorption collision. Thus the fraction of neutrons actually passing through dA is $\exp(-\Sigma_a s)\, d\Omega/4\pi$, and $P(\mathbf{r})$ is simply the integral of this quantity over all directions; therefore,

$$P(\mathbf{r}) = \frac{1}{4\pi} \int_\Omega e^{-\Sigma_a s}\, d\Omega \tag{7.217}$$

In the case of a uniform distribution of isotropic sources throughout V, we obtain for P_{esc} the expression

$$P_{esc} = \frac{1}{4\pi V} \int_V \int_\Omega e^{-\Sigma_a s}\, d\Omega\, d\mathbf{r} \tag{7.218}$$

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This result may be further developed by interchanging the order of integration. To compute the volume element $d\mathbf{r}$ we picture the volume V as consisting of a series of tubes of cross section $(\boldsymbol{\Omega} \cdot \mathbf{n}_i)\, dA$ and length s_0, as shown in Fig. 7.9. The symbol $\mathbf{n}_i$ denotes the inward directed normal to the surface dA (and $\mathbf{n}$ the outward directed normal so that $\mathbf{n}_i \equiv -\mathbf{n}$). Then the volume element may be written

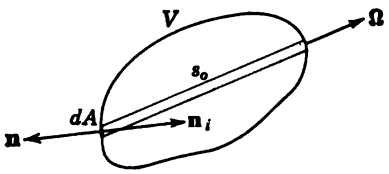

FIG. 7.9 Volume element.

$$d\mathbf{r} \equiv (\boldsymbol{\Omega} \cdot \mathbf{n}_i)\, dA\, ds \tag{7.219}$$

Substitution into (7.218) yields

$$P_{esc} = \frac{1}{4\pi V} \int_A \int_{\boldsymbol{\Omega}} \int_0^{s_0} e^{-\Sigma_a s} (\boldsymbol{\Omega} \cdot \mathbf{n}_i)\, ds\, d\boldsymbol{\Omega}\, dA$$

or

$$P_{esc} = \frac{1}{4\pi V \Sigma_a} \int_A \int_{\boldsymbol{\Omega}} (1 - e^{-\Sigma_a s_0})(\boldsymbol{\Omega} \cdot \mathbf{n}_i)\, d\boldsymbol{\Omega}\, dA \tag{7.220}$$

The general problem of computing the escape probability from a volume V is closely related to the problem of computing the average flux in the volume were it to be immersed in an infinite medium supporting a uniform isotropic flux. This relationship has considerable practical importance; moreover, the two problems are analytically equivalent. To demonstrate this equivalence, consider a body V of the shape and material assumed in the calculation of the escape probability to be located in an isotropic bath of neutrons such that at great distances from V the total track length is ϕ_∞. Then the number of neutrons incident upon the body per unit time per unit solid angle from direction $\boldsymbol{\Omega}_0$ through the surface element dA is

$$\phi_\infty \left(\frac{\boldsymbol{\Omega}_0}{4\pi}\right) \cdot \mathbf{n}\, dA \tag{7.221}$$

Now if $\phi(\mathbf{r},\boldsymbol{\Omega})$ is the directed flux at $\mathbf{r}$ in V, then the total track length at $\mathbf{r}$ is given by (7.59). Further let $\phi(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_0,\boldsymbol{\Omega}_0)$ be the directed flux at $\mathbf{r}$ due to a unit source at $\mathbf{r}_0$ radiating neutrons in direction $\boldsymbol{\Omega}_0$, and let $\mathcal{S}(\mathbf{r}_0,\boldsymbol{\Omega}_0)$ be the source strength. Then

$$\phi(\mathbf{r}) = \int_{\boldsymbol{\Omega}} \phi(\mathbf{r},\boldsymbol{\Omega})\, d\boldsymbol{\Omega} = \int_{\boldsymbol{\Omega}} d\boldsymbol{\Omega} \int_{\boldsymbol{\Omega}_0} \phi(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_0,\boldsymbol{\Omega}_0)\, \mathcal{S}(\mathbf{r}_0,\boldsymbol{\Omega}_0)\, d\boldsymbol{\Omega}_0 \tag{7.222}$$

If we use for $\mathcal{S}$ the relation (7.221) and define $\mathcal{S}$ to be the neutrons released per unit area per unit time at the surface, then the above relation becomes

$$\phi(\mathbf{r}) = \frac{\phi_\infty}{4\pi} \int_{\boldsymbol{\Omega}} d\boldsymbol{\Omega} \int_{\boldsymbol{\Omega}_0} \boldsymbol{\Omega}_0 \cdot \mathbf{n}\, \phi(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_0,\boldsymbol{\Omega}_0)\, d\boldsymbol{\Omega}_0\, dA \tag{7.223}$$

This result may be written in an alternate form by the application of the

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reciprocity theorem,[1] which states: The directed flux at $\mathbf{r}$ in direction $\boldsymbol{\Omega}$ due to unit source at $\mathbf{r}_0$ radiating in direction $\boldsymbol{\Omega}_0$ is the same as the directed flux at $\mathbf{r}_0$ in direction $-\boldsymbol{\Omega}_0$ due to unit source at $\mathbf{r}$ radiating in direction $-\boldsymbol{\Omega}$. Therefore

$$\phi(\mathbf{r},\boldsymbol{\Omega};\mathbf{r}_0,\boldsymbol{\Omega}_0) = \phi(\mathbf{r}_0,-\boldsymbol{\Omega}_0;\mathbf{r},-\boldsymbol{\Omega}) \tag{7.224}$$

Substitution into (7.223) yields

$$\phi(\mathbf{r}) = \frac{\phi_\infty}{4\pi}\int_{\boldsymbol{\Omega}} d\boldsymbol{\Omega}\int_{\boldsymbol{\Omega}_0}\boldsymbol{\Omega}_0\cdot\mathbf{n}\,\phi(\mathbf{r}_0,-\boldsymbol{\Omega}_0;\mathbf{r},-\boldsymbol{\Omega})\,d\boldsymbol{\Omega}_0\,dA \tag{7.225}$$

which gives the total track length at $\mathbf{r}$ inside V produced by sources on the surface A arising from the presence of the isotropic bath in the surrounding space. If we reverse the sign of the variables of integration $\boldsymbol{\Omega}$ and $\boldsymbol{\Omega}_0$, then $\phi(\mathbf{r})/\phi_\infty$ is immediately identified as the fraction of neutrons born at $\mathbf{r}$ which escape through A; thus,

$$P(\mathbf{r}) = \frac{\phi(\mathbf{r})}{\phi_\infty} \tag{7.226}$$

and by (7.216), we have

$$P_{esc} = \frac{\bar{\phi}}{\phi_\infty} \tag{7.227}$$

which states that the average escape probability from a volume V containing a uniform distribution of isotropic sources may be obtained by computing the average flux distribution in the same volume if it were placed in a uniform field ϕ_∞.

The calculation of escape probabilities is often most conveniently carried out by means of the *chord method* developed by Dirac.[2] For this analysis we refer to Fig. 7.10, which shows an arbitrary convex volume V. As before, $\mathbf{n}_i$ is the inward directed normal at dA and $s(\boldsymbol{\Omega})$ is the chord length from dA in the direction $\boldsymbol{\Omega}$. Let us assume now that the number of chords of length between s and $s + ds$ in the solid angle $d\boldsymbol{\Omega}$ is *proportional* to $\boldsymbol{\Omega}\cdot\mathbf{n}_i$, the cosine of the angle between the inward directed normal at dA and $\boldsymbol{\Omega}$. Note that the direction $\boldsymbol{\Omega}$ in $\boldsymbol{\Omega}\cdot\mathbf{n}_i$ corresponds to the direction in $s(\boldsymbol{\Omega})$. On this basis the total number of chords in V with lengths between s and $s + ds$ is proportional to the integral

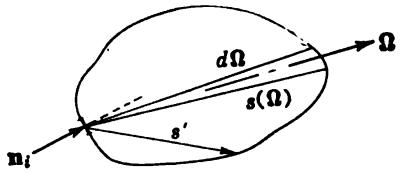

FIG. 7.10 Chords through volume V.

$$\int_A dA\int_{\boldsymbol{\Omega}=\boldsymbol{\Omega}(s)}\boldsymbol{\Omega}\cdot\mathbf{n}_i\,d\boldsymbol{\Omega}$$

[1] *Ibid.*, p. 16, for proof of this theorem.

[2] P.A.M. Dirac, "Approximate Rate of Neutron Multiplication for a Solid of Arbitrary Shape and Uniform Density," British Report M.S.D. 5, Part I, General Theory, 1943.

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where the directions Ω over which the integration is performed are selected such that $s \leq s' \leq s + ds$. If this relation is divided by the total number of chords in V, using the cosine weighting mentioned above for each direction Ω, the fraction $h(s)\,ds$ of all chords with lengths between s and $s + ds$ in V is given by

$$h(s)\,ds \equiv \frac{\int_A dA \int_{\Omega=\Omega(s)} \Omega \cdot \mathbf{n}_i\, d\Omega}{\int_A dA \int_\Omega \Omega \cdot \mathbf{n}_i\, d\Omega} \tag{7.228}$$

where the only restriction on the region of integration in the denominator is $\Omega \cdot \mathbf{n}_i > 0$.

The denominator of this expression is easily evaluated. Thus, for $d\Omega \equiv d\mu\, d\psi$ in the usual notation, we have

$$\int_A dA \int_\Omega \Omega \cdot \mathbf{n}_i\, d\Omega = \int_A dA \int_0^1 \mu\, d\mu \int_0^{2\pi} d\psi = \pi A \tag{7.229}$$

The average chord length $\bar{s}$ through V may then be defined by

$$\bar{s} \equiv \int_s s\, h(s)\, ds \tag{7.230}$$

The application of (7.228) and (7.229) yields

$$\bar{s} = \frac{1}{\pi A} \int_s s \int_A dA \int_{\Omega=\Omega(s)} \Omega \cdot \mathbf{n}_i\, d\Omega \tag{7.231}$$

Interchanging the order of integration

$$\bar{s} = \frac{1}{\pi A} \int_\Omega d\Omega \int_A s\, \Omega \cdot \mathbf{n}_i\, dA = \frac{V}{\pi A} \int_\Omega d\Omega = \frac{4V}{A} \tag{7.232}\star$$

since integration over all values of s extends the Ω integration to all possible directions from dA. This result was previously cited in (5.295).

The expression for $\bar{s}$ may be used to reduce the formula (7.220) for computing the escape probability, which we now write in the form

$$\begin{aligned} P_{esc} &= \frac{1}{4\pi V \Sigma_a} \int_A dA \int_\Omega (1 - e^{-\Sigma_a s})\, (\Omega \cdot \mathbf{n}_i)\, d\Omega \\ &= \frac{1}{4\pi V \Sigma_a} \int_A dA \int_{s_{\min}}^{s_{\max}} \int_{\Omega=\Omega(s)} (1 - e^{-\Sigma_a s})(\Omega \cdot \mathbf{n}_i)\, d\Omega \end{aligned} \tag{7.233}$$

where we have separated the integration over Ω into two parts. First we integrate over all directions leading to chord length s, then over all chord lengths from the smallest to the largest. Now, it follows from (7.228) and (7.231) that

$$\frac{4\pi V}{\bar{s}}\, h(s)\, ds = \int_A dA \int_{\Omega=\Omega(s)} \Omega \cdot \mathbf{n}_i\, d\Omega$$

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The substitution of this relation into (7.233) yields

$$P_{esc} = \frac{1}{4\pi V \Sigma_a} \int_{s_{min}}^{s_{max}} (1 - e^{-\Sigma_a s}) \int_A dA \int_{\Omega = \Omega(s)} \Omega \cdot \mathbf{n}_i \, d\Omega$$

and

$$P_{esc} = \frac{1}{\Sigma_a \bar{s}} \int_{s_{min}}^{s_{max}} (1 - e^{-\Sigma_a s}) h(s) \, ds \qquad (7.234)\star$$

An important application of this general result is given in Chap. 10 in connection with the problem of computing the resonance-escape probability for fuel lumps in a heterogeneous reactor.

The evaluation of the chord-distribution function h is easily demonstrated when the volume V has a simple shape. As an example, consider a slab of width a (see Fig. 7.11). Let chords of length s make angle θ with the x axis, and let $\mu \equiv \cos\theta$. Then for any chord s' with inclination θ', we have $\Omega \cdot \mathbf{n}_i = \cos\theta' \equiv \mu'$, and with $d\Omega \equiv d\mu' \, d\psi'$, (7.228) reduces to

$$h(\mu) \, d\mu = \frac{1}{\pi A} \int_A dA \int_{\mu}^{\mu + d\mu} \mu' \, d\mu' \int_0^{2\pi} d\psi' \simeq 2\mu \, d\mu$$

or

$$h(s) \, ds = \frac{2a^2}{s^3} \, ds \qquad (7.235)$$

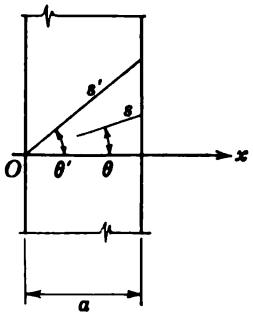

FIG. 7.11 Chord through slab.

keeping in mind that μ implies chords of length s. The result (7.229) has been used in evaluating the general relation for (7.228). For this case the average chord length is shown from (7.230) to be $\bar{s} = 2a$.

c. Multiplying Media with External Source. We examine here media which exhibit both *elastic-scattering* and *fission* interactions between neutrons and nuclei in addition to the absorption property considered in Sec. 7.4b. In the present development we introduce an alternate formulation of the one-velocity transport equation (7.15) which combines into a single integral term both the scattering and fission processes. This is accomplished by defining a new quantity c which is given by

$$c(\mathbf{r}) \equiv \frac{\nu \Sigma_f(\mathbf{r}) + \Sigma_s(\mathbf{r})}{\Sigma_t(\mathbf{r})} \qquad (7.236)\star$$

where the various cross sections and ν have the usual definitions. According to (7.236), c is the average number of neutrons produced *per collision*. The neutron-balance condition may be written in terms of this number. For the steady-state system it takes the form [cf. Eq. (7.15)]

$$\Omega \cdot \nabla \phi(\mathbf{r},\Omega) + \Sigma_t(\mathbf{r}) \, \phi(\mathbf{r},\Omega) = \Sigma_t(\mathbf{r}) \, c(\mathbf{r}) \int_{\Omega'} \phi(\mathbf{r},\Omega') \, f^\star(\Omega;\Omega') \, d\Omega' + S(\mathbf{r},\Omega) \qquad (7.237)\star$$

where $f^\star(\Omega;\Omega') \, d\Omega \equiv$ probability that, when a neutron with direction Ω' suffers a collision, it yields a secondary (product) neutron with direction in $d\Omega$ about Ω (7.238)

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Note that this function describes the *general* collision process consisting of scatterings and fissions. The average yield per collision corresponding to $f^\star$ is the quantity c of (7.236). It follows that the integral of (7.237) multiplied by $c\Sigma_t$ gives the number of neutrons which emerge from *all* collisions with direction in unit solid angle about $\mathbf{\Omega}$. In the case of a nonmultiplying medium $c = \Sigma_s/\Sigma_t$ and $f^\star(\mathbf{\Omega};\mathbf{\Omega}') = f(\mathbf{\Omega};\mathbf{\Omega}')$, as given by (7.8). Finally, it should be mentioned that the source function S which appears in (7.237) does not include the contributions from fission and therefore differs from the function S given in Eq. (7.15).

In the present calculation we assume that all collisions are isotropic in (L); therefore,

$$f^\star(\mathbf{\Omega};\mathbf{\Omega}') \equiv \frac{1}{4\pi} \tag{7.239}$$

The integral which appears in (7.237) now reduces to the expression for the total flux $\phi(\mathbf{r})$ [cf. Eq. (7.59)], and (7.237) may be written

$$\mathbf{\Omega} \cdot \nabla\phi(\mathbf{r},\mathbf{\Omega}) + \Sigma_t(\mathbf{r})\, \phi(\mathbf{r},\mathbf{\Omega}) = \frac{1}{4\pi}\, \Sigma_t(\mathbf{r})\, c(\mathbf{r})\, \phi(\mathbf{r}) + S(\mathbf{r},\mathbf{\Omega}) \tag{7.240}$$

If this equation is compared to (7.195), we observe immediately that the two relations are of the same form, the only difference being due to the Σ_t which appears at the left of (7.240) (in place of the Σ_a), and the addition of a source term at the right which augments the external sources S. Thus the solution to (7.240) may be obtained by applying the result (7.205). In place of the source function S which appears in (7.205) we use the two-term source of (7.240) and modify the definition of the optical thickness β so that it is given in terms of the total cross section Σ_t in place of Σ_a. In this way we obtain the solution

$$\phi(\mathbf{r},\mathbf{\Omega}) = \frac{1}{4\pi} \int_0^\infty \Sigma_t(\mathbf{r} - \mathbf{R})\, c(\mathbf{r} - \mathbf{R})\, \phi(\mathbf{r} - \mathbf{R}) e^{-\beta(\mathbf{r}-\mathbf{R};\mathbf{r})}\, dR + \int_0^\infty S(\mathbf{r} - \mathbf{R},\mathbf{\Omega}) e^{-\beta(\mathbf{r}-\mathbf{R};\mathbf{r})}\, dR \tag{7.241}\star$$

The expression for the total flux is obtained from the general result (7.207). Again noting that the source consists of the two functions on the right-hand side of (7.240) and that β is defined in terms of Σ_t, we have

$$\phi(\mathbf{r}) = \int_V \left[S(\mathbf{r}',\mathbf{\Omega}') + \frac{1}{4\pi} c(\mathbf{r}')\, \Sigma_t(\mathbf{r}')\, \phi(\mathbf{r}') \right] \frac{e^{-\beta(\mathbf{r}';\mathbf{r})}\, d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^2} \tag{7.242}$$

where $\mathbf{\Omega}' \equiv \mathbf{R}/R$ [as used previously in Eqs. (7.206) and (7.207)]. Both of these relations are integral equations and may be solved by means of a Neumann series[1] which is developed by successive approximations. The

[1] See, for example, H. Margenau and G. M. Murphy, "The Mathematics and Physics of Chemistry," pp. 504–505, D. Van Nostrand Company, Inc., Princeton, N.J., 1943.

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method, although straightforward, is laborious, especially if the convergence of the series is slow so that several terms are required. The method consists of making successive estimates of the function which appears in (7.242). For example, a first estimate of ϕ is obtained by ignoring the ϕ integral at the right of (7.242). The resulting expression for ϕ involves only the integral of the function S, which is specified. This solution may then be used in the ϕ integral of (7.242) and a second estimate for ϕ obtained. Higher orders of accuracy are obtained by continuing this process. When $\phi(\mathbf{r})$ has been computed to the desired accuracy, it may then be used in (7.241) to obtain the directed flux $\phi(\mathbf{r},\mathbf{\Omega})$.

The solution to the general time-dependent system is easily developed from the above results. In the notation of the present method, the nonstationary one-velocity transport equation is

$$\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},\mathbf{\Omega},t) + \mathbf{\Omega}\cdot\nabla\phi(\mathbf{r},\mathbf{\Omega},t) + \Sigma_t(\mathbf{r})\,\phi(\mathbf{r},\mathbf{\Omega},t) = \frac{1}{4\pi}\Sigma_t(\mathbf{r})\,c(\mathbf{r})\,\phi(\mathbf{r},t) \tag{7.243}$$

wherein we have omitted the external source term S to avoid undue complication. The solution to (7.243) is obtained by seeking a number α such that $\phi(\mathbf{r},\mathbf{\Omega},t)$ may be given in the form

$$\phi(\mathbf{r},\mathbf{\Omega},t) \equiv \phi(\mathbf{r},\mathbf{\Omega})e^{\alpha t} \tag{7.244}$$

The technique by which this is accomplished is discussed under the section on the Serber-Wilson method. If we substitute (7.244) into (7.243), we obtain immediately

$$\mathbf{\Omega}\cdot\nabla\phi(\mathbf{r},\mathbf{\Omega}) + \mathbf{\Sigma}(\mathbf{r})\,\phi(\mathbf{r},\mathbf{\Omega}) = \frac{1}{4\pi}\Sigma_t(\mathbf{r})\,c(\mathbf{r})\,\phi(\mathbf{r}) \tag{7.245}\star$$

where we have defined

$$\mathbf{\Sigma} \equiv \Sigma_t + \frac{\alpha}{v} \tag{7.246}\star$$

This quantity may be called an "effective-removal" cross section and is related to the actual-removal (total) cross section by the addition to Σ_t of a "time-absorption" term α/v. If solutions of the form (7.244) exist, then we have succeeded in reducing the time-dependent problem to an equivalent steady-state problem which describes the neutron economy in a system with total-removal cross section $\mathbf{\Sigma}$ and a source term identical to that of the actual system.

d. Source-free Infinite Multiplying Media. Of particular interest in reactor technology are multiplying media which contain no external neutron sources (i.e., sources other than the fission reactions). For these problems, the transport equation takes the form

$$\mathbf{\Omega}\cdot\nabla\phi(\mathbf{r},\mathbf{\Omega}) + \Sigma_t(\mathbf{r})\,\phi(\mathbf{r},\mathbf{\Omega}) = \Sigma_t(\mathbf{r})\,c(\mathbf{r})\int_{\mathbf{\Omega}'}\phi(\mathbf{r},\mathbf{\Omega}')\,f^\star(\mathbf{\Omega};\mathbf{\Omega}')\,d\mathbf{\Omega}' \tag{7.247}$$

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If the distribution of neutrons emerging from collisions is isotropic in (L), then the solution to this equation is

$$\phi(\mathbf{r},\mathbf{\Omega}) = \frac{1}{4\pi}\int_0^\infty \Sigma_t(\mathbf{r}-\mathbf{R})\, c(\mathbf{r}-\mathbf{R})\, \phi(\mathbf{r}-\mathbf{R}) e^{-\beta(\mathbf{r}-\mathbf{R};\mathbf{r})}\, dR \quad (7.248)\star$$

The total flux is given by

$$\phi(\mathbf{r}) = \frac{1}{4\pi}\int_V \frac{c(\mathbf{r}')\,\Sigma_t(\mathbf{r}')\,\phi(\mathbf{r}') e^{-\beta(\mathbf{r}';\mathbf{r})}\, d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \quad (7.249)$$

If the medium in question is *homogeneous*, then β reduces to the form (7.211), and the above solutions may be written

$$\phi(\mathbf{r},\mathbf{\Omega}) = \frac{c\Sigma_t}{4\pi}\int_0^\infty \phi(\mathbf{r}-\mathbf{R}) e^{-\Sigma_t R}\, dR \quad (7.250)$$

$$\phi(\mathbf{r}) = \frac{c\Sigma_t}{4\pi}\int_V \frac{\phi(\mathbf{r}') e^{-\Sigma_t|\mathbf{r}-\mathbf{r}'|}\, d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \quad (7.251)\star$$

We now demonstrate some of the properties of these functions which will be useful for later applications.

1. *Elementary Solution for $\phi(\mathbf{r})$ in Infinite Media.* The elementary solution to Eq. (7.251) for the *infinite medium* is $e^{i\mathbf{B}\cdot\mathbf{r}}$. The quantity $\mathbf{B}$ is an eigenvector, and its magnitude is determined by substituting $\phi(\mathbf{r}) = e^{i\mathbf{B}\cdot\mathbf{r}}$ into (7.251). Thus

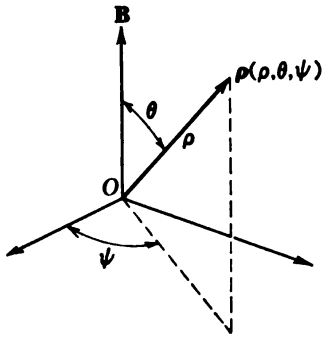

FIG. 7.12 Eigenvector B and coordinate system.

$$e^{i\mathbf{B}\cdot\mathbf{r}} = \frac{c\Sigma_t}{4\pi}\int_V \frac{e^{-\Sigma_t|\mathbf{r}-\mathbf{r}'|+i\mathbf{B}\cdot\mathbf{r}'}\, d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \quad (7.252)$$

If we define $\boldsymbol{\varrho} \equiv \mathbf{r}-\mathbf{r}'$, then this equation may be written

$$1 = \frac{c\Sigma_t}{4\pi}\int_V \frac{e^{-\Sigma_t\rho+i\mathbf{B}\cdot\boldsymbol{\varrho}}\, d\boldsymbol{\varrho}}{\rho^2} \quad (7.253)$$

where we have canceled out the common factor $e^{i\mathbf{B}\cdot\mathbf{r}}$. To evaluate the integral we define

$$\boldsymbol{\varrho} \equiv \boldsymbol{\varrho}(\rho,\theta,\psi) \qquad d\boldsymbol{\varrho} \equiv \rho^2\, d\rho\, d\mu\, d\psi$$

with $\mu \equiv \cos\theta$ (see Fig. 7.12). Then (7.253) yields

$$1 = \frac{c\Sigma_t}{4\pi}\int_0^\infty d\rho \int_{-1}^1 d\mu \int_0^{2\pi} e^{-\Sigma_t\rho+iB\rho\mu}\, d\psi = \frac{c\Sigma_t}{4\pi}\left[\frac{4\pi}{B}\tan^{-1}\left(\frac{B}{\Sigma_t}\right)\right]$$

Thus $e^{i\mathbf{B}\cdot\mathbf{r}}$ is a solution to (7.251) for an infinite medium, provided that

$$\left(\frac{\Sigma_t}{B}\right)\tan^{-1}\left(\frac{B}{\Sigma_t}\right) = \frac{1}{c} \quad (7.254)\star$$

2. *General Solution for $\phi(\mathbf{r})$ for Infinite Media.* The general solution to Eq. (7.251) is obtained by superimposing the elementary solutions

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$e^{i\mathbf{B}\cdot\mathbf{r}}$. If we define the eigenvector $\mathbf{B}$ by the relation

$$\mathbf{B} \equiv B\mathbf{u}$$

where $\mathbf{u}$ is a unit vector (a direction), then the general solution may be obtained by integration over all directions $\mathbf{u}$; thus

$$\phi(\mathbf{r}) = \int_{\mathbf{u}} a(\mathbf{u}) e^{iB\mathbf{u}\cdot\mathbf{r}}\, d\mathbf{u} \qquad (7.255)\star$$

The eigenvalue B is computed from (7.254), and the arbitrary function $a(\mathbf{u})$ is analogous to the arbitrary constant which always appears in the steady-state solution to the diffusion equation for a multiplying medium. Its value can be determined in these cases only by specifying the initial condition of the system.

3. $\phi(\mathbf{r})$ *Is Solution to the Stationary-wave Equation.* If we compute the Laplacian of $\phi(\mathbf{r})$ using (7.251) and (7.255), we obtain

$$\nabla^2\phi(\mathbf{r}) = \int_{\mathbf{u}} a(\mathbf{u})\nabla^2 e^{iB\mathbf{u}\cdot\mathbf{r}}\, d\mathbf{u} = -B^2 \int_{\mathbf{u}} a(\mathbf{u}) e^{iB\mathbf{u}\cdot\mathbf{r}}\, d\mathbf{u} \qquad (7.256)$$

Thus the solution (7.255) for the infinite medium satisfies the differential equation

$$\nabla^2\phi(\mathbf{r}) + B^2\phi(\mathbf{r}) = 0$$

We recognize this relation as the appropriate neutron-balance equation for a multiplying medium at steady state in the one-velocity approximation [cf. Eq. (5.161)]. Solutions of the transport equation are therefore also solutions of the "diffusion equation" (properly called the stationary-wave or Helmholtz equation). Conversely, solutions of the diffusion equation will also satisfy the transport equation exactly in the case of infinite media. Solutions of the diffusion equation for finite geometries, on the other hand, do not; however, if these solutions are applied in regions far from boundaries, the transport equation can be approximately satisfied. In these regions the angular distribution of the flux is *nearly isotropic* (which is a limitation of the diffusion theory), and the diffusion theory results should give a good approximation to the spatial distribution of the neutrons. Thus solutions to the diffusion equation are the *asymptotic* solutions to the transport equation.

4. *Directed Flux* $\phi(\mathbf{r},\mathbf{\Omega})$ *in Infinite Media.* The general solution for $\phi(\mathbf{r},\mathbf{\Omega})$ in an infinite homogeneous medium is obtained by substituting the general expression (7.255) into (7.250); it is easy to show that integration over R yields

$$\phi(\mathbf{r},\mathbf{\Omega}) = \frac{c\Sigma_t}{4\pi}\int_0^\infty dR \int_{\mathbf{u}} a(\mathbf{u}) e^{iB\mathbf{u}\cdot(\mathbf{r}-\mathbf{R})-\Sigma_t R}\, d\mathbf{u} = \frac{c}{4\pi}\int_{\mathbf{u}} \frac{a(\mathbf{u})e^{iB\mathbf{u}\cdot\mathbf{r}}\, d\mathbf{u}}{1 + iB\mathbf{u}\cdot\mathbf{\Omega}/\Sigma_t} \qquad (7.257)\star$$

The application of this result to the infinite-slab problem is demonstrated in the analysis which follows.

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e. Infinite-slab Reactor. An approximate solution for the angular distribution of the neutron flux in an infinite slab of multiplying material may be derived from the general solutions for infinite media developed in Sec. 7.4d. The present formulation of the problem follows the work of A. H. Wilson.[1]

Consider a uniform infinite slab of half-width a oriented with respect to the x axis, as shown in Fig. 7.13. The problem is to determine $\phi(\mathbf{r},\Omega)$ throughout this medium. The general solution for $\phi(\mathbf{r},\Omega)$ in an infinite medium is given by (7.257); however, this solution does not apply directly to the present problem since one dimension of the medium is finite. Nevertheless, it is possible to develop an approximate solution to the slab problem from the results for the infinite medium.

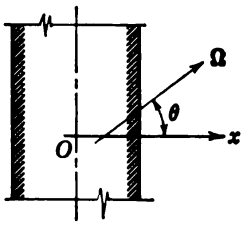

FIG. 7.13 Infinite-slab reactor.

The directed flux in an infinite medium wherein the spatial distribution of neutrons is given by a single variable x, and therefore by a single angle θ, has been given by Wilson. In this special case $\phi(x,\theta)$ is symmetric in x and θ, and the elementary solution for $\phi(x,\theta)$ may be written in the form

$$\phi(x,\theta) = \frac{1}{2}\left(\frac{e^{iBx}}{1 + \frac{iB}{\Sigma_t}\cos\theta} + \frac{e^{-iBx}}{1 - \frac{iB}{\Sigma_t}\cos\theta}\right) = \frac{\cos Bx + \frac{B}{\Sigma_t}\cos\theta \sin Bx}{1 + \frac{B^2}{\Sigma_t^2}\cos^2\theta} \tag{7.258}$$

As usual, the eigenvalue B is to be computed from (7.254); this may be demonstrated by substituting (7.258) into (7.250). The above expression, then, is an exact solution to the transport equation. However, it has no direct physical application for two reasons: (1) it applies only to the *infinite* medium; and (2) not all the solutions (7.258), for the various values of B from (7.254), are positive. In spite of these facts, (7.258) can be used to construct physically useful solutions. The technique is to replace the actual finite body by an infinite medium for which (7.258) gives mathematically exact solutions and so construct the solution that the regions of the infinite space, wherein $\phi < 0$, are outside the boundaries of the real system. This approach is analogous, then, to the method of images which was used for the Fermi age solution to the slab problem (see Sec. 6.4b). In general, this method cannot yield an exact solution to the medium with a finite dimension since it is not possible to satisfy

[1] A. H. Wilson, The General Properties of the Transport Equation and Its Use in Determining the Multiplication in Bodies Having Spherical Symmetry, A.E.R.E. Harwell, M. S. 105A, Dec. 7, 1944.

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the boundary condition completely. However, an approximate solution can be obtained which can be useful for actual applications.

The boundary condition we should like to satisfy is [cf. Eq. (7.97)]

$$\phi(a,\theta) = 0 \qquad \frac{\pi}{2} \leq \theta \leq \pi \tag{7.259}$$

Inasmuch as this condition can be satisfied only in part, we select instead the less restrictive requirement that there be no neutrons returning to the slab along directions *normal* to the faces; thus, we ask that

$$\phi(a,\pi) = 0 \tag{7.260}\star$$

If this condition is applied to the solution (7.258), the following equation is obtained:

$$\cot Ba = \frac{B}{\Sigma_t} \tag{7.261}$$

This equation is to be solved simultaneously with the eigenvalue relation (7.254). The joint solution yields the criticality requirements for the slab. It is convenient to write this pair of equations in the form

$$\begin{aligned} \cot Ba &= \frac{B}{\Sigma_t} \qquad (a) \\ \frac{\tan (B/c\Sigma_t)}{B/c\Sigma_t} &= c \qquad (b) \end{aligned} \tag{7.262}\star$$

As in the diffusion-theory problems, the specification of the cross sections (i.e., fuel concentration) yields the size a, and vice versa. For example, if the fuel concentration is specified, then Σ_t and c [from (7.236)] may be computed, and Eq. (7.262b) yields B. The substitution of this result into (7.262a) gives the half-width of the slab a.

Since boundary condition (7.260) satisfies the correct relation (7.259) only at $\theta = \pi$, it is of interest to determine the directed flux $\phi(a,\theta)$ for other directions. Wilson has performed this calculation for the case $c = 4/\pi$ and $Ba = \pi/4$. The resulting values of $\phi(a,\theta)$ are shown by the broken curve in Fig. 7.14 to a relative scale. The representation of the physical situation by the approximate condition (7.260) is evidently not too poor. As is to be expected, $\phi(a,\theta) \neq 0$ for any value of θ different from π; however, over a sizable angle $\phi(a,\theta) \ll \phi(a,\pi/2)$.

There is an alternate condition to (7.260) which yields somewhat better values of B. This condition corresponds to the diffusion-theory requirement that the partial current from the vacuum be zero [see (5.57)]. The appropriate statement in terms of the directed flux $\phi(\mathbf{r},\Omega)$ is [cf. (7.64)]

$$dA \int_{in} \phi(\mathbf{a},\Omega)(\Omega \cdot \mathbf{s})\, d\Omega = 0 \tag{7.263}\star$$

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where dA is an element of area on the boundary surface **a**, **s** is the outward pointing normal to dA, and the integration extends over all Ω directed into the medium from the surrounding space. In the slab problem, $\phi = \phi(x,\theta)$, and (7.263) reduces to

$$\int_{\pi/2}^{\pi} \phi(a,\theta) \cos\theta \sin\theta \, d\theta = 0 \tag{7.264}$$

where we have used the differential $d\Omega \equiv \sin\theta \, d\theta \, d\psi$. This condition

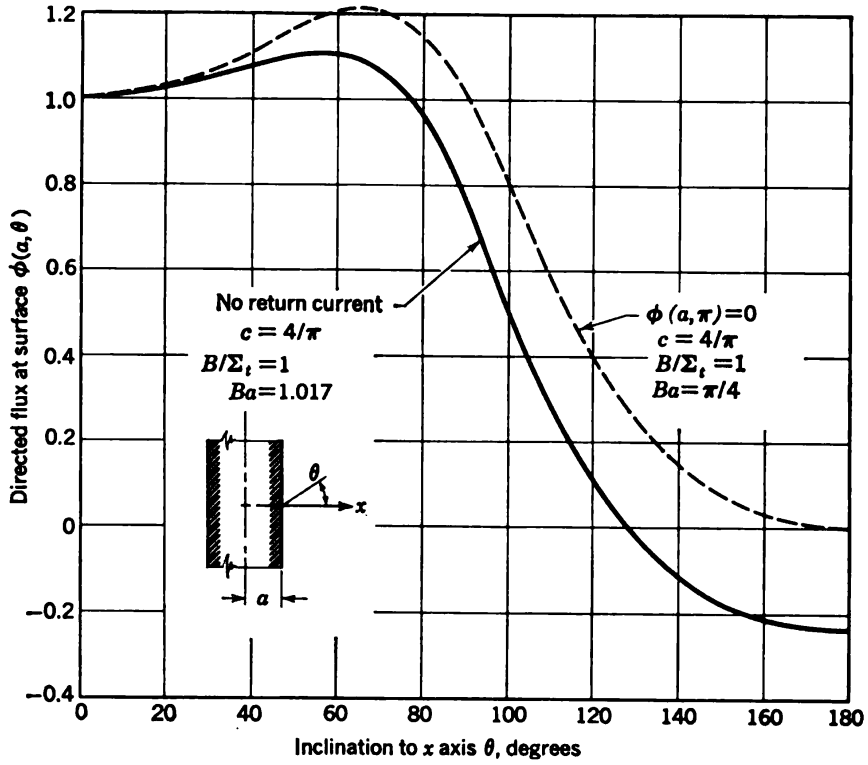

FIG. 7.14 Directed flux at surface of slab.

will yield $\phi(a,\theta) < 0$ for some values of θ; but the effect of this on the estimate of the inward current will be compensated for by the contributions from the remaining directions.

The substitution of the infinite-medium solution (7.258) into (7.264) yields, after integration,

$$\cot Ba = \frac{B}{\Sigma_t}\left\{\frac{1 - (\Sigma_t/B)\tan^{-1}(B/\Sigma_t)}{\frac{1}{2}\ln[1 + (B/\Sigma_t)^2]}\right\} \tag{7.265}\star$$

A comparison of this result with (7.261) reveals that the right-hand side of (7.265) is always less than B/Σ_t; thus these two conditions give different values of a for a given set of cross sections. This discrepancy is well demonstrated by the sample computation for $c = 4/\pi$ and $B/\Sigma_t = 1$. In the case of the boundary condition $\phi(a,\pi) = 0$, Eq. (7.261) yields $Ba = 0.785$, whereas the no-return current condition (7.265) yields

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$Ba = 1.017$. Thus the two conditions presented above would give noticeably different estimates of the slab thickness for a critical system. This discrepancy is due to the fact that the neutrons in each case are emitted from the slab in a nearly isotropic distribution. Figure 7.14 shows this tendency for both cases (the solid-line curve refers to the no-return current condition). Only if the neutron emission is mainly along the outward normal to the surface will the two methods agree well. This is the case for spherical geometries, an example of which follows.

f. Bare Spherical Reactor. We consider here a second demonstration of the technique of constructing approximate solutions for the directed flux in a finite system from the results for the infinite medium. In this example we analyze a homogeneous multiplying sphere of radius a. It was shown previously that the directed flux in an infinite homogeneous medium could be written in the form

$$\phi(\mathbf{r},\mathbf{\Omega}) = \frac{c\Sigma_t}{4\pi} \int_0^\infty \phi(\mathbf{r} - \mathbf{R}) e^{-\Sigma_t R}\, dR \tag{7.250}$$

The elementary solution to this equation is $e^{i\mathbf{B}\cdot\mathbf{r}}$, where the eigenvalue B is computed from (7.254). It was also shown that this solution satisfied the stationary-wave (diffusion) equation (see Sec. 7.4d). Now, the only solution of the diffusion equation (5.161) which is spherically symmetric and finite everywhere is [cf. Eq. (5.166)]

$$\phi(r) = \frac{A \sin Br}{r} \tag{7.266}$$

This function will also satisfy the integral equation (7.250); thus, we may use this relation for the total flux $\phi(\mathbf{r})$ under the integral sign. The result is

$$\begin{aligned} \phi(\mathbf{r},\mathbf{\Omega}) &= \frac{c\Sigma_t A}{4\pi} \int_0^\infty \frac{\sin\,(B|\mathbf{r} - \mathbf{R}|) e^{-\Sigma_t R}\, dR}{|\mathbf{r} - \mathbf{R}|} \\ &= \frac{c\Sigma_t A}{4\pi} \int_0^\infty \frac{\sin\,[B(r^2 + R^2 - 2rR\mu)^{\frac{1}{2}}] e^{-\Sigma_t R}\, dR}{(r^2 + R^2 - 2rR\mu)^{\frac{1}{2}}} \end{aligned} \tag{7.267}$$

where μ is the cosine of the angle subtended by the two vectors $\mathbf{r}$ and $\mathbf{R}$ (i.e., $\mathbf{r} \cdot \mathbf{R} = rR\mu$). Note that in the present system $\phi(\mathbf{r},\mathbf{\Omega}) \to \phi(r,\theta)$.

All the statements which have so far been made apply to the infinite homogeneous medium. We now assume that the solution (7.267) is valid in the interior of the specified sphere. If (7.267) were an exact solution for this finite system, then we could satisfy the boundary condition $\phi(a,\theta) = 0$ for $\pi/2 \leq \theta \leq \pi$. Since this is not the case we attempt to satisfy this requirement in part. As in the slab problem, if we ask that $\phi > 0$ everywhere within the sphere, then the appropriate condition to be satisfied is

$$\phi(a,\pi) = 0$$

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which is identical to (7.260). The application of this approximate boundary condition to (7.267) yields the equation

$$\frac{c\Sigma_t A}{4\pi}\int_0^\infty \frac{e^{-\Sigma_t R}\sin B(a+R)\,dR}{a+R} = \frac{c\Sigma_t A e^{\Sigma_t a}}{4\pi}\int_a^\infty \frac{e^{-\Sigma_t s}\sin Bs\,ds}{s} = 0 \tag{7.268}$$

This equation gives the eigenvalue B which satisfies condition (7.260). The integral which appears in this relation is more conveniently written in the form

$$\begin{aligned}\int_a^\infty \frac{e^{-\Sigma_t s}\sin Bs\,ds}{s} &= \frac{1}{2i}\int_a^\infty [e^{-s(\Sigma_t - iB)} - e^{-s(\Sigma_t + iB)}]\frac{ds}{s} \\ &= \frac{1}{2i}[E_1(\Sigma_t a - iBa) - E_1(\Sigma_t a + iBa)] = -\operatorname{Im} E_1(\Sigma_t a + iBa)\end{aligned} \tag{7.269}$$

where $E_1(z)$ is the exponential integral [see Eq. (5.306)] and Im $E_1(z)$ denotes the imaginary part of $E_1(z)$. The exponential integrals of complex argument have been tabulated and will be discussed in detail in connection with the Serber-Wilson method.

From the above results we find, then, that the condition $\phi(a,\pi) = 0$ yields the requirement

$$-\operatorname{Im} E_1(\Sigma_t a + iBa) = 0 \tag{7.270}\star$$

This condition is to be used in conjunction with the eigenvalue relation (7.254). As in the case of the set (7.262), (7.270) together with (7.254) yield either the size or the fuel concentration, given the other.

The boundary condition which requires that there be no return current from the surrounding vacuum [see (7.263)] may also be applied to the present problem. Wilson has carried out this calculation, and the detailed analytical statement of this condition is given in the previously mentioned report.[1] He finds, however, that the result agrees with (7.270) to within a few per cent. Since the resulting expression for the no-return current condition is quite complicated, it is suggested that the more simple form (7.270) be used.

We conclude the present section with a few remarks about the calculation of the directed flux $\phi(r,\theta)$ within the finite sphere. These calculations have been performed by Wilson, and the reference report presents several series representations of this function which are suitable for computational purposes. These include special expansions for both $B/\Sigma_t \gg 1$ and $B/\Sigma_t \ll 1$, as well as a general solution which is given in terms of Bessel functions and Legendre functions of the second kind.

[1] *Ibid.*

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7.5 Applications of Spherical-harmonics Method

In this section we illustrate the application of the spherical-harmonics method by considering two problems of some practical interest. Both problems deal with steady-state one-dimensional systems, and the calculation is carried out on the basis of the one-velocity model developed in Sec. 7.2f. In the present applications the general time-dependent relations given in (7.84) reduce to the following set of differential equations:

$$-\Sigma_t\phi_n(x) + \Sigma_s\eta_n\phi_n(x) + S_n(x) = \left(\frac{n+1}{2n+1}\right)\frac{d}{dx}\phi_{n+1}(x) + \left(\frac{n}{2n+1}\right)\frac{d}{dx}\phi_{n-1}(x) \quad (7.271)\star$$

where $n = 0, 1, \ldots$. Our immediate objective in each case will be to obtain solutions of this set of equations which satisfy the particular boundary conditions that define the two problems. Since the method we will use is independent of the order of the approximation (i.e., the value of n), we will truncate the set (7.271) at $n = 3$. This yields the so-called P_3 approximation which is the next more accurate model after the diffusion theory. It is of interest to note in this connection that only the odd-order approximations yield higher degrees of accuracy. For example, if $n = 1$ is the first-order approximation (P_1), $n = 3$ is the second order. The approximation corresponding to $n = 2$ yields the same order differential equation (second) as does the $n = 1$ approximation. Thus the $n = 2$ calculation yields no new information about the angular distribution of the flux, and so we must go to $n = 3$ for a more accurate model. In the same way $n = 4$ is of the same order as $n = 3$, and so the next better approximation after P_3 is P_5, and so on.

The P_3 approximation of the one-velocity transport model is obtained from Eqs. (7.271) by retaining all relations which involve the harmonics ϕ_n for $n \leq 3$. These are

$$\begin{aligned}
-\Sigma_t\phi_0 + \Sigma_s\eta_0\phi_0 + S_0 &= \frac{d}{dx}\phi_1 && (a)\\
-\Sigma_t\phi_1 + \Sigma_s\eta_1\phi_1 + S_1 &= \frac{2}{3}\frac{d}{dx}\phi_2 + \frac{1}{3}\frac{d}{dx}\phi_0 && (b)\\
-\Sigma_t\phi_2 + \Sigma_s\eta_2\phi_2 + S_2 &= \frac{3}{5}\frac{d}{dx}\phi_3 + \frac{2}{5}\frac{d}{dx}\phi_1 && (c)\\
-\Sigma_t\phi_3 + \Sigma_s\eta_3\phi_3 + S_3 &= \frac{3}{7}\frac{d}{dx}\phi_2 && (d)
\end{aligned} \quad (7.272)$$

where we have neglected the ϕ_4 term in (d).

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In the first calculation which follows we compute in detail the angular distribution of the neutron flux in the vicinity of an interface between two dissimilar media. This example demonstrates the application of the interface-boundary condition (7.100). The second example deals with a slab of multiplying material, and in this calculation the emphasis is placed on the specification of the vacuum-interface condition and on the estimation of the critical width of the slab by means of the various models which have been developed.

a. Angular Distribution of Neutrons at an Interface. Consider two semi-infinite media of different nuclear properties which have the plane $x = 0$ as a common boundary. There exists throughout this entire space a uniform one-velocity isotropic neutron source. The problem is to compute and compare the distribution of the one-velocity directed flux in the two media according to the P_3 approximation.

For this application the general steady-state solution to the transport equation in the P_3 approximation (7.272) reduces to the somewhat simpler form.

$$\begin{aligned} \frac{d\phi_1}{dx} + \Sigma_0\phi_0 &= S_0 && (a) \\ \frac{2}{3}\frac{d\phi_2}{dx} + \frac{1}{3}\frac{d\phi_0}{dx} + \Sigma_1\phi_1 &= 0 && (b) \\ \frac{3}{5}\frac{d\phi_3}{dx} + \frac{2}{5}\frac{d\phi_1}{dx} + \Sigma_2\phi_2 &= 0 && (c) \\ \frac{3}{7}\frac{d\phi_2}{dx} + \Sigma_3\phi_3 &= 0 && (d) \end{aligned} \tag{7.273}$$

in which we have defined

$$\Sigma_n \equiv \Sigma_t - \Sigma_s\eta_n \tag{7.274}$$

and have recognized that for an isotropic source $S_n = 0$ for all $n > 0$. Since the above equations describe the directed flux in a homogeneous medium which contains a constant source, the application of these equations to a two-medium system requires that a separate solution be obtained for each region and the two results matched at the common surface $x = 0$.

We solve the four simultaneous equations in (7.273) by eliminating the three harmonics ϕ_1, ϕ_2, and ϕ_3 and thereby obtaining a single fourth-order differential equation in ϕ_0. The solution of the fourth-order equation may then be substituted into (7.273) and corresponding expressions obtained for the higher harmonics. The procedure is straightforward and will not be discussed in detail here. Suffice it to say that the calculation may be carried out by first eliminating ϕ_3 from (7.273*c*) by means of (7.273*d*), then ϕ_1 and finally ϕ_2 by performing several differentiations of the Eqs. (7.273*a*), (7.273*b*), and (7.273*c*). The resulting expression for

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ϕ_0 may be written in the form

$$\alpha \frac{d^4\phi_0}{dx^4} - \beta \frac{d^2\phi_0}{dx^2} + \gamma\phi_0 = \Sigma_1\Sigma_2 S_0 \tag{7.275}$$

where

$$\alpha \equiv \frac{3}{35\Sigma_3} \qquad \beta \equiv \frac{\Sigma_2}{3} + \frac{9\Sigma_0\Sigma_1}{35\Sigma_3} + \frac{4\Sigma_0}{15\Sigma_1} \qquad \gamma \equiv \Sigma_0\Sigma_1\Sigma_2 \tag{7.276}$$

Equation (7.275) is an ordinary differential equation with constant coefficients. Its solution is easily shown to be

$$\phi_0(x) = ae^{mx} + be^{-mx} + ce^{nx} + de^{-nx} + \frac{S_0}{\Sigma_0} \tag{7.277}$$

wherein a, b, c, and d are constants to be established from the interface condition, and the roots of the characteristic equation, m and n, are given by

$$\begin{aligned} m &\equiv \left[\frac{\beta}{2\alpha}\left(1 + \sqrt{1 - \frac{4\alpha\gamma}{\beta^2}}\right)\right]^{\frac{1}{2}} \\ n &\equiv \left[\frac{\beta}{2\alpha}\left(1 - \sqrt{1 - \frac{4\alpha\gamma}{\beta^2}}\right)\right]^{\frac{1}{2}} \end{aligned} \tag{7.278}$$

The solutions for the various harmonics are obtained by substituting Eq. (7.277) into the set, Eqs. (7.273). Thus Eq. (7.273a) yields $\phi_1(x)$, Eq. (7.273b), $\phi_2(x)$, and Eq. (7.273d), $\phi_3(x)$. The results are

$$\phi_1(x) = -\Sigma_0\left(\frac{a}{m}e^{mx} - \frac{b}{m}e^{-mx} + \frac{c}{n}e^{nx} - \frac{d}{n}e^{-nx}\right) + f \tag{7.279}$$

$$\phi_2(x) = \tfrac{1}{2}Mae^{mx} + \tfrac{1}{2}Mbe^{-mx} + \tfrac{1}{2}Nce^{nx} + \tfrac{1}{2}Nde^{-nx} - \tfrac{3}{2}f\Sigma_1 x - \tfrac{3}{2}\Sigma_1 g \tag{7.280}$$

$$\phi_3(x) = \frac{3}{14\Sigma_3}(-amMe^{mx} + bmMe^{-mx} - cnNe^{nx} + dnNe^{-nx} + 3\Sigma_1 f) \tag{7.281}$$

where f and g are constants of integration and

$$M \equiv \left(\frac{3\Sigma_0\Sigma_1}{m^2} - 1\right) \qquad N \equiv \left(\frac{3\Sigma_0\Sigma_1}{n^2} - 1\right) \tag{7.282}$$

If the solutions (7.279) through (7.281) are used in Eq. (7.273c), then a comparison of the coefficients of like functions of x reveals that both f and g must be identically zero.

The above relations are the general solutions which apply to each of the two media in question. Specific solutions are obtained for each medium by requiring that the flux remain finite for all x. This immediately identifies several of the integration constants, and it may be seen that for the medium which occupies the space $x \leq 0$ (which we denote

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by the index M), Eqs. (7.277) through (7.281) reduce to the set

$$\phi_0^{(M)}(x) = ae^{m_M x} + ce^{n_M x} + \frac{S_0}{\Sigma_0^{(M)}} \tag{7.283}$$

$$\phi_1^{(M)}(x) = -\Sigma_0^{(M)}\left(\frac{a}{m_M}e^{m_M x} + \frac{c}{n_M}e^{n_M x}\right) \tag{7.284}$$

$$\phi_2^{(M)}(x) = \tfrac{1}{2}M_M a e^{m_M x} + \tfrac{1}{2}N_M c e^{n_M x} \tag{7.285}$$

$$\phi_3^{(M)}(x) = -\frac{3}{14\Sigma_3^{(M)}}(am_M M_M e^{m_M x} + cn_M N_M e^{n_M x}) \tag{7.286}$$

If we denote by the index F the medium which occupies the space $x \geq 0$, then the appropriate solutions are

$$\phi_0^{(F)}(x) = be^{-m_F x} + de^{-n_F x} + \frac{S_0}{\Sigma_0^{(F)}} \tag{7.287}$$

$$\phi_1^{(F)}(x) = \Sigma_0^{(F)}\left(\frac{b}{m_F}e^{-m_F x} + \frac{d}{n_F}e^{-n_F x}\right) \tag{7.288}$$

$$\phi_2^{(F)}(x) = \tfrac{1}{2}M_F b e^{-m_F x} + \tfrac{1}{2}N_F d e^{-n_F x} \tag{7.289}$$

$$\phi_3^{(F)}(x) = \frac{3}{14\Sigma_3^{(F)}}(bm_F M_F e^{-m_F x} + dn_F N_F e^{-n_F x}) \tag{7.290}$$

The parameters m_i, n_i, M_i, and N_i are computed from (7.278) and (7.282), using the appropriate cross sections for the particular medium i.

The constants a, b, c, and d are obtained by the application of the interface condition (7.101), which, in the present notation, may be written

$$\phi_M(0,\mu) = \phi_F(0,\mu) \qquad -1 \leq \mu \leq 1 \tag{7.291}$$

The P_3 approximation for the directed flux $\phi(x,\mu)$ is obtained by terminating the series [cf. Eq. (7.74)]

$$\phi(x,\mu) \equiv \sum_{s=0}^{\infty}\left(\frac{2s+1}{2}\right)\phi_s(x)\,P_s(\mu) \tag{7.292}$$

at $s = 3$; thus, introducing the definition of the functions $P_s(\mu)$, we have

$$\phi(x,\mu) \simeq \tfrac{1}{2}\phi_0(x) + \tfrac{3}{2}\phi_1(x)\mu + \tfrac{5}{4}\phi_2(x)(3\mu^2 - 1) + \tfrac{7}{4}\phi_3(x)(5\mu^3 - 3\mu) \tag{7.293}\star$$

If we apply this approximation to each side of (7.291), then for all values of s we must have that

$$\phi_s^{(M)}(0) = \phi_s^{(F)}(0) \tag{7.294}$$

since μ is an independent variable. The application of the condition (7.294) to Eqs. (7.283) through (7.290) yields the following four simul-

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taneous algebraic equations in a, b, c, and d:

$$\begin{gathered} a - b + c - d = S_0\left[\frac{1}{\Sigma_0^{(F)}} - \frac{1}{\Sigma_0^{(M)}}\right] \\ \left[\frac{\Sigma_0^{(M)}}{m_M}\right]a + \left[\frac{\Sigma_0^{(F)}}{m_F}\right]b + \left[\frac{\Sigma_0^{(M)}}{n_M}\right]c + \left[\frac{\Sigma_0^{(F)}}{n_F}\right]d = 0 \\ M_M a - M_F b + N_M c - N_F d = 0 \\ \left[\frac{m_M M_M}{\Sigma_3^{(M)}}\right]a + \left[\frac{m_F M_F}{\Sigma_3^{(F)}}\right]b + \left[\frac{n_M N_M}{\Sigma_3^{(M)}}\right]c + \left[\frac{n_F N_F}{\Sigma_3^{(F)}}\right]d = 0 \end{gathered} \tag{7.295}$$

The solution of these equations for the four unknowns when substituted into the harmonic relations (7.283) through (7.290) gives a complete description of the directed flux through the two regions M and F.

We illustrate the above method by means of a sample computation. Let

$$\begin{aligned} &\Sigma_a^{(M)} = 0.002\Sigma_t^{(M)} && \Sigma_a^{(F)} = \Sigma_s^{(F)} = \tfrac{1}{2}\Sigma_t^{(F)} \\ &\Sigma_s^{(M)} = 0.998\Sigma_t^{(M)} && \bar{\mu}_0^{(M)} = 0.5 \qquad \bar{\mu}_0^{(F)} = 0 \\ &\frac{S_0}{\Sigma_a^{(M)}} = 10 && \frac{S_0}{\Sigma_a^{(F)}} = 0.5 \end{aligned} \tag{7.296}$$

If we assume that $\mathfrak{y}_n = 0$ for $n > 1$, then it follows that

$$\begin{gathered} \Sigma_0 = \Sigma_t - \Sigma_s\mathfrak{y}_0 = \Sigma_a \qquad \Sigma_1 = \Sigma_t - \Sigma_s\mathfrak{y}_1 = \Sigma_t\left(1 - \overline{\mu}_0\frac{\Sigma_s}{\Sigma_t}\right) \\ \Sigma_2 = \Sigma_t \qquad \Sigma_3 = \Sigma_t \end{gathered} \tag{7.297}$$

These data may be used to compute the parameters m, n, M, and N. The results are:

$$\begin{aligned} m_M &= 1.974\Sigma_t^{(M)} & n_M &= 0.05477\Sigma_t^{(M)} \\ M_M &= -0.9921 & N_M &= 0.00003333 \\ m_F &= 2.441\Sigma_t^{(F)} & n_F &= 0.9876\Sigma_t^{(F)} \\ M_F &= -0.7481 & N_F &= 0.5382 \end{aligned}$$

The substitution of these values into Eqs. (7.295) yields for the coefficients

$$\begin{aligned} a &= -0.09324 & b &= 0.2517 \\ c &= -8.634 & d &= 0.5212 \end{aligned}$$

The solutions for the harmonics $\phi_n^{(i)}(x)$ are then

$$\begin{aligned} \phi_0^{(M)}(x) &= 10 - 0.09324e^{1.974\Sigma_t^{(M)}x} - 8.634e^{0.05477\Sigma_t^{(M)}x} \\ \phi_1^{(M)}(x) &\cong 0.3153e^{0.05477\Sigma_t^{(M)}x} \\ \phi_2^{(M)}(x) &= 0.04625e^{1.974\Sigma_t^{(M)}x} \\ \phi_3^{(M)}(x) &= -0.03913e^{1.974\Sigma_t^{(M)}x} \end{aligned} \tag{7.298}$$

and

$$\begin{aligned} \phi_0^{(F)}(x) &= 0.5 + 0.2517e^{-2.441\Sigma_t^{(F)}x} + 0.5212e^{-0.9876\Sigma_t^{(F)}x} \\ \phi_1^{(F)}(x) &= 0.05156e^{-2.441\Sigma_t^{(F)}x} + 0.2635e^{-0.9876\Sigma_t^{(F)}x} \\ \phi_2^{(F)}(x) &= -0.09414e^{-2.441\Sigma_t^{(F)}x} + 0.1402e^{-0.9876\Sigma_t^{(F)}x} \\ \phi_3^{(F)}(x) &= -0.09849e^{-2.441\Sigma_t^{(F)}x} + 0.05936e^{-0.9876\Sigma_t^{(F)}x} \end{aligned} \tag{7.299}$$

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The corresponding solutions for the P_1 approximation (diffusion theory) are of interest for purposes of comparison. The appropriate differential equations are obtained from (7.271); these equations were developed in Sec. 7.2f.

$$-D\phi_0''(x) + \Sigma_a\phi_0(x) = S_0 \tag{7.87}$$

$$\phi_1(x) = -D\phi_0'(x) \tag{7.88}$$

The general solutions for ϕ_0 and ϕ_1 are

$$\phi_0(x) = Ae^{\varkappa x} + Ce^{-\varkappa x} + \frac{S_0}{\Sigma_a} \tag{7.300}$$

$$\phi_1(x) = -A\varkappa De^{\varkappa x} + C\varkappa De^{-\varkappa x} \tag{7.301}$$

where $\varkappa^2 \equiv \Sigma_a/D$. In the region M these reduce to

$$\phi_0^{(M)}(x) = Ae^{\varkappa_M x} + \frac{S_0}{\Sigma_a^{(M)}} \tag{7.302}$$

$$\phi_1^{(M)}(x) = -AD_M\varkappa_M e^{\varkappa_M x} \tag{7.303}$$

and in the region F,

$$\phi_0^{(F)}(x) = Ce^{-\varkappa_F x} + \frac{S_0}{\Sigma_a^{(F)}} \tag{7.304}$$

$$\phi_1^{(F)}(x) = CD_F\varkappa_F e^{-\varkappa_F x} \tag{7.305}$$

We compute the constants A and C by the application of the boundary condition (7.294). This yields

$$A = \frac{S_0[(1/\Sigma_a^{(F)}) - (1/\Sigma_a^{(M)})]}{1 + D_M\varkappa_M/D_F\varkappa_F} \quad \text{and} \quad C = -\left(\frac{D_M\varkappa_M}{D_F\varkappa_F}\right)A \tag{7.306}$$

The angular distribution for the flux $\phi(x,\mu)$ is given by

$$\phi(x,\mu) = \tfrac{1}{2}\phi_0(x) + \tfrac{3}{2}\phi_1(x)\mu \tag{7.93}$$

The appropriate solutions to these equations for the system defined by the constants (7.296) is found to be

$$\phi_0^{(M)}(x) = 10 - 8.728e^{0.05537\Sigma_t^{(M)}x}$$

$$\phi_1^{(M)}(x) = 0.3153e^{0.05537\Sigma_t^{(M)}x} \tag{7.307}$$

and

$$\phi_0^{(F)}(x) = 0.5 + 0.7722e^{-1.225\Sigma_t^{(F)}x}$$

$$\phi_1^{(F)}(x) = 0.3153e^{-1.225\Sigma_t^{(F)}x} \tag{7.308}$$

Table 7.1 lists $\phi(x,\mu)/\phi_0(x)$ for various x and μ according to the P_3 approximations given by (7.298) and (7.299). The numbers in the parenthesis are the corresponding values obtained from the diffusion-theory relations (7.307) and (7.308).

These results have been used to compute the total flux $\phi_0(x)$ throughout the vicinity of the interface. Figure 7.15 shows this distribution at various stations measured from the interface $x = 0$ in units of *mean free paths* $[1/\Sigma_t^{(i)}]$ for each region. The particular choice of constants given

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in (7.296) represents reasonably well the relative properties of a moderator (scattering) (M) type material and an absorber (F) type material. It is of interest to note the relative suppression of the flux in the absorber

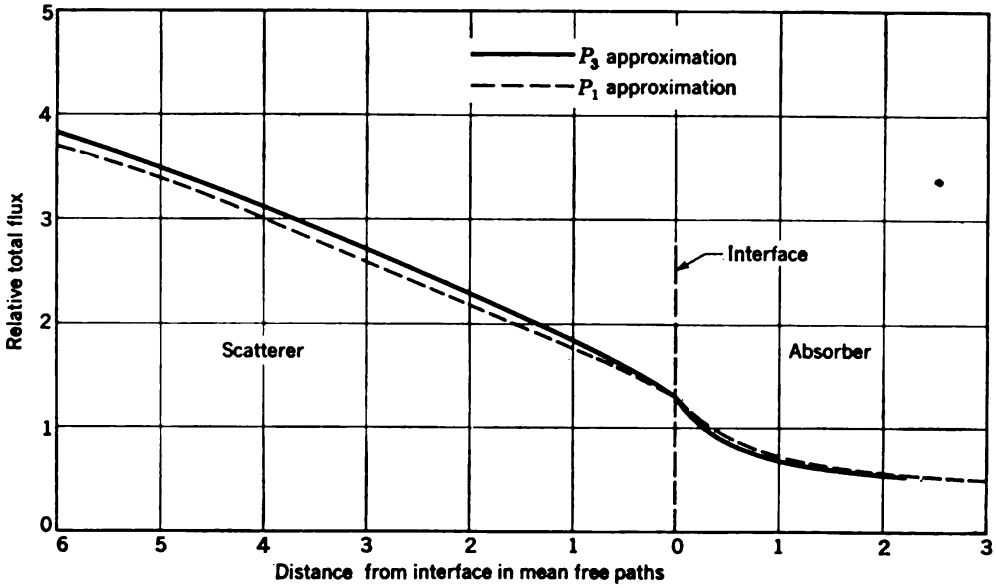

FIG. 7.15 Total neutron flux in the vicinity of the interface between absorbing and scattering materials.

region as compared to the scattering region. Note also that the relatively higher neutron density in the scatterer results in a net flow of neutrons into the absorber region.

TABLE 7.1 ANGULAR DISTRIBUTION OF FLUX AT INTERFACE ACCORDING TO P_1 AND P_3 APPROXIMATIONS

Distance from interface	$\mu = -1$	$\mu = -\frac{1}{2}$	$\mu = 0$	$\mu = \frac{1}{2}$	$\mu = 1$
$\lambda_t^{(M)}$	0.2724 (0.2431)	0.3708 (0.3716)	0.4956 (0.5000)	0.6269 (0.6284)	0.7453 (0.7568)
$\frac{1}{2}\lambda_t^{(M)}$	0.2660 (0.1954)	0.3352 (0.3477)	0.4862 (0.5000)	0.6579 (0.6523)	0.7890 (0.8046)
0	0.3268 (0.1282)	0.2558 (0.3141)	0.4546 (0.5000)	0.7215 (0.6859)	0.8548 (0.8718)
$\frac{1}{2}\lambda_t^{(F)}$	0.3379 (0.2209)	0.3441 (0.3605)	0.4190 (0.5000)	0.6154 (0.6395)	0.9859 (0.7791)
$\lambda_t^{(F)}$	0.3725 (0.3088)	0.4022 (0.4044)	0.4231 (0.5000)	0.5593 (0.5956)	0.9348 (0.6911)

Figure 7.16 shows the relative distributions of the directed flux at various angles to the x axis. Sketches a through e in the figure reveal that, at all stations in the neighborhood of the interface, there is a

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relatively larger number of neutrons moving in the directions $0 \lesssim \mu \leq 1$; i.e., the majority of the neutrons are headed into the interior regions of the absorber. Only at relatively large distances from the interface in both media (several mean free paths) does the distribution become isotropic. This is shown in Fig. 7.16*f*. These results demonstrate well

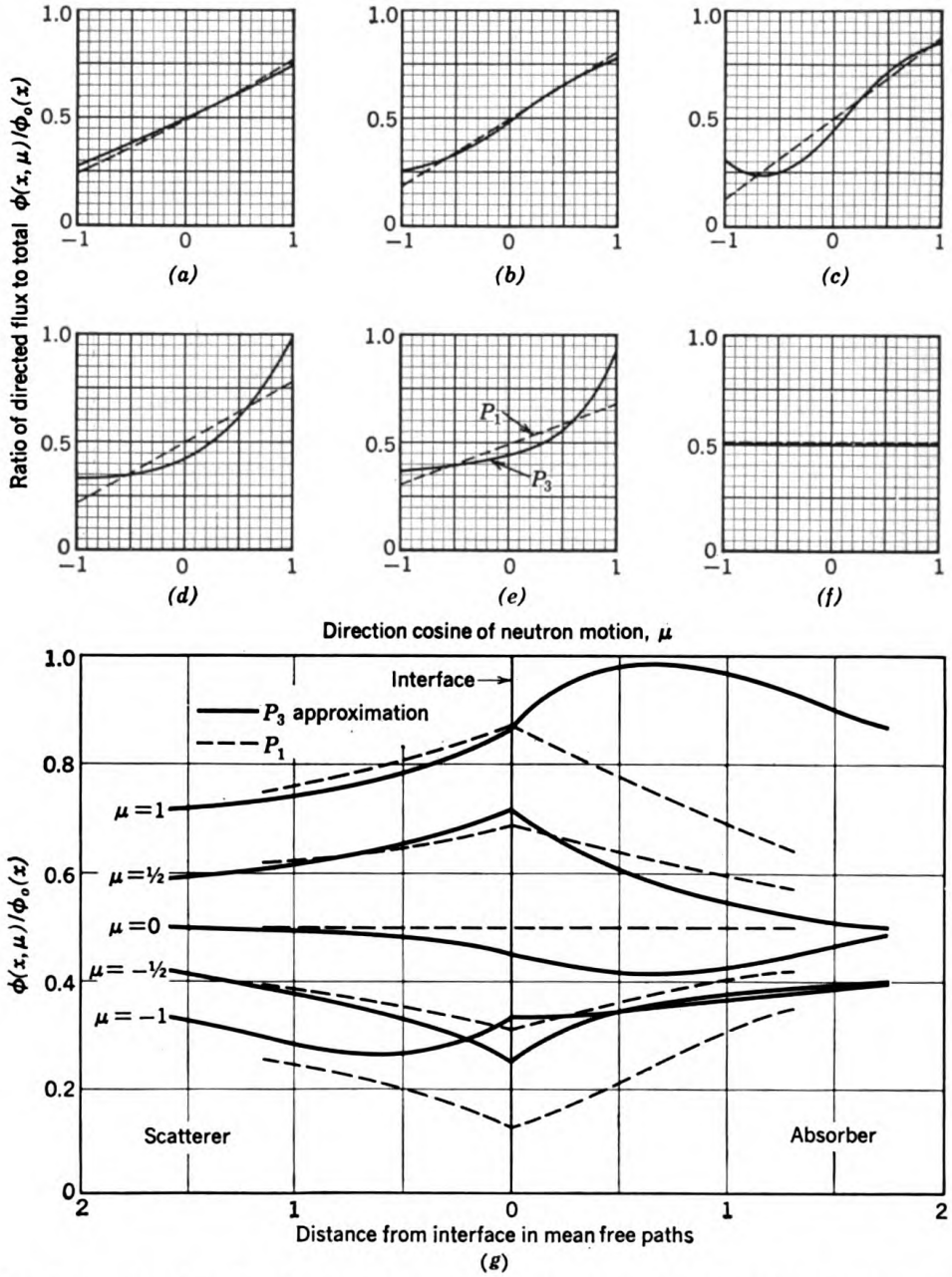

FIG. 7.16 Angular distribution of neutron flux around the interface between an absorber and a scatterer (MFP, mean free path): (*a*) scatterer, 1 MFP; (*b*) scatterer, $\frac{1}{2}$ MFP; (*c*) interface; (*d*) absorber, $\frac{1}{2}$ MFP; (*e*) absorber, 1 MFP; (*f*) scatterer or absorber, ∞ MFP; (*g*) composite sketch of directed flux for several values of μ.

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the assertion made previously on the validity of the diffusion theory, namely, that the flux distribution according to this model gives an accurate representation of the flux only in regions far removed from boundaries.

Figure 7.16*g* is a composite sketch of the directed flux for several values of μ. It allows a comparison of the relative density of neutrons moving in a particular direction to the total neutron population at each station.

b. Infinite-slab Reactor. Consider an infinite slab of multiplying material with width $2a$ operating at steady state. The neutron sources in this system are the fission reactions, and on the basis of the one-velocity model we find that the number of neutrons produced per unit volume per unit time is given by the expression

$$\nu\Sigma_f \int_{-1}^{1} \phi(x,\mu)\, d\mu = \nu\Sigma_f \phi_0(x) \equiv S(x) \tag{7.309}$$

The appropriate set of equations which describe this system in the P_3 approximation are obtained from (7.273), using the definition (7.309) for the neutron sources. Thus,

$$\begin{aligned} \frac{d\phi_1}{dx} + \Sigma_0'\phi_0 &= 0 \\ \frac{2}{3}\frac{d\phi_2}{dx} + \frac{1}{3}\frac{d\phi_0}{dx} + \Sigma_1\phi_1 &= 0 \\ \frac{3}{5}\frac{d\phi_3}{dx} + \frac{2}{5}\frac{d\phi_1}{dx} + \Sigma_2\phi_2 &= 0 \\ \frac{3}{7}\frac{d\phi_2}{dx} + \Sigma_3\phi_3 &= 0 \end{aligned} \tag{7.310}$$

where the Σ_n for all $n > 0$ are given by (7.274), and the symbol Σ_0' is defined by

$$\Sigma_0' \equiv \Sigma_t - \Sigma_s\eta_0 - \nu\Sigma_f = \Sigma_a - \nu\Sigma_f \tag{7.311}$$

The general solution to Eqs. (7.310) is given by the relations (7.277) through (7.281). Note that in the present notation, however, the symbol Σ_0 is replaced by Σ_0'. Because of the symmetry of the present problem, some of the arbitrary constants which appear in the general solution can be eliminated. For this purpose we introduce the following condition at the center of the slab ($x = 0$):

$$\int_{-1}^{1} P_n(\mu)\, \phi(0,\mu)\, d\mu = 0 \qquad n = 1, 3, 5, \ldots \tag{7.312}\star$$

This is analogous to the requirement (7.98) which is imposed at the surface of the slab. The substitution of the expansion (7.292) into (7.312) yields

$$\sum_{s=0}^{\infty} \left(\frac{2s+1}{2}\right) \phi_s(0) \int_{-1}^{1} P_s(\mu)\, P_n(\mu)\, d\mu = \left(\frac{2n+1}{2}\right) \phi_n(0) = 0 \tag{7.313}$$

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Thus it follows that

$$\phi_1(0) = 0 \qquad \phi_3(0) = 0 \tag{7.314}$$

If these requirements are imposed on the relations (7.279) and (7.281), we obtain the pair of equations

$$Mm(b-a) + Nn(d-c) = 0 \qquad \frac{1}{m}(b-a) + \frac{1}{n}(d-c) = 0 \tag{7.315}$$

which constitute a condition on the quantities $(b-a)$ and $(d-c)$. Since the determinant of these equations does not vanish, the only acceptable solution to (7.315) is

$$b - a = 0 \qquad d - c = 0 \tag{7.316}$$

From this result it follows immediately that the general solution for $\phi_0(x)$ may be written in the form [see (7.277)]

$$\phi_0(x) = A \cosh mx + C \cosh nx \tag{7.317}$$

and the Eqs. (7.310) yield

$$\phi_1(x) = -\frac{A\Sigma_0'}{m} \sinh mx - \frac{C\Sigma_0'}{n} \sinh nx \tag{7.318}$$

$$\phi_2(x) = \tfrac{1}{2}AM \cosh mx + \tfrac{1}{2}CN \cosh nx \tag{7.319}$$

$$\phi_3(x) = -\frac{3}{14\Sigma_3}(mMA \sinh mx + nNC \sinh nx) \tag{7.320}$$

The arbitrary constants A and C are obtained by applying the outer boundary condition (7.98). For the case $n = 1$, this yields at $x = a$

$$\sum_{s=0}^{3}\left(\frac{2s+1}{2}\right)\phi_s(a)\int_{-1}^{0} P_1(\mu)\,P_s(\mu)\,d\mu = \phi_0(a) - 2\phi_1(a) + \frac{5}{4}\phi_2(a) = 0 \tag{7.321}$$

and for $n = 3$,

$$\sum_{s=0}^{3}\left(\frac{2s+1}{2}\right)\phi_s(a)\int_{-1}^{0} P_3(\mu)\,P_s(\mu)\,d\mu = -\phi_0(a) + 5\phi_2(a) - 8\phi_3(a) = 0 \tag{7.322}$$

The application of these two conditions to the expressions (7.317) through (7.320) yields two equations in the unknowns A and C, and these may be reduced to the form

$$\frac{C}{A} = -\frac{\cosh ma + (2\Sigma_0'/m)\sinh ma + (5M/8)\cosh ma}{\cosh na + (2\Sigma_0'/n)\sinh na + (5N/8)\cosh na} \tag{7.323}$$

$$\frac{1 + \frac{5}{8}M + \frac{2\Sigma_0'}{m}\tanh ma}{1 + \frac{5}{8}N + \frac{2\Sigma_0'}{n}\tanh na} = \frac{-1 + \frac{5}{2}M + \frac{12mM}{7\Sigma_3}\tanh ma}{-1 + \frac{5}{2}N + \frac{12nN}{7\Sigma_3}\tanh na} \tag{7.324}$$

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where (7.324) may be identified as the *criticality equation* for the infinite-slab reactor in the P_3 approximation.

Consider as a numerical example of the slab problem the system defined by the following set of parameters:

$$\begin{gathered}\Sigma_a = 0.2\Sigma_t \qquad \Sigma_s = 0.8\Sigma_t \qquad \Sigma_f = 0.1\Sigma_t \\ \nu = 2.46 \qquad \bar{\mu}_0 = 0.5 \\ \Sigma_0' = -0.046\Sigma_t \qquad \Sigma_1 = 0.6\Sigma_t \qquad \Sigma_2 = \Sigma_3 = \Sigma_t\end{gathered} \tag{7.325}$$

These data yield for the quantities m, n, M, and N the values

$$\begin{aligned} m &= 1.937\Sigma_t \qquad & n &= 0.2965i\Sigma_t \\ M &= -1.022 \qquad & N &= -0.05820 \end{aligned}$$

The substitution of these results into the criticality equation (7.324) gives the half-width a of the infinite-slab reactor which will be critical with the composition (7.325). This solution is found to be

$$a\Sigma_t = 4.196 \qquad P_3 \text{ approximation} \tag{7.326}$$

The value of A/C is computed from (7.323), and the result yielded is $A/C = -0.00002898$. Thus the total flux according to (7.317) is

$$\phi_0(x) = C[\cos(0.2965\Sigma_t x) - 0.00002898 \cosh(1.937\Sigma_t x)] \tag{7.327}$$

This system may also be analyzed on the basis of the P_1 approximation. The total flux in this case is given by $\phi_0(x) = A \cos Bx$. In order to obtain a consistent comparison of the P_1 and P_3 calculations we discard the usual extrapolated boundary condition $\phi_0(\tilde{a}) = 0$ in favor of the no-return current condition $j_-(a) = 0$. From the diffusion-theory expression for j_- given in Eq. (5.49) we obtain the condition

$$\tan Ba = \frac{1}{2DB} \tag{7.328}$$

And, from the expression for the buckling (5.126), we have

$$\left(\frac{B}{\Sigma_t}\right)^2 = 3\left(1 - \bar{\mu}_0 \frac{\Sigma_s}{\Sigma_t}\right)\left(\nu \frac{\Sigma_f}{\Sigma_t} - \frac{\Sigma_a}{\Sigma_t}\right) \tag{7.329}$$

using $D = 1/3(\Sigma_t - \bar{\mu}_0\Sigma_s)$. The substitution of the data (7.325) into these relations gives

$$\frac{B}{\Sigma_t} = 0.2878 \qquad Ba = 1.261 \qquad a\Sigma_t = 4.382$$

for the P_1 approximation. The expression for the flux is

$$\phi_0(x) = A \cos(0.2878\Sigma_t x)$$

It is of interest also to compute the half-width a on the basis of the integral-equation technique developed in Sec. 7.4e. For the boundary

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condition (7.260) $\phi(a,\pi) = 0$, we have the requirements (7.262). In the present problem

$$c = \frac{\nu\Sigma_f + \Sigma_s}{\Sigma_t} = 1.046$$

and from (7.262*b*) we obtain $B/\Sigma_t = 0.3781$. The substitution of this result into (7.262*a*) yields the critical width $Ba = 1.209$, or in mean free paths,

$$\Sigma_t a = Ba\left(\frac{\Sigma_t}{B}\right) = 3.198 \tag{7.330}$$

If we apply instead the no-return current condition (7.264), the criticality equation is given by (7.265). For $B/\Sigma_t = 0.3781$, we obtain $Ba = 1.343$. The corresponding solution in mean free paths is

$$\Sigma_t a = 3.552 \tag{7.331}$$

We summarize these results below:

	P_1	P_3	$\phi(a,\pi) = 0$	No return current
$\Sigma_t a$	4.382	4.196	3.198	3.552

It is of special interest to note that the diffusion-theory methods, even in the P_3 approximation, yield significantly different solutions from those of the integral (transport) theory model. This is to be expected in the present application since the system in question is relatively narrow

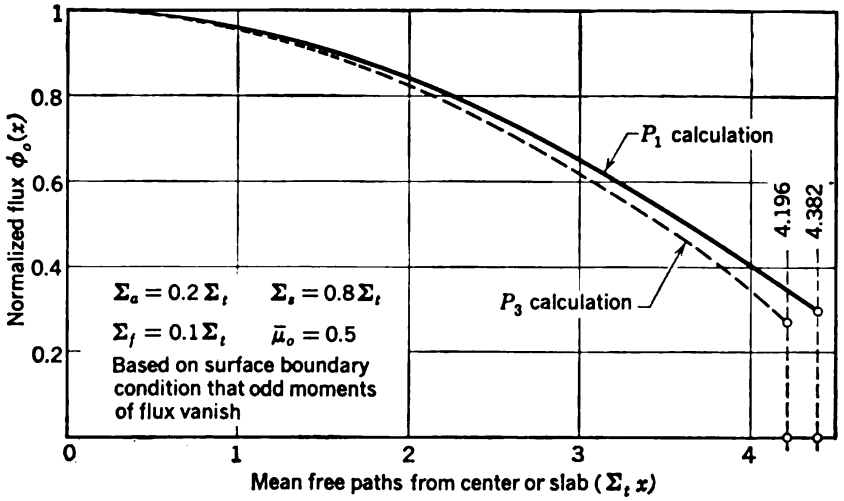

FIG. 7.17 Total flux distribution in an infinite bare-slab reactor by P_1 and P_3 approximations.

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(roughly seven mean free paths). Note, however, that the P_3 calculation agrees more closely with the integral-equation results than does the P_1 calculation. Finally, it should be recognized that, if a lower fuel concentration had been selected, the slab width a would have been larger and all four solutions would be expected to agree more closely.

The total flux distribution in the slab reactor defined by (7.325) is shown in Fig. 7.17 for both the P_1 and the P_3 approximations.

7.6 Computational Methods for the Neutron Age

The purpose of this section is to display three computational methods for the neutron age that are especially useful for hand calculation. These methods are based on various approximate models which involve some rather gross idealizations of the physical system. It is convenient to discuss these at this point since they serve well to demonstrate the application of the Fermi age model and the transport-theory methods developed in the preceding sections.

The particular utility of the first model, which was developed by Fermi and others, is in application to water-metal mixtures. The second model is a rationale for extending these results by similarity techniques, and the third is useful for materials of large nuclear mass. It should be pointed out that each of these methods was designed primarily as a fast hand tool. With the advent of the fast computing machine there is perhaps less interest in these methods; nevertheless, the development of the analytical models is helpful for discussing some of the physical features of each system.

a. Fermi-Marshak Method for Hydrogenous Mixtures. A very instructive and useful application of the general Boltzmann equation (7.110) is the calculation of the neutron age in hydrogenous systems. This problem was first considered by Fermi,[1] and later by Ornstein and Uhlenbeck[2] and by Marshak.[3] The present treatment, which draws principally from the work of Marshak, is based on several simplifying assumptions about the physical situation that render the method especially attractive for computing the neutron age in water-metal systems. Other more accurate models for describing the neutron population in general hydrogenous systems are developed and discussed in considerable detail in Chap. 12. Special attention is given there to provide a complete mathematical treatment along with a comparison of the analytical and physical features of each model. The analysis which follows, on the other hand, is intended primarily as a demonstration of

[1] E. Fermi, *Ricerca sci.*, **7** (2), 13 (1936).

[2] L. S. Ornstein and G. E. Uhlenbeck, *Physica*, **4**, 478 (1937); and R. E. Marshak, MT-17 Series T.I.B. (1943).

[3] R. E. Marshak, *Revs. Mod. Phys.*, **19**, 3 (1947), esp. pp. 200–201 and 209–211.

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the transport-theory methods and to derive a suitable formula for computing the age. For these reasons some of the purely mathematical steps are omitted from the analysis and left for detailed discussion in Chap. 12.

We know from our earlier study of the Fermi age model that the age $\tau(u)$ is proportional to the quantity $\overline{r^2}(u)$, the mean radial distance squared at which fast neutrons from a specified source reach lethargy u; the constant of proportionality in this relation is dependent upon the nature of the source [see, for example, Eq. (6.52)]. Our ultimate objective, then, is to compute $\overline{r^2}(u)$. For this purpose we introduce the following model:

(1) The slowing-down medium is infinite, homogeneous, and isotropic
(2) Neutrons of lethargy $u = 0$ appear in this medium at a constant rate from a single plane source of unit strength located at $x = 0$
(3) The nuclei of all nonhydrogenous materials in this medium are of infinite mass
(4) Elastic scattering is isotropic in the center-of-mass systems of coordinates
(5) The absorption cross section of the medium is small compared to the scattering cross section

(7.332)

The system defined by these assumptions is evidently one-dimensional and at steady state, and in this case the general Boltzmann relation reduces to a somewhat simpler form. The appropriate equation is obtained from (7.116); in terms of the lethargy variable this may be written

$$\mathbf{\Omega} \cdot \nabla\phi(\mathbf{r},u,\mathbf{\Omega}) + \Sigma_t(u)\,\phi(\mathbf{r},u,\mathbf{\Omega}) = S(\mathbf{r},u,\mathbf{\Omega}) + \frac{1}{2\pi}\int_{u'} \Sigma_s(u')\,du' \int_{\mathbf{\Omega}'} \phi(\mathbf{r},u',\mathbf{\Omega}')\,\delta(u,\mu_0;u',\mathbf{\Omega}')\,d\mathbf{\Omega}' \quad (7.333)$$

where we have used the definition (7.114) and the transformation (7.127). In the one-dimensional system the transport term reduces to the derivative as given in (7.78), and $\phi(\mathbf{r},u,\mathbf{\Omega}) \to \phi(x,u,\mathbf{\Omega})$ and $S(\mathbf{r},u,\mathbf{\Omega}) \to S(x,u,\mathbf{\Omega})$. A more convenient pair of functions is defined by the integrals

$$\begin{aligned} \phi(x,u,\mu) &\equiv \int_0^{2\pi} \phi(x,u,\mathbf{\Omega})\,d\psi \\ S(x,u,\mu) &\equiv \int_0^{2\pi} S(x,u,\mathbf{\Omega})\,d\psi \end{aligned} \quad (7.334)$$

after (7.76). For an isotropic plane source producing monoenergetic neutrons at lethargy $u = 0$, the function $S(x,u,\mathbf{\Omega}) = (1/4\pi)\,\delta(x)\,\delta(u)$. Therefore,

$$S(x,u,\mu) = \tfrac{1}{2}\,\delta(x)\,\delta(u) \quad (7.335)$$

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If Eq. (7.333) is integrated over the interval of ψ, and the expressions (7.78), (7.334), and (7.335) are applied, the result is

$$\mu \frac{\partial}{\partial x} \phi(x,u,\mu) + \Sigma_t(u)\, \phi(x,u,\mu) = \tfrac{1}{2}\, \delta(x)\, \delta(u) + \frac{1}{2\pi} \int_{u'} \Sigma_s(u')\, du' \int_{\Omega'} \phi(x,u',\mu')\, \vartheta(u,\mu_0;u',\Omega')\, d\Omega' \quad (7.336)$$

Equation (7.336) gives a complete description of the neutron distribution in the system specified by the model (7.332). If the solution of this equation were obtained, the resulting expression for $\phi(x,u,\mu)$ could be used to compute the quantity $\overline{x^2}(u)$, the mean-square perpendicular distance from the source plane at which neutrons reach lethargy u; this calculation could be carried out by using a technique similar to the one employed in Sec. 6.2d. The immediate problem, then, is to determine the relationship between $\overline{r^2}(u)$ and $\overline{x^2}(u)$.

Let us define a function $f(u;x,y,z)$ such that

$$f(u;x,y,z)\, dx\, dy\, dz \equiv \begin{array}{l}\text{probability that neutron of lethargy}\\ \text{zero released at origin reaches}\\ \text{lethargy } u \text{ in } dx\, dy\, dz \text{ at station } (x,y,z)\end{array} \quad (7.337)$$

Since x, y, and z are independent variables, this function may be written in the form

$$f(u;x,y,z) \equiv f_1(u;x)\, f_2(u;y)\, f_3(u;z) \quad (7.338)$$

where the f_i are the frequency functions for the individual variables. We use this form to compute $\overline{r^2}(u)$; thus,

$$\overline{r^2}(u) \equiv \int r^2 f(u;x,y,z)\, dx\, dy\, dz \quad (7.339)$$

$$\overline{r^2}(u) \equiv \int_{-\infty}^{\infty} f_3(u;z)\, dz \int_{-\infty}^{\infty} f_2(u;y)\, dy \int_{-\infty}^{\infty} f_1(u;x)(x^2+y^2+z^2)\, dx$$

wherein we have used $r^2 \equiv x^2 + y^2 + z^2$. Since

$$\overline{x^2}(u) \equiv \int_{-\infty}^{\infty} x^2 f_1(u;x)\, dx \quad (7.340)$$

it is obvious that

$$\overline{r^2}(u) \equiv \overline{x^2}(u) + \overline{y^2}(u) + \overline{z^2}(u) \equiv 3\overline{x^2}(u) \quad (7.341)$$

Thus the age may be computed from the relation

$$\tau(u) = \tfrac{1}{6}\overline{r^2}(u) = \tfrac{1}{2}\overline{x^2}(u) \quad (7.342)\star$$

The quantity $\overline{x^2}(u)$ in turn is obtained from the definition

$$\overline{x^2}(u) \equiv \frac{\int_{-\infty}^{\infty} x^2\, dx \int_{\Omega} \phi(x,u,\mu)\, d\Omega}{\int_{-\infty}^{\infty} dx \int_{\Omega} \phi(x,u,\mu)\, d\Omega} \quad (7.343)\star$$

which corresponds to the relation (6.50) [see also (6.51)].

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The calculation of the age requires therefore the solution of Eq. (7.336) for the flux $\phi(x,u,\mu)$. In carrying out the calculation (7.343), it is actually more convenient to work with the Fourier transform of the collision density than with the flux proper. This technique is generally useful for computations of this type, and we will have occasion to apply it to other problems later, specifically to the calculation of the age using the Christy-Wheeler kernel and in the analysis of hydrogenous systems (Chap. 12). We develop the appropriate relations required for this technique in the analysis which follows. Since our primary interest here is to demonstrate an application of the Boltzmann equation, we will not examine all the details of the mathematical analysis but will instead concentrate attention on the reactor-physics aspects of the problem. Thus the present analysis is to serve as an outline, and the reader is referred to the report by Marshak and to Chap. 12 for further analytical details.

In accord with the remarks above we replace the flux ϕ in Eq. (7.336) by the collision density F, where

$$F(x,u,\mu) \equiv \Sigma_t(u)\phi(x,u,\mu) \tag{7.344}$$

We use also the mean free path

$$\lambda_t(u) \equiv \frac{1}{\Sigma_t(u)} \simeq \frac{1}{\Sigma_s(u)} \tag{7.345}$$

the approximation being based on assumption (5), Eq. (7.332). The substitution of these definitions into (7.336) yields

$$\lambda_t(u)\mu \frac{\partial}{\partial x} F(x,u,\mu) + F(x,u,\mu) \simeq \frac{1}{2}\,\delta(u)\,\delta(x) + \frac{1}{2\pi}\int_0^u du' \int_{\Omega'} F(x,u',\mu')\,\mathfrak{s}(u,\mu_0;u',\Omega')\,d\Omega' \tag{7.346}$$

We compute next the Fourier transform $\mathfrak{F}\{\ \}$ of this equation. If we call

$$\mathfrak{F}\{F(x,u,\mu)\} \equiv \Phi(\xi,u,\mu) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x,u,\mu)e^{i\xi x}\,dx \tag{7.347}$$

then, the transform of (7.346) is

$$[1 - i\lambda_t(u)\xi\mu]\Phi(\xi,u,\mu) = \frac{\delta(u)}{2\sqrt{2\pi}} + \frac{1}{2\pi}\int_0^u du' \int_{\Omega'} \Phi(\xi,u',\mu')\,\mathfrak{s}(u,\mu_0;u',\Omega')\,d\Omega' \tag{7.348}$$

Our purpose is to obtain an approximate solution to this equation which may be used to compute $\overline{x^2}(u)$. This may be accomplished by reducing the above form to a set of coupled equations through the use of suitable series expansions for the functions Φ and $\mathfrak{s}$. The procedure is similar to

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that followed in Secs. 7.2 and 7.3. We define, therefore, the following expansions in spherical harmonics:

$$\Phi(\xi,u,\mu) \equiv \sum_{m=0}^{\infty} \left(\frac{2m+1}{2}\right) \Phi_m(\xi,u)\, P_m(\mu) \tag{7.349}$$

$$\mathfrak{s}(u,\mu_0;u',\Omega') \equiv \sum_{\alpha=0}^{\infty} \sum_{\beta=-\alpha}^{\alpha} \mathfrak{s}_\alpha(u;u',\Omega')\, Y_\alpha^\beta(\Omega)\, Y_\alpha^{\beta*}(\Omega') \tag{7.350}$$

where (7.350) corresponds to the expansion for $\mathfrak{q}(v,\mu_0;\mathbf{v}')$ given in (7.119). The coefficients Φ_m and $\mathfrak{s}_\alpha$ are obtained in the usual way by applying the orthogonality properties of the spherical harmonics.

The substitution of these expressions into (7.348) yields

$$\begin{aligned}[1 - i\lambda_t(u)\xi\mu] \sum_{m=0}^{\infty} \left(\frac{2m+1}{2}\right) \Phi_m(\xi,u)\, P_m(\mu) &= \frac{\delta(u)}{2\sqrt{2\pi}} \\ &+ \frac{1}{2\pi} \sum_{m=0}^{\infty} \left(\frac{2m+1}{2}\right) \int_0^u \Phi_m(\xi,u') \sum_{\alpha=0}^{\infty} \mathfrak{s}_\alpha(u;u',\Omega')\, du' \\ &\times \sum_{\beta=-\alpha}^{\alpha} Y_\alpha^\beta(\Omega) \int_{\Omega'} Y_\alpha^{\beta*}(\Omega')\, P_m(\mu')\, d\Omega'\end{aligned} \tag{7.351}$$

The integral over Ω' is easily evaluated with the aid of the orthogonality properties of the spherical harmonics, if we note that $P_n(\mu) \equiv Y_n^0(\Omega)/H_n^0$. With this reduction, (7.351) takes the form

$$\begin{aligned}&[1 - i\lambda_t(u)\xi\mu] \sum_{m=0}^{\infty} \left(\frac{2m+1}{2}\right) \Phi_m(\xi,u)\, P_m(\mu) \\ &= \frac{\delta(u)}{2\sqrt{2\pi}} + \frac{1}{2\pi} \sum_{m=0}^{\infty} \left(\frac{2m+1}{2}\right) P_m(\mu) \int_0^u \Phi_m(\xi,u')\, \mathfrak{s}_m(u;u',\Omega')\, du'\end{aligned} \tag{7.352}$$

This relation reduces to an infinite set of coupled equations in Φ_m if we multiply through by various $P_n(\mu)$ and integrate over all μ. Thus, for the case $n = 0$, we obtain

$$\Phi_0(\xi,u) - i\lambda_t(u)\, \xi\Phi_1(\xi,u) = \frac{\delta(u)}{\sqrt{2\pi}} + \frac{1}{2\pi} \int_0^u \Phi_0(\xi,u')\, \mathfrak{s}_0(u;u',\Omega')\, du' \tag{7.353}$$

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and for $n > 0$,

$$\Phi_n(\xi,u) - \frac{i\xi\lambda_t(u)}{2n+1}\left[(n+1)\Phi_{n+1}(\xi,u) + n\Phi_{n-1}(\xi,u)\right] = \frac{1}{2\pi}\int_0^u \Phi_n(\xi,u')\,\mathfrak{s}_n(u;u',\Omega')\,du' \quad (7.354)$$

The systems of interest to the present application are hydrogenous mixtures of various substances. It is convenient from the standpoint of the neutron physics to separate such mixtures into hydrogen nuclei and all other nuclei. If we denote by the index H the quantities that refer to the characteristics of the hydrogen nuclei and by the index O the properties of all other materials, then the scattering function $\mathfrak{s}$ may be written

$$\mathfrak{s} \equiv \frac{\Sigma_s^{(H)}\mathfrak{s}^{(H)}}{\Sigma_s} + \frac{\Sigma_s^{(O)}\mathfrak{s}^{(O)}}{\Sigma_s} \qquad \text{with } \Sigma_s \equiv \Sigma_s^{(H)} + \Sigma_s^{(O)} \quad (7.355)$$

Let $$\Lambda \equiv \frac{\Sigma_s^{(H)}}{\Sigma_s}$$

then $$1 - \Lambda \equiv \frac{\Sigma_s^{(O)}}{\Sigma_s} \quad (7.356)$$

and $$\mathfrak{s} = \Lambda\mathfrak{s}^{(H)} + (1-\Lambda)\mathfrak{s}^{(O)} \quad (7.357)$$

But, we know from our previous study of the Boltzmann equation that for isotropic scattering in the (C) system [see Eqs. (7.131), (7.135), and (7.138)]

$$\mathfrak{s}(u,\mu_0;u',\Omega') = \frac{\delta(\mu_0 - \zeta)e^{u'-u}}{1-\alpha} \quad (7.358)$$

In the case of hydrogen, this yields

$$\mathfrak{s}^{(H)} = \delta(\mu_0 - \zeta_H)e^{u'-u} \quad (7.359)$$

where from (7.134)

$$\zeta_H = e^{\frac{1}{2}(u'-u)} \quad (7.360)$$

By assumption (3) of Eq. (7.332) all other materials have infinite mass; therefore the scattering from these nuclei is isotropic in (L), and the change in lethargy is taken to be zero. It follows that the appropriate scattering function must have the form

$$\mathfrak{s}^{(O)} = \tfrac{1}{2}\,\delta(u' - u) \quad (7.361)$$

Note that the integral of this function over all u and μ_0 is unity. If this function and (7.359) are used in $\mathfrak{s}$, the scattering function for a hydrogenous mixture may be written in the form

$$\mathfrak{s}(u,\mu_0;u',\Omega') = \Lambda(u')\,\delta(\mu_0 - \zeta_H)e^{u'-u} + \tfrac{1}{2}[1 - \Lambda(u')]\,\delta(u' - u) \quad (7.362)$$

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The coefficients $\mathfrak{s}_n$ may be computed from the integral relation (7.130). For example,

$$\begin{aligned} \mathfrak{s}_0 &= \Lambda(u')e^{u'-u} + [1 - \Lambda(u')]\,\delta(u' - u) \\ \mathfrak{s}_1 &= \tfrac{1}{2}\Lambda(u')e^{\frac{1}{2}(u'-u)} \end{aligned} \tag{7.363}$$

These functions may now be substituted into Eqs. (7.353) and (7.354) and the various harmonics Φ_n obtained from a simultaneous solution of a finite set of these relations which yields an acceptable approximation to the transform function Φ.

The relationship of the function Φ to the mean-square distance $\overline{x^2}(u)$ is now demonstrated. To begin with we note that the formula (7.343) may be written in terms of the collision density F; thus

$$\overline{x^2}(u) = \frac{\int_{-\infty}^{\infty} x^2\,dx \int_{\Omega} F(x,u,\mu)\,d\Omega}{\int_{-\infty}^{\infty} dx \int_{\Omega} F(x,u,\mu)\,d\Omega} \tag{7.364}$$

The function F, however, is the inverse Fourier transform of Φ; i.e.,

$$F(x,u,\mu) = \mathfrak{F}^{-1}\{\Phi(\xi,u,\mu)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\xi,u,\mu)\,e^{-i\xi x}\,d\xi \tag{7.365}$$

Moreover, Φ is given by the expansion (7.349); it follows, then, that the integral in the denominator of (7.364) reduces to

$$\begin{aligned} \int_{\Omega} F(x,u,\mu)\,d\Omega &= \frac{1}{\sqrt{2\pi}} \int_{\Omega} d\Omega \int_{-\infty}^{\infty} \sum_{m=0}^{\infty} \left(\frac{2m+1}{2}\right) \Phi_m(\xi,u)\,P_m(\mu)e^{-i\xi x}\,d\xi \\ &= \frac{1}{2\sqrt{2\pi}\,H_0^0} \int_{-\infty}^{\infty} \Phi_0(\xi,u)e^{-i\xi x}\,d\xi \end{aligned} \tag{7.366}$$

wherein we have applied the definition (7.28). The integral which appears in the last equation is the first of a set of functions called the *moments* of Φ_0. The general definition for the mth moment is

$$F_0^{(m)}(x,u) \equiv \frac{(-1)^m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \xi^m \Phi_0(\xi,u)e^{-i\xi x}\,d\xi \tag{7.367}$$

Note that

$$F_0^{(0)}(x,u) \equiv F_0(x,u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi_0(\xi,u)e^{-i\xi x}\,d\xi \tag{7.368}$$

is the Fourier inversion of the coefficient $\Phi_0(\xi,u)$. Evidently,

$$\int_{\Omega} F(x,u,\mu)\,d\Omega = \frac{1}{2H_0^0} F_0(x,u) \tag{7.369}$$

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In an analogous fashion we introduce also

$$\Phi_0^{(m)}(\xi,u) = \frac{i^m}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^m F_0(x,u) e^{i\xi x}\, dx \tag{7.370}$$

and observe that

$$\Phi_0^{(0)}(\xi,u) \equiv \Phi_0(\xi,u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_0(x,u) e^{i\xi x}\, dx \tag{7.371}$$

which is the Fourier transform of $F_0(x,u)$. If these relations are used in the integrals of (7.364), we obtain the results

$$\begin{aligned} \int_{-\infty}^{\infty} dx \int_{\Omega} F(x,u,\mu)\, d\Omega &= \frac{1}{2H_0^0} \int_{-\infty}^{\infty} F_0(x,u)\, dx \\ &= \frac{\sqrt{2\pi}}{2H_0^0}\, \Phi_0(0,u) \end{aligned} \tag{7.372}$$

$$\begin{aligned} \int_{-\infty}^{\infty} x^2\, dx \int_{\Omega} F(x,u,\mu)\, d\Omega &= \frac{1}{2H_0^0} \int_{-\infty}^{\infty} x^2 F_0(x,u)\, dx \\ &= -\frac{\sqrt{2\pi}}{2H_0^0}\, \Phi_0^{(2)}(0,u) \\ &= -\frac{\sqrt{2\pi}}{2H_0^0} \left[\frac{\partial^2}{\partial \xi^2} \Phi_0(\xi,u)\right]_{\xi=0} \end{aligned} \tag{7.373}$$

The expression (7.364) may then be written

$$\overline{x^2}(u) = 2\tau(u) = -\left[\frac{(\partial^2/\partial\xi^2)\Phi_0(\xi,u)}{\Phi_0(\xi,u)}\right]_{\xi=0} \tag{7.374}\star$$

Thus we find that the calculation of the age requires a knowledge of the zeroth-order harmonic of the Fourier transform of the collision density. This function $\Phi_0(\xi,u)$ may be obtained from the solution of Eqs. (7.353) and (7.354). Since only Φ_0 and Φ_1 are involved, one needs to solve only the relations

$$\begin{aligned} \Phi_0(\xi,u) - i\lambda_t(u)\, \xi\, \Phi_1(\xi,u) &= \frac{\delta(u)}{\sqrt{2\pi}} + \frac{1}{2\pi} \int_0^u \Phi_0(\xi,u')\, \mathfrak{s}_0(u;u',\Omega')\, du' \\ \Phi_1(\xi,u) - \frac{1}{3}\, i\xi\lambda_t(u)\, \Phi_0(\xi,u) &= \frac{1}{2\pi} \int_0^u \Phi_1(\xi,u')\, \mathfrak{s}_1(u;u',\Omega')\, du' \end{aligned} \tag{7.375}$$

and substitute the resulting expression for Φ_0 into (7.374).

The remainder of this analysis is primarily of a mathematical nature and of little interest to the present purpose. The details of this calculation have been carried out by Marshak and are presented in the reference report. In this calculation Marshak uses the coefficients for the scattering function given in (7.363) and obtains an approximate solution for Φ_0 from Eqs. (7.375). This result, when applied to (7.374), yields for the age of a neutron of lethargy u, which first appears in the system

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at lethargy $a < u$, the following expression:

$$\tau(u;a) = \frac{1}{3}\left[\frac{\lambda_t^2(a)}{\Lambda(a)} + \frac{\lambda_t^2(u)}{\Lambda(u)} + I_1(u)\, I_2(a) + I_1(u)\, I_3(u;a) + I_5(u;a) + I_2(a) I_4(u;a) + I_6(u;a) - I_4(u;a) I_7(a)\right] \quad (7.376)\star$$

This equation applies when $u \neq a$. The various functions in (7.376) are defined:

$$\begin{aligned} I_1(a) &\equiv \lambda_t(a) e^{-h(a)} \\ I_2(a) &\equiv \lambda_t(a) e^{h(a)} \\ I_3(u;a) &\equiv \int_a^u I_2(w)\, dw \\ I_4(u;a) &\equiv \int_a^u I_1(w)\, dw \\ I_5(u;a) &\equiv \int_a^u \frac{\lambda_t^2(w)\, dw}{\Lambda(w)} \\ I_6(u;a) &\equiv \int_a^u I_1(w)\, dw \int_0^w I_2(s)\, ds \\ I_7(a) &\equiv \int_0^a I_2(w)\, dw \end{aligned} \quad (7.377)$$

and

$$h(a) \equiv \frac{a}{2} + \int_0^a [1 - \Lambda(w)]\, dw \quad (7.378)$$

Age calculations[1,2] for fission neutrons in various metal-water mixtures of interest to reactor design have been carried out using the Fermi-Marshak equation (7.376). These results are given in Fig. 7.18 for aluminum, stainless steel, and zirconium as a function of the metal-to-water volume ratio.

b. Equivalent Mean-free-path Method for Water-Metal Mixtures. An essential feature of the model used in the Fermi-Marshak calculation of the age in hydrogenous systems was the assumption that all nonhydrogenous materials in the mixture have infinite mass. Thus this model attributes the entire mechanism of neutron slowing down to the scattering collisions with hydrogen nuclei. The heavier materials in the mixture serve primarily to "spread out" the neutron population, and the amount of this spreading-out effect at any particular energy is determined by the relative magnitude of the macroscopic scattering cross section of these materials and of the hydrogen at that energy. Since the macroscopic scattering cross section of the hydrogen in most metal-water systems of interest to reactor design is appreciably larger than the cross section of the

[1] L. W. Nordheim, G. Nordheim, and H. Soodak, "Slowing Down of Neutrons in H_2O and in Graphite," Metallurgical Project CP-1251 (A-1827), Jan. 20, 1944.

[2] Nancy M. Dismuke and M. Ruth Arnette, "Age to Thermal Energy (0.025 ev) of Fission Neutrons in H_2O-Al Mixtures," Mon P-219, Dec. 5, 1956.

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nonhydrogenous materials, $\Sigma_s^{(O)}$, over the entire slowing-down range, the *details* of the energy structure of $\Sigma_s^{(O)}$ affect little the age of the mixture. And, in first approximation, the spreading-out effect of the nonhydrogenous materials may be described by an appropriate energy–average

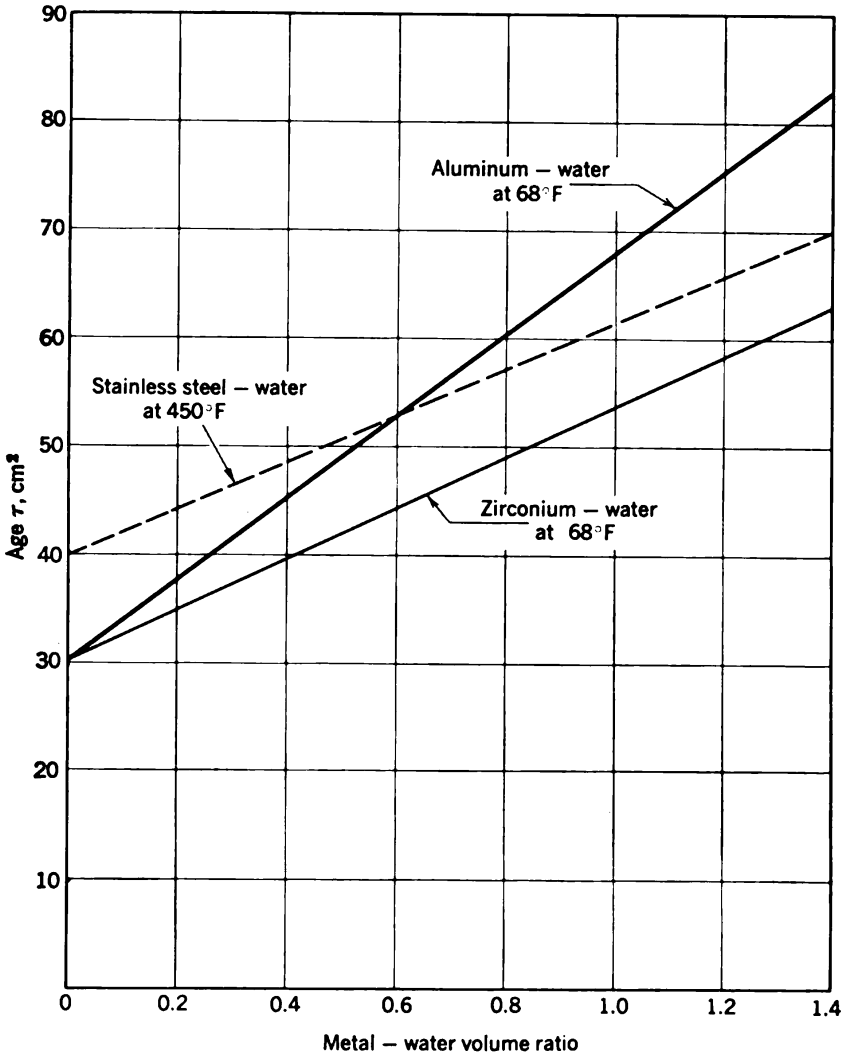

FIG. 7.18 Age of fission neutrons in water mixtures by Fermi-Marshak method.

scattering cross section. Thus the principal factors which determine the age in metal-water mixtures are the volume fraction of water and the average scattering cross section of the metal.

To illustrate this point further, consider the water mixtures of two different metals *a* and *b*. Let the average scattering cross section of metal *a* be greater than that of metal *b*. If the volume fraction of water is the same for each system, then we would expect that the age in system *a* would be greater than the age in system *b*. However, if the two metals are not too dissimilar [i.e., if $\Sigma_s^{(a)}$ is not much different from

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$\Sigma_s^{(b)}$], it is clear that there exists some volume fraction of water $v_w^{(a)}$ for system a, where $v_w^{(a)} > v_w^{(b)}$, such that the ages in the two systems are identical. If one could somehow determine this value of $v_w^{(a)}$, then a knowledge of the age in system b could be used to estimate the age in system a.

A clue as to how two different metal-water mixtures might be compared is obtained from the Fermi-Marshak expression for the age. We observe that this result is completely specified by the mean free path in the mixture. Thus, if as argued previously, the details of the energy structure of the metal cross section are not a primary factor in the determination of the age, then any two systems with the same mixture mean free path would exhibit the same age. One might argue, then, that the comparison could be based on a representative value of λ_t.

Let us defer for the present the question of how one is to determine an appropriate value of λ_t and inquire first into the details of the comparison. If $\lambda_t^{(i)}$ denotes the mean free path in a water mixture of metal i, then by the usual relations

$$\lambda_t^{(i)} \simeq \left[\frac{v_i \rho_i \sigma_s^{(i)} \mathcal{A}}{A_i} + \frac{v_w \rho_w^{(i)} \sigma_s^{w(i)} \mathcal{A}}{A_w} \right]^{-1} \tag{7.379}$$

where for the present systems $\lambda_t \simeq \lambda_s$, and

$\rho_i \equiv$ density‡ of metal i
$\rho_w^{(i)} \equiv$ density‡ of water in system containing metal i
$v_i \equiv$ volume fraction of metal i
$v_w^{(i)} \equiv$ volume fraction of water in system containing metal i

The other symbols in the equation have the usual definitions. The volume fractions are related by (3.26); thus,

$$v_i + v_w^{(i)} = 1 \tag{7.380}$$

Now consider water mixtures of two different metals a and b. If the age for a given slowing-down range is to be the same in the two systems, then by our arguments above we require that

$$\lambda_t^{(a)} = \lambda_t^{(b)} \tag{7.381}$$

For the $\lambda_t^{(i)}$ we use expression (7.379), and introducing also the relation (7.380) for the $v_w^{(i)}$, (7.381) yields the condition

$$\frac{v_a \rho_a \sigma_s^{(a)}}{A_a} + \frac{\rho_w^{(a)} \sigma_s^{w(a)}}{A_w} (1 - v_a) = \frac{v_b \rho_b \sigma_s^{(b)}}{A_b} + \frac{\rho_w^{(b)} \sigma_s^{w(b)}}{A_w} (1 - v_b) \tag{7.382}$$

Suppose that the age has been determined accurately by some method for system a and that these data are available as a function of the volume fractions of metal and water as shown in Fig. 7.18. These results may

‡ These densities refer to the pure material at the specified temperature.

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then be used to estimate the age in system b, since given a volume fraction v_b of metal for system b, we can compute from (7.382) the volume fraction v_a for system a which would have roughly the same age. Using this value of v_a we could read off immediately from the age-versus-volume fraction curve for system a the age for system b which contains a volume fraction v_b of metal.

The expression for v_a in terms of v_b obtained from (7.382) may be written in the form

$$v_a = \frac{v_b(\theta_b - \chi) + (\chi - 1)}{\theta_a - 1} \tag{7.383}$$

where we have defined

$$\chi \equiv \frac{\rho_w^{(b)}\sigma_s^{w(b)}}{\rho_w^{(a)}\sigma_s^{w(a)}} \qquad \theta_i \equiv \left[\frac{\rho_i\sigma_s^{(i)}}{A_i}\right]\left[\frac{A_w}{\rho_w^{(a)}\sigma_s^{w(a)}}\right] \tag{7.384}$$

It is sometimes more convenient to work with the metal-to-water volume fraction V_i, where

$$V_i \equiv \frac{v_i}{v_w^{(i)}} \qquad \text{and} \qquad v_i \equiv \frac{V_i}{1 + V_i} \tag{7.385}$$

Equation (7.383) may be written in terms of this ratio; thus,

$$V_a = \frac{V_b(\theta_b - 1) + (\chi - 1)}{V_b(\theta_a - \theta_b) + (\theta_a - \chi)} \tag{7.386}$$

This relation, or (7.383), may be used to relate one metal-water system to another. We observe that the comparison is based on some suitable value of the microscopic scattering cross sections of the water and metal in each system and on the densities of the various constituents. Evidently this technique may be used to extrapolate data from one metal-water system to another as well as from one temperature to another.

The selection of representative scattering cross sections to be used in (7.384) may be based on flux-averaged values. If we assume as a rough approximation that the flux is $1/E$, then we may compute the σ_s from the relations

$$\sigma_s \equiv \frac{\displaystyle\int_{E_1}^{E_0} \sigma_s(E)\,\frac{dE}{E}}{\displaystyle\int_{E_1}^{E_0} \frac{dE}{E}} = \frac{1}{u_1 - u_0}\int_{u_0}^{u_1} \sigma_s(u)\,du \tag{7.387}\star$$

where $E_1 < E_0$ and $(E_0 \to E_1)$ is the slowing-down range of interest.

The equivalent mean-free-path technique has been applied to several systems of engineering interest, and the agreement with more accurate methods is surprisingly good. Table 7.2 lists some values for the age to thermal of *fission* neutrons in stainless steel–water mixtures at 450°F for various metal-water ratios. Column 2 gives the age as computed

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from the Fermi-Marshak relation, column 4 the age as estimated by the present method using Al-H_2O data, and column 6 the age estimated from Zr-H_2O data. Columns 3 and 5 list the metal-water ratios for the aluminum and zirconium mixtures which correspond to the stainless-water ratios

TABLE 7.2 AGE OF FISSION NEUTRONS IN STAINLESS STEEL–WATER MIXTURES AT 450°F

Volume ratio of stainless to water	τ, cm² Fermi-Marshak	From Al-H_2O data		From Zr-H_2O data	
		Al-H_2O ratio	τ	Zr-H_2O ratio	τ
(1)	(2)	(3)	(4)	(5)	(6)
0	40.0	0.332	41.2	0.472	41.5
0.3	46.5	0.500	49.0	0.779	48.5
0.6	53.0	0.652	54.5	1.038	54.5
1.0	61.5	0.799	60.0	1.346	62.0

listed in the first column. The scattering cross sections for these calculations were computed from (7.387). These are listed in Table 7.3, along with the appropriate densities:

TABLE 7.3

SS-H_2O, 450°F	Al-H_2O, 68°F	Zr-H_2O, 68°F
$\rho_w^{(ss)} = 0.834$ g/cm³	$\rho_w^{(Al)} = 1.0$	$\rho_w^{(Zr)} = 1.0$
$\rho_{ss} = 7.75$	$\rho_{Al} = 2.7$	$\rho_{Zr} = 6.43$
$\sigma_s^{w(ss)} = 39.3$ barns	$\sigma_s^{w(Al)} = 42.7$	$\sigma_s^{w(Zr)} = 42.7$
$\sigma_s^{(ss)} = 6.8$	$\sigma_s^{(Al)} = 1.5$	$\sigma_s^{(Zr)} = 9.0$

c. Age Calculations by Christy-Wheeler Kernel. The calculation of the age on the basis of the continuous slowing-down model has two serious shortcomings. The first arises in connection with the scattering properties of materials at high energies, and the second because of the use of the slowing-down density in place of the flux in the calculation of $\overline{r^2}$.

The first of these considerations is important when the scattering cross section is small at high energies and/or exhibits forward direction tendency [see, for example, Eq. (4.31) and related discussion]. In both instances the neutron will travel large distances from its point of origin before the scatterings become sufficiently frequent and isotropic to justify the use of the continuous slowing-down model. Usually after the first collision the neutron energy is markedly lower, and these considerations become less important. Thus it is possible to improve the continuous slowing-down model by computing from the actual source distribution the distribution of first scattering collisions and using these

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as the starting points for the slowing-down process. This approach is another extension of the first-collision source model introduced previously to illustrate the kernel method (Sec. 5.7).

The second correction is an attempt to improve on the use of the slowing-down density in the calculation of $\overline{r^2}$ by introducing a more accurate statement in terms of the neutron flux. If the usual coupling relation [for example, Eq. (4.93)] were exact, then the use of either the flux or the slowing-down density would yield the same result; since this is not the case, a correction is required. The distinction becomes important when determining the end point of the neutron trajectory involved in the $\overline{r^2}(u)$ calculation. If this is to be represented by an absorption reaction at the specified lethargy, which is the phenomenon observed in making age measurements with the use of absorber foils, then the neutron flux $\phi(\mathbf{r},u)$ is required. Since in general a neutron reaches lethargy u at some point $\mathbf{r}'$ different from $\mathbf{r}$, a relation is required between $\phi(\mathbf{r},u)$ and $q(\mathbf{r}',u)$ which accounts for local spatial variations in the neutron distribution.

As in our earlier work, we compute the age from the value of $\overline{r^2}(u)$ for a point source of fast neutrons in an infinite medium. For this purpose we select a point source of unit strength at the origin of a suitable coordinate system in the medium. The calculation of $\overline{r^2}(u)$ is based on the continuous slowing-down model which has been modified to include the first and last flight corrections proposed above. Consider, therefore, a source for the continuous slowing-down process which consists of the first-scattering centers of neutrons emitted from a point source at the origin. If the scattering cross section of zero lethargy (source) neutrons is $\Sigma_s(0)$, then the number of first-scattering collisions per unit time in $d\mathbf{r}_0$ at $\mathbf{r}_0$ is given by

$$S(\mathbf{r}_0)\,d\mathbf{r}_0 \equiv \frac{\Sigma_s(0)e^{-\Sigma_s(0)r_0}\,d\mathbf{r}_0}{4\pi r_0^2} \tag{7.388}$$

These are the sources for the slowing-down process. For each neutron scattered at $\mathbf{r}_0$,

$$G(|\mathbf{r}' - \mathbf{r}_0|)\,d\mathbf{r}' \equiv \frac{p(u)e^{-|\mathbf{r}'-\mathbf{r}_0|^2/4\tau(u)}\,d\mathbf{r}'}{[4\pi\tau(u)]^{\frac{3}{2}}} \tag{7.389}$$

slow past lethargy u per unit time in volume $d\mathbf{r}'$ around $\mathbf{r}'$. The first scattering sources then give rise to a slowing-down density at $\mathbf{r}'$:

$$q(\mathbf{r}',u) = \int_\infty G(|\mathbf{r}' - \mathbf{r}_0|)S(\mathbf{r}_0)\,d\mathbf{r}_0$$

In order to determine the flux $\phi(\mathbf{r},u)$ we must first convert the expression for $q(\mathbf{r}',u)$ into a source strength. This is accomplished by multiplying q by du/ξ, where ξ is the average gain in lethargy per collision; this product gives the number of neutrons actually occupying the interval du. To obtain the flux at $\mathbf{r}$ we integrate this source with the transport kernel;

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thus,

$$\phi(\mathbf{r},u)\,du = \int_\infty \left[q(\mathbf{r}',u)\,\frac{du}{\xi}\right] T(|\mathbf{r}-\mathbf{r}'|)\,d\mathbf{r}'$$

where

$$T(|\mathbf{r}-\mathbf{r}'|) \equiv \frac{e^{-\Sigma_s(u)|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|^2} \tag{7.390}$$

Note that the above expression for ϕ also follows immediately from Eq. (7.242) for the case of a homogeneous, nonmultiplying medium with no direct external sources at lethargy u. It is of further interest to note that this relation for ϕ reduces to the usual coupling equation when q is spatially independent [cf. Eq. (4.93)]. For the present case

$$\phi(\mathbf{r},u) = \frac{1}{\xi}\int_\infty T(|\mathbf{r}-\mathbf{r}'|)\int_\infty G(|\mathbf{r}'-\mathbf{r}_0|)S(\mathbf{r}_0)\,d\mathbf{r}_0 \tag{7.391}$$

This expression has been referred to as the *Christy-Wheeler* kernel.

It will be more convenient in this calculation to work with the neutron flux $\phi(x,u)$ from a plane source than with the function (7.391). From the function $\phi(x,u)$ we can compute $\overline{x^2}(u)$ and then, with the aid of (7.341), obtain $\overline{r^2}(u)$. An expression for $\phi(x,u)$, analogous to (7.391), is

$$\phi(x,u) = \frac{1}{\xi}\int_{-\infty}^{\infty} T(|x-x'|)\,dx' \int_{-\infty}^{\infty} G(|x'-x_0)S(x_0)\,dx_0 \tag{7.392}$$

It should be recognized that in this relation the function G is the plane Gaussian kernel [see, for example, the form (6.28)], and T and S are the one-velocity plane transport kernels. The explicit forms of these functions are not required for this calculation; if desired for other purposes, these may be obtained from the integral relation (7.191) by introducing a suitable delta-function representation[1] of the source [see Sec. 5.2d].

The quantity $\overline{x^2}(u)$ may be obtained, formally, by substituting the expression for $\phi(x,u)$ from (7.392) into the relation

$$\overline{x^2}(u) \equiv \frac{\int_{-\infty}^{\infty} x^2\phi(x,u)\,dx}{\int_{-\infty}^{\infty}\phi(x,u)\,dx} \tag{7.393}$$

As in the case of the Fermi-Marshak model, it will be more convenient to evaluate the above expression by introducing the Fourier transform of the neutron flux. Therefore, let

$$\mathfrak{F}\{\phi(x,u)\} \equiv \Theta(\xi,u) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi(x,u)e^{i\xi x}\,dx \tag{7.394}$$

[1] These kernels have been computed in the Case-de Hoffmann-Placzek paper. See p. 4 in the reference report.

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Now, the nth moment of $\Theta(\xi,u)$ is given by [cf. Eqs. (7.367) and (7.370)]

$$\Theta^{(n)}(\xi,u) \equiv \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^n \phi(x,u) e^{i\xi x}\, dx \tag{7.395}$$

It follows, then, that

$$\begin{aligned} \int_{-\infty}^{\infty} \phi(x,u)\, dx &= \sqrt{2\pi}\, \Theta^{(0)}(0,u) \\ \int_{-\infty}^{\infty} x^2 \phi(x,u)\, dx &= -\sqrt{2\pi}\, \Theta^{(2)}(0,u) \end{aligned} \tag{7.396}$$

Thus (7.393) may be written [cf. Eq. (7.374)]

$$\overline{x^2}(u) = -\left[\frac{\dfrac{\partial^2}{\partial \xi^2}\Theta(\xi,u)}{\Theta(\xi,u)}\right]_{\xi=0} \tag{7.397}$$

The calculation of $\overline{x^2}(u)$ requires, therefore, the evaluation of $\Theta(\xi,u)$. This may be obtained from (7.392). If $g(\xi)$, $s(\xi)$, and $\theta(\xi)$ denote the Fourier transforms of the functions G, S, and T, respectively, then, since (7.392) is the convolution[1] of these functions, i.e.,

$$T \star G \star S \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} T(|x - x'|)\, dx' \int_{-\infty}^{\infty} G(|x' - x_0|)\, S(x_0)\, dx_0 \tag{7.398}$$

it follows that

$$\mathfrak{F}\{T \star G \star S\} = \theta(\xi)\, g(\xi)\, s(\xi) \tag{7.399}$$

and

$$\mathfrak{F}\{\phi(x,u)\} = \Theta(\xi,u) = \frac{2\pi}{\xi}\, \theta(\xi)\, g(\xi)\, s(\xi) \tag{7.400}$$

The numerator of (7.397) may be computed from this expression. For this purpose we require

$$\frac{\xi}{2\pi} \frac{\partial^2}{\partial \xi^2} \Theta(\xi,u) = \theta'' g s + g'' s \theta + s'' g \theta + 2(s' g' \theta + s g' \theta' + s' g \theta') \tag{7.401}$$

where the primes denote differentiation with respect to ξ. The derivatives indicated here may be expressed as the various moments of the transforms of the functions T, G, and S, as defined in Eq. (7.395). If we note that all the odd moments of these transforms vanish at $\xi = 0$, it is seen that

$$\frac{\xi}{2\pi}\left[\frac{\partial^2}{\partial \xi^2}\Theta(\xi,u)\right]_{\xi=0} = \theta''(0)\, g(0)\, s(0) + g''(0)\, s(0)\, \theta(0) + s''(0)\, g(0)\, \theta(0)$$

The application of this result and (7.400) to (7.397) yields

$$\overline{x^2}(u) = -\frac{g''(0)}{g(0)} - \frac{s''(0)}{s(0)} - \frac{\theta''(0)}{\theta(0)} \tag{7.402}$$

[1] See, for example, I. N. Sneddon, "Fourier Transforms," pp. 23–27, McGraw-Hill Book Company, Inc., New York, 1951.

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The terms at the right are the $\overline{x^2}(u)$ values for the kernels G, S, and T, respectively; for example,

$$-\frac{g''(0)}{g(0)} \equiv \overline{x_G^2}(u)$$

Thus
$$\overline{x^2}(u) = \overline{x_G^2}(u) + \overline{x_S^2}(u) + \overline{x_T^2}(u) \tag{7.403}$$

But, $\overline{r^2}(u) = 3\overline{x^2}(u)$; therefore,

$$\overline{r^2}(u) = \overline{r_G^2}(u) + \overline{r_S^2}(u) + \overline{r_T^2}(u) \tag{7.404}$$

The first term at the right is immediately recognized as the $\overline{r^2}(u)$ computed for a point source on the basis of the Fermi age model [see Eqs. (6.49) through (6.52)], since[1]

$$\overline{r_G^2}(u) \equiv \frac{\int r^2 G\, d\mathbf{r}}{\int G\, d\mathbf{r}} = 6\tau_G(u) \tag{7.405}$$

The last two terms are easily computed; thus

$$\begin{aligned} \overline{r_S^2}(u) &= \frac{\int r^2 S\, d\mathbf{r}}{\int S\, d\mathbf{r}} = \frac{\int_0^\infty r^2 e^{-\Sigma_s(0)r}\, dr}{\int_0^\infty e^{-\Sigma_s(0)r}\, dr} = \frac{2}{[\Sigma_s(0)]^2} \\ \overline{r_T^2}(u) &= \frac{2}{[\Sigma_s(u)]^2} \end{aligned} \tag{7.406}$$

The substitution of these results into (7.404) yields an expression for $\overline{r^2}(u)$. If we define the age $\tau(u)$ for this system by the relation $\tau(u) = \frac{1}{6}\overline{r^2}(u)$, then we obtain

$$\tau(u) = \tau_G(u) + \frac{1}{3[\Sigma_s(0)]^2} + \frac{1}{3[\Sigma_s(u)]^2} \tag{7.407}\star$$

This expression for the age offers some improvement over the expression

TABLE 7.4 AGE OF FISSION NEUTRONS IN BERYLLIUM AND GRAPHITE

Material	Fermi age	Christy-Wheeler	Experiment[a]
Beryllium.....	79.7 cm²	84.0	97.2
Graphite.....	352	364	364

[a] U.S. Atomic Energy Commission, "Reactor Handbook: Physics," p. 485, McGraw-Hill Book Company, Inc., New York, 1955.

$\tau_G(u)$ obtained from the Fermi model and should yield reasonably good results for nonhydrogenous materials. Some sample results are given in Table 7.4.

[1] The symbol $\tau_G(u)$ denotes the age based on the Fermi continuous slowing-down model.

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d. Age of Fission Neutrons. Inasmuch as the primary source of neutrons in a reactor is the fission reactions, it is of particular interest to develop a suitable expression for the *average* age of neutrons which appear according to a specified spectrum $\chi(u)$. Various methods have been developed for computing the age of neutrons slowing through a particular lethargy interval, and three of these have been discussed in this section. The method selected for a specific application is determined, of course, by the material composition of the system. For our present purpose we require only that there be an acceptable method for computing the age $\tau(u;u')$ for the interval (u,u'). These results may then be used to compute the average age $\bar{\tau}(u)$ corresponding to the spectrum $\chi(u)$.

TABLE 7.5 HISTOGRAM OF FISSION SPECTRUM
(u = 0 corresponds to 10 Mev)

Δu_n	χ_n^*	Δu_n	χ_n^*
0 → 0.5	0.021	3.5 → 4	0.035
0.5 → 1	0.104	4 → 4.5	0.0151
1 → 1.5	0.207	4.5 → 5	0.0093
1.5 → 2	0.231	5 → 5.5	0.0056
2 → 2.5	0.183	5.5 → 6	0.0034
2.5 → 3	0.117	6 → 6.5	0.0014
3 → 3.5	0.067	6.5 → 7	0.0002

Consider, therefore, a normalized function $\chi(u)$ which describes the distribution of neutrons appearing from fission sources. If $\tau(u;u')$ is the age of neutrons of lethargy u which first appeared from χ at lethargy u', then we define the average age to u for the spectrum from 0 to u by the relation

$$\bar{\tau} \equiv \frac{\int_0^u \tau(u;u')\,\chi(u')\,du'}{\int_0^u \chi(u')\,du'} \tag{7.408}$$

If the integrals are to be performed numerically, this expression may be approximated by

$$\bar{\tau} \simeq \frac{\sum_{n=1}^{N} \tau(u;u_n)\,\chi_n\,\Delta u_n}{\sum_{n=1}^{N} \chi_n\,\Delta u_n} \tag{7.409}$$

where the quantities u_n may be defined by the relation

$$u_n \equiv \sum_{m=1}^{n-1} \Delta u_m + \frac{\Delta u_n}{2} \qquad n > 1 \qquad u_1 \equiv \frac{\Delta u_1}{2}$$

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In many instances it will be convenient to work with the spectrum $\chi(u)$ in the form of a histogram. One that has been used frequently is given in Table 7.5 (see also Fig. 7.19). It should be recognized that the quantity χ_n^* listed in this table is the *fraction* of neutrons which appear

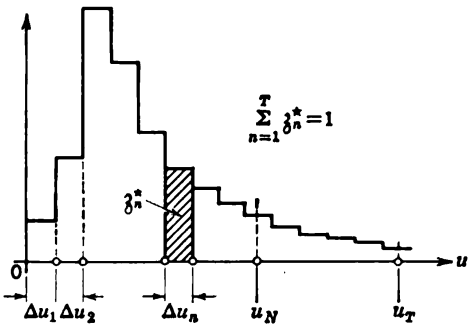

FIG. 7.19 Histogram of fission spectrum.

from the fission spectrum in the indicated lethargy interval. Thus the χ_n^* and χ_n are related by the equation

$$\chi_n^* \equiv \chi_n \, \Delta u_n$$

In this notation the average age to thermal of fission neutrons is given by

$$\bar{\tau}(u_{th}) = \sum_{n=1}^{T} \tau(u;u_n)\chi_n^* \tag{7.410}$$

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CHAPTER 8

REFLECTED REACTORS

8.1 Introduction

a. Reflector Savings. The bare reactors discussed in the preceding chapters can easily be improved from the viewpoint of neutron economy by the addition of a surrounding nonmultiplying medium which serves to reflect back some of the neutrons escaping from the core and thereby allows them further opportunity to cause fissions. The combination of materials to be used in a reflector depends, of course, upon the usual requirements of materials compatibility, economy, strength, shielding, etc. On the basis of neutron conservation alone, there are two major benefits to be obtained from the addition of reflectors:

(1) Neutron escapes from the core are reduced
(2) Neutrons are moderated in the reflector (8.1)

In the models which have been used for a bare reactor, the vacuum has been treated as a perfect absorber for neutrons; thus, the addition of any material whatsoever, no matter how thin or how highly absorbing, to the outer surface of the core will be of some advantage, since it will always return some fraction (albeit small) of all neutrons incident upon it from the core. In our work with the diffusion-theory model we found it useful to describe the reflective properties of a medium by means of the albedo. This concept is especially appropriate in the consideration of multiregion reactor systems. Since the albedo is the fraction of neutrons incident on the surface of a medium which are reflected back, its value depends upon the thickness of the medium as well as upon its absorption cross section. It is clear that, although a thin reflector will always return some neutrons to the core, it will also allow a certain fraction to diffuse through and thereby be lost to the system. As the thickness is increased, the fraction of neutrons lost by escape through the outer surface of the reflector is decreased. It is neither profitable nor practical, however, to employ reflectors of very large thickness. Although the albedo increases with increasing thickness (Fig. 8.1), little is gained by extending the thickness beyond a few diffusion lengths.

The second advantage offered by a reflector is derived from its moderating properties. This characteristic is especially important in dealing

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with thermal (or partially thermal) reactors. In all reactor cores, fission neutrons appear at all energies, and if the nuclear properties of the system are designed so as to exploit the thermal characteristics of the fuel, it is necessary to introduce a moderating material into the system. Depending upon the engineering requirements of the reactor, it may not always be possible or desirable to mix these low-mass materials directly with the fuel. In these cases the reflector regions may be composed of moderating materials, and much of the slowing down allowed to occur there. Thus reflectors in many reactor systems may serve as major sources of thermal neutrons. This arrangement has some advantage over systems in which the fuel and moderator are intimately mixed, since the slowing down

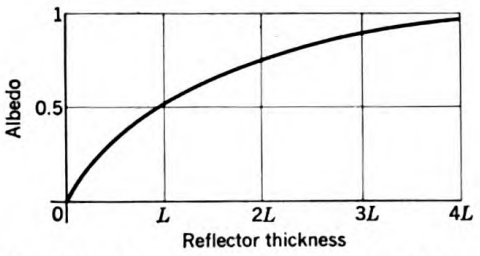

FIG. 8.1 Albedo as a function of reflector thickness.

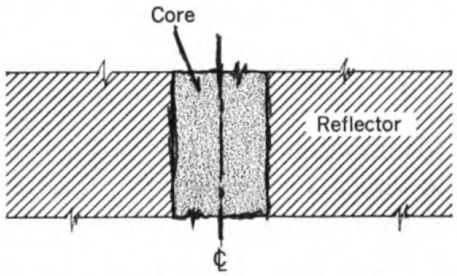

FIG. 8.2 Slab reactor with infinite reflectors.

through the resonance range of the fuel takes place in regions which contain no fuel. This generally results in an increase in the resonance-escape probability.

The moderating properties of a reflector are not required (or desired), of course, in fast reactor systems. There, the principal concern is to minimize the amount of low-nuclear-mass materials so as to realize the "fastest" neutron spectrum possible. Thus the primary function of a reflector in fast reactors is to reduce leakage losses.

There is a third advantage to be gained from the use of an outer reflector on a reactor, namely, that the reflector may also serve as part of the radiation shield (or the shield may in part serve as a reflector). This effect has not been included in (8.1) since for the purposes of this study the essential features of multiregion systems are embodied in the first two considerations.

A convenient measure of merit of a reflector is the so-called *reflector savings*. The reflector savings gives the decrease in the critical core size of a bare reactor which could be realized by the addition of a particular reflector. To illustrate this idea consider, for example, an infinite-slab reactor of half-width a. In the one-velocity approximation, the criticality equation is given by

$$\nu\Sigma_f = \Sigma_a + DB^2 \tag{5.126}$$

where $B = \pi/2\tilde{a}$. This relation defines the critical fuel concentration

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for the dimension a. Now assume that this slab is reflected by two semi-infinite media of graphite as shown in Fig. 8.2. The albedo for a semi-infinite medium is given by (5.116), and for the properties of graphite listed in Table 5.1, this relation gives approximately 0.92; that is, 92 per cent of all neutrons leaving the faces of the slab are reflected back. Thus the leakage losses expressed by the DB^2 term in (5.126) are considerably reduced, and the multiplying slab with the proposed reflector configuration could be maintained at criticality with a markedly lower concentration of fuel. On the other hand, if we wish to retain the original (bare-slab) fuel concentration, the width of the core in the reflected system could be reduced. It is this difference in critical size which we call the reflector savings; thus if a_0 is the critical size of a bare system and a_R that of a reflected system with the same fuel concentration, then we define

$$\text{Reflector savings} \equiv a_0 - a_R \tag{8.2}\star$$

This number is a measure of the amount of core material saved by the addition of a reflector. The reflector savings depends, of course, on the relative nuclear properties of the core and reflector as well as on the thickness of the reflector itself. In dealing with reflected configurations, the number a_0 is commonly called the *equivalent bare* dimension and, as previously demonstrated in the sample calculation of Sec. 6.6, it may be extremely useful in determining the nuclear properties of complicated multiregion reactors.

b. Analytical Models. The general problem of treating the slowing down and diffusion of neutrons in the reflected system is analytically difficult, even in the case of a highly thermal reactor in which the principal interest is centered in the diffusion of thermal neutrons. The principal difficulties are involved in making the following calculations:

1. Losses of neutrons by absorption and leakage as they slow down to thermal: the presence of the reflector introduces new absorbing materials and a new spatial dependence for the flux, leading to complicated expressions for the absorption and leakage probabilities.

2. Spatial distribution of the neutrons which enter the thermal group: because of migration of the neutrons during the slowing-down process, the spatial dependence of the slowing-down density at thermal will not be the same as that of the thermal fissions; moreover, there will be an apparent source of thermal neutrons in the reflector, since some neutrons are thermalized there.

In all the preceding chapters the focus of attention was on the development of fundamental concepts and suitable models for describing the neutron population in elementary reactor geometries. The two basic models so far examined, the diffusion model and the Fermi age, were applied to bare reactors. In the present chapter our purpose is to show how these methods may be extended, with suitable modifications, to

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more practical configurations, which, as we have already observed, invariably involve multiregion media. These modifications are complicated by the two primary considerations listed above. The nature of these problems is well displayed in the case of the continuous slowing-down diffusion model.

It will be recalled that in developing the age-diffusion model some difficulty was encountered in working with boundary conditions involving the energy variable. This problem was circumvented by eliminating the energy variable from the boundary condition statement for the outer surface of the reactor. If the continuous slowing-down diffusion model is applied to practical systems involving several regions of dissimilar composition, it is clear that a further complication arises in establishing workable relations for the interface conditions. We observed above that because of this dissimilarity in material composition the spatial forms of the neutron flux at various energies differ. Thus it is not possible to describe the spatial distribution of neutrons of all energies by means of a single function, and so analytical solutions are not generally possible for these systems. One way in which this difficulty may be overcome involves the use of *energy groups.* In this method the entire slowing-down range is divided into various energy intervals, and an appropriate differential equation of the one-velocity diffusion type is used to describe the spatial distribution of the neutrons of each group. But, since the various groups are coupled through the slowing-down-density term, this approach frequently yields a formidable set of simultaneous equations which are best handled by means of fast computing machines. The computational techniques involved in this method are discussed at some length in Sec. 8.8.

It is clear from the preceding discussion that a very accurate description of the neutron population in an actual reactor requires extensive computational effort. For the present, our attention will be directed to less exacting models which are amenable to hand calculation. We begin with the application of the one-velocity diffusion model to multiregion media. Since this model cannot account for the distribution of neutrons in slowing down, its use in analyzing reflected configurations serves to determine only the reduction in neutron escapes that can be achieved by a particular reflector. It is important to note, however, that many of the methods and results developed in the one-velocity treatment are directly applicable to the multigroup model.

The second basically different approach to the multiregion problem is the *two-group* model. This model attempts to describe the distribution of neutrons in both space and energy by separating all neutrons into thermal neutrons and nonthermal (fast) neutrons. The spatial distribution of neutrons in each group is given by a one-velocity diffusion relation in much the same manner as in the multigroup treatment; however, the

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two-group model developed here differs markedly from the general multigroup model in one essential feature. The two-group treatment is an exact analytical solution of a particular set of differential equations which describe an idealized reactor system, whereas the solution from the multigroup method is an iterated (numerical) result. Although the two-group model would appear to be very crude, at first glance, it has proved to be extremely useful and effective for preliminary analyses of complex systems.

8.2 The One-velocity Model

a. General Equations. We have noted that in reflected reactors, which are dependent at least in part on thermal fissions, the reflector serves as an important source of thermal neutrons. It is clear that, on the basis of this fact alone, the spatial distribution of the slowing-down density at thermal cannot be proportional, everywhere, to the distribution of thermal fissions. Since this is a fundamental premise of the one-velocity model, it is to be expected that the application of this model to reflected systems will yield at best a poor approximation to reality in most cases of practical interest. In fact, the most that can be said with this model about the distribution of thermal fissions is that it is proportional to the thermal flux in multiplying regions and zero everywhere else. However, there is some advantage to be gained from this application since this model will at least describe the reflective characteristics of the various nonmultiplying regions. Furthermore, it often yields a useful first approximation to the neutron distribution, and the analytical treatment of the one-velocity model will be helpful in developing the multigroup methods.

Throughout the analysis which follows we assume that the reflected reactor consists of a central core of multiplying material which is internally homogeneous, and surrounding the core, although not necessarily covering all its surface, is another internally homogeneous medium which is nonmultiplying and which we call the reflector. Thus all the reactors considered here are two-region systems. This does not limit our analysis, however, since all the fundamental ideas and techniques involved are well demonstrated by these systems and are more generally applicable. The analysis of systems having three or more different regions can be carried out by the same methods. These calculations are merely more laborious.

The basic relations which give the distribution of neutrons in the core (C) and reflector (R) of a one-velocity system may be written

$$-D_C\nabla^2\phi_C(\mathbf{r}) + \Sigma_a^{(C)}\phi_C(\mathbf{r}) = \nu\Sigma_f\phi_C(\mathbf{r}) \qquad (a)$$
$$-D_R\nabla^2\phi_R(\mathbf{r}) + \Sigma_a^{(R)}\phi_R(\mathbf{r}) = 0 \qquad (b) \qquad (8.3)$$

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The boundary conditions to be satisfied by the functions ϕ_C and ϕ_R are assumed to be:

(1) Symmetry in the core, or nonsingularity of the flux at the core center
(2) Continuity of the neutron flux at all points of the core-reflector interface
(3) Continuity of the net current at all points of the core-reflector interface
(4) The flux must vanish on all extrapolated outer surfaces of the assembly

(8.4)

Throughout the remainder of this section we will be concerned with the solution of Eqs. (8.3), using the conditions (8.4) for three basic reactor geometries. For convenience in notation we write the differential equations in the form

$$\nabla^2\phi_C(\mathbf{r}) + B^2\phi_C(\mathbf{r}) = 0 \qquad B^2 \equiv \frac{1}{D_C}[\nu\Sigma_f - \Sigma_a^{(C)}] \tag{a}$$
$$\nabla^2\phi_R(\mathbf{r}) - \kappa_R^2\phi_R(\mathbf{r}) = 0 \qquad \kappa_R^2 \equiv \frac{\Sigma_a^{(R)}}{D_R} \tag{b}$$

(8.5)

b. Spherical Geometry. The case of a spherical core (radius R_0) and a spherical shell reflector (outer radius R_1) is particularly simple if spherical symmetry is assumed. In this case the general solutions of Eqs. (8.5) are

$$\begin{aligned}\phi_C(\mathbf{r}) = \phi_C(r) &= A_C\frac{\sin Br}{r} + C_C\frac{\cos Br}{r}\\ \phi_R(\mathbf{r}) = \phi_R(r) &= A_R\frac{\sinh \kappa_R r}{r} + C_R\frac{\cosh \kappa_R r}{r}\end{aligned} \tag{8.6}$$

Since the function $\cos Br/r$ is singular at $r = 0$, we take $C_C \equiv 0$. Also, by condition (4) in (8.4), we have for the present case,

$$\phi_R(\tilde{R}_1) = 0 \qquad \text{with } \tilde{R}_1 \equiv R_1 + 0.71\lambda_{\text{tr}}^{(R)} \tag{8.7}$$

If we apply this condition to ϕ_R, and discard the cosine term in ϕ_C, it is easily shown that the solutions take the form

$$\begin{aligned}\phi_C(r) &= A\frac{\sin Br}{r}\\ \phi_R(r) &= C\frac{\sinh \kappa_R(\tilde{R}_1 - r)}{r}\end{aligned} \tag{8.8}$$

The continuity conditions (2) and (3) in (8.4) yield

$$\begin{aligned}\phi_C(R_0) &= \phi_R(R_0)\\ D_C\phi_C'(R_0) &= D_R\phi_R'(R_0)\end{aligned} \tag{8.9}$$

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When the results (8.8) are used in the first of these relations, an equation between A and C is obtained; thus

$$C = \frac{A \sin BR_0}{\sinh \varkappa_R(\tilde{R}_1 - R_0)} \tag{8.10}$$

If Eqs. (8.8) are used in (8.9), and the flux relation is divided by the current relation, the constants A and C cancel out and the following equation results:

$$BR_0 \cot BR_0 - 1 = -\frac{D_R}{D_C}[\varkappa_R R_0 \coth \varkappa_R(\tilde{R}_1 - R_0) + 1] \tag{8.11}$$

In order that the system operate at steady state, the above relationship must be satisfied by the parameters of the core and reflector. This relation is, in fact, the *criticality condition* for the one-velocity two-region spherical reactor. The computational procedure involved in the application of this relation is straightforward and follows closely the methods used in bare-reactor calculations. The only difference, of course, is that the size and material properties are here related by a transcendental equation. To illustrate how it may be used, consider the following situation: A spherically symmetric reactor of core radius R_0 and outer reflector radius R_1 is to be analyzed. The reflector material has been selected, and its constants are $\varkappa_R$ and D_R. The problem is to determine the critical concentration of fuel in the core. The usual procedure is first to assume a value for D_C. This is easily done, since D_C is determined in large part by the nonfuel constituents of the core (presumably known) and only slightly affected by the fuel concentration. Given D_C, then, the right-hand side of (8.11) is completely determined, and the left-hand expression may be solved for the corresponding value of B. The substitution of this result into the B^2 equation of (8.5) yields the fuel concentration.

Having determined the quantity B, the flux relations (8.8) are now completely defined. Note that, as in the bare-reactor results, the expressions for the flux include an arbitrary constant A (or C) which is obtained from the initial conditions. (It is convenient in some cases to evaluate this constant in terms of the total power output of the reactor.) Of some interest here is the comparison of the results for the reflected sphere and for the bare sphere. These are given in Table 8.1.

Before discussing the analogous treatment of other geometries, some of the physical consequences of the addition of the reflector will be considered. The results which follow are developed for the special case of the sphere, but similar relations can be obtained for the cylinder and the rectangular-block reactors. We begin with a study of the reflected-sphere system in which the *extrapolated* thickness $(\tilde{R}_1 - R_0)$ of the reflector is small compared to both the core radius R_0 and the diffusion length in the reflector L_R. (Admittedly the diffusion theory postulated

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above does not apply well in this case, but the *qualitative* behavior is correctly presented.)

Now, for small values of the argument, the hyperbolic cotangent is given by the expansion

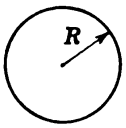
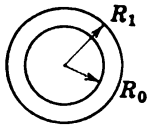
$$\coth x = \frac{1}{x} + \frac{x}{3} + \cdots \tag{8.12}$$

If we assume that $D_C \simeq D_R$, then the application of the first term in the above expression to the criticality equation (8.11) yields

$$BR_0 \cot BR_0 \simeq -\frac{R_0}{\tilde{R}_1 - R_0} \tag{8.13}$$

But, by assumption, the right-hand side is a large number; thus $BR_0 \simeq \pi$.

TABLE 8.1 COMPARISON OF BARE- AND REFLECTED-SPHERE REACTORS

	Bare sphere	Reflected sphere
Geometry		
Flux in core	$\frac{A' \sin Br}{r}$	$\frac{A \sin Br}{r}$
Definition of B^2	$\frac{1}{D}(\nu\Sigma_f - \Sigma_a)$	$\frac{1}{D_C}[\nu\Sigma_f - \Sigma_a^{(C)}]$
Condition for steady state	$B = \frac{\pi}{\tilde{R}}$	$BR_0 \cot BR_0 - 1 = -\frac{D_R}{D_C}[\kappa_R R_0 \coth \kappa_R(\tilde{R}_1 - R_0) + 1]$

We require, therefore, an expression for $\cot x$ in the case $x \equiv \pi - \varepsilon$, where $\varepsilon \ll 1$. This is

$$\cot(\pi - \varepsilon) \simeq -\frac{1}{\varepsilon} \tag{8.14}$$

The use of this relation in (8.13) gives

$$\frac{BR_0}{BR_0 - \pi} \simeq -\frac{R_0}{\tilde{R}_1 - R_0} \quad \text{and} \quad B \simeq \frac{\pi}{\tilde{R}_1} \simeq \frac{\pi}{\tilde{R}_0} \tag{8.15}$$

when $\tilde{R}_1 - R_0 \to 0$. Thus in the limit for very thin reflectors the transcendental relation (8.11) goes over into the bare-reactor result. (Note that, since we have assumed $D_C \simeq D_R$, the extrapolation distance is the same for core and reflector.)

The approximations introduced above may be used to estimate the reflector savings realized by adding a very thin reflector to a bare spherical

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reactor. Let the critical radius of the bare system be R_{00}; then by definition

$$\text{Reflector savings} = R_{00} - R_0 \tag{8.16}$$

where R_0 is the core radius of the reflected reactor having the same core composition. If we apply the requirement that $\varkappa_R(\tilde{R}_1 - R_0) \ll 1$, then the criticality relation takes the approximate form

$$BR_0 \cot BR_0 - 1 \simeq -\frac{D_R}{D_C}\left(\frac{R_0}{\tilde{R}_1 - R_0} + 1\right) \tag{8.17}$$

Since, by assumption, the reflector thickness is also small compared to R_0, the right-hand side of (8.17) is very large, and it follows from the approximation (8.14) that the above relation may be written

$$\frac{BR_0}{BR_0 - \pi} \simeq -\frac{D_R}{D_C}\left(\frac{R_0}{\tilde{T}} + 1\right) \qquad \text{with } \tilde{T} \equiv \tilde{R}_1 - R_0 \tag{8.18}$$

If we use the assumption that $\tilde{T}/R_0 \ll 1$, then the solution of this equation for R_0 yields

$$R_0 \simeq \frac{\pi}{B} - \frac{D_C\tilde{T}}{D_R} \tag{8.19}$$

We know from our relations for the bare spherical reactor that $R_{00} = \pi/B$; therefore, the substitution of (8.19) into (8.16) gives

$$\text{Reflector savings} \simeq \frac{D_C\tilde{T}}{D_R} \qquad \text{thin reflector} \tag{8.20}\star$$

In many cases of interest $D_C \simeq D_R$; thus, the addition of a reflector of extrapolated thickness $\tilde{T} \equiv \tilde{R}_1 - R_0$ to a bare spherical reactor allows the critical core radius to be reduced by the amount $\tilde{T}$. The linear relationship (8.20) does not hold, of course, if the inequalities $\varkappa_R(\tilde{R}_1 - R_0) \ll 1$ and $(\tilde{R}_1 - R_0)/R_0 \ll 1$ are not satisfied. In these cases, the critical radius for the reflected system must be computed from (8.11) and the result substituted into (8.16) to obtain the savings. It should be borne in mind also that the result (8.20) applies specifically to spherical systems, although this expression is frequently used as a general relation for other geometries when making quick first estimates for the effect of a thin reflector.

The general approach used above may also be applied to a system having a thick reflector. In this case we impose the restriction that $\varkappa_R(\tilde{R}_1 - R_0) \gg 1$ and furthermore that $\varkappa_R R_0 \gg 1$; i.e., both the reflector thickness and the core size are to be much larger than the diffusion length of the reflector material. Because of the former requirement, $\coth \varkappa_R(\tilde{R}_1 - R_0) \simeq 1$, and (8.11) is approximated by

$$BR_0 \cot BR_0 \simeq -\frac{D_R}{D_C}(\varkappa_R R_0 + 1) + 1 \tag{8.21}$$

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By the latter requirement, the right-hand side of this relation is large and negative, and therefore the approximation (8.14) applies for the left-hand side. It follows that

$$R_0 \simeq \frac{\pi}{B} - \frac{D_C L_R}{D_R}$$

and by (8.16)

$$\text{Reflector savings} \simeq \frac{D_C L_R}{D_R} \qquad \text{thick reflector} \tag{8.22}\star$$

An accurate calculation for the reflector savings based on the solution of the criticality equation for the one-velocity model will yield a curve of the shape shown in Fig. 8.3. The approximations (8.20) and (8.22) apply for the extreme values of reflector thickness.

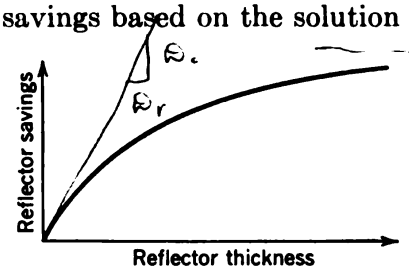

FIG. 8.3 Reflector savings.

The problem of the spherical one-velocity reflected reactor may also be studied by means of the albedo concept. Thus let us denote the albedo of the spherical shell reflector by α, where

$$\alpha \equiv \frac{\text{fraction of neutrons incident upon the reflector from the core which are returned to the core}}{} \tag{8.23}$$

This number is a function of the inner and outer radii of the shell (R_1 and R_0) as well as the (one-speed) transport and absorption cross sections of the material of the shell. In general, α is also dependent upon the angular distribution of the incident neutrons; in the present analysis we select the distribution which is consistent with diffusion theory.

The one-velocity flux in the core which satisfies the differential equation (8.5a) must now meet the boundary conditions

$$\begin{aligned} &(1)\ \phi_C(r) \neq \infty \qquad \text{for } r \leq R_0 \\ &(2)\ j^{(C)}_{-}(R_0) = \alpha j^{(C)}_{+}(R_0) \end{aligned} \tag{8.24}$$

The solution of the core equation which satisfies condition (1) in (8.24) is given in (8.8). The substitution of this expression into condition (2) in (8.24) yields the relation

$$\frac{D_C}{R_0}(1 - BR_0 \cot BR_0) = \frac{1}{2}\left(\frac{1-\alpha}{1+\alpha}\right) \tag{8.25}\star$$

which corresponds to (8.11). Regarding R_0 and D_C as being fixed, (8.25) has the limiting forms [using (8.14)]

$$\begin{aligned} &\alpha \to 0 \qquad B \to \frac{\pi}{R_0 + 2D_C} \qquad \text{for } \frac{R_0}{2D_C} \gg 1 \\ &\alpha \to 1 \qquad BR_0 \cot BR_0 = 1 \end{aligned}$$

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If the concentration of materials in the core is held constant, the reactor may be kept critical by reducing the radius R_0 as the albedo of the reflector is increased; this behavior of the critical radius is illustrated schematically in Fig. 8.4. The albedo of the reflector may be increased by (1) increasing the reflector thickness, (2) increasing the macroscopic scattering cross section of the reflector, and (3) decreasing the macroscopic absorption cross section of the reflector.

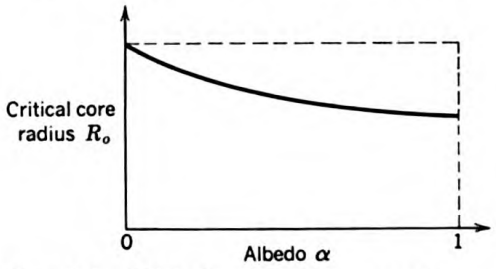

FIG. 8.4 Critical radius versus reflector albedo for fixed core composition.

c. Cylindrical Geometry. The general problem of a finite cylinder of multiplying material *completely* surrounded by a reflector cannot be easily solved analytically (see Fig. 8.5). For the present we will consider two less general cases. The completely reflected reactor is treated in Sec. 8.6 by means of an approximate method which entails an iterative procedure.

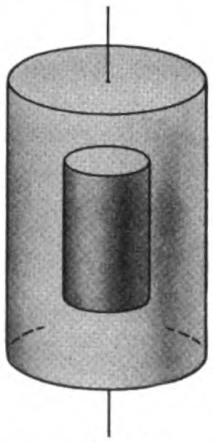

FIG. 8.5 Completely reflected cylindrical reactor.

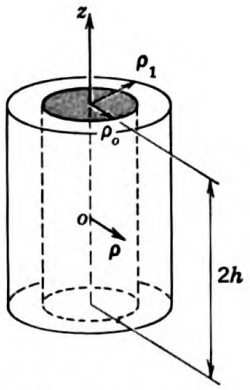

FIG. 8.6 Cylindrical reactor with side reflector.

1. *Cylindrical Core with Reflector on the Curved Surface.* The geometry and dimensions of the system in question are shown in Fig. 8.6. The appropriate forms of the differential equations (8.5) for the case of axial symmetry are

$$\begin{aligned} &\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\phi_C(\rho,z) + \frac{\partial^2}{\partial z^2}\phi_C(\rho,z) + B^2\phi_C(\rho,z) = 0 \qquad (a) \\ &\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}\phi_R(\rho,z) + \frac{\partial^2}{\partial z^2}\phi_R(\rho,z) - \kappa_R^2\phi_R(\rho,z) = 0 \qquad (b) \end{aligned} \tag{8.26}$$

The solutions to these equations are to satisfy the boundary conditions (8.4). In each of the equations (8.26) the flux is separable into a ρ and

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a z dependent component; that is,

$$\begin{aligned} \phi_C(\rho,z) &\equiv F_C(\rho)\, G_C(z) \\ \phi_R(\rho,z) &\equiv F_R(\rho)\, G_R(z) \end{aligned} \tag{8.27}$$

The substitution of these relations into (8.26) yields the equations

$$\begin{aligned} &\frac{1}{\rho F_C(\rho)} \frac{d}{d\rho} [\rho F'_C(\rho)] + \frac{G''_C(z)}{G_C(z)} \equiv -B^2 \qquad (a) \\ &\frac{1}{\rho F_R(\rho)} \frac{d}{d\rho} [\rho F'_R(\rho)] + \frac{G''_R(z)}{G_R(z)} \equiv \varkappa_R^2 \qquad (b) \end{aligned} \tag{8.28}$$

Since ρ and z are independent variables, the above relations can be satisfied only if each term is a constant; therefore, we set

$$\begin{aligned} \frac{1}{\rho F_C(\rho)} [\rho F'_C(\rho)]' &\equiv -\lambda_C^2 & \frac{1}{\rho F_R(\rho)} [\rho F'_R(\rho)]' &\equiv +\lambda_R^2 \\ \frac{G''_C(z)}{G_C(z)} &\equiv -\mu_C^2 & \frac{G''_R(z)}{G_R(z)} &\equiv -\mu_R^2 \end{aligned} \tag{8.29}$$

It follows from (8.28) that

$$\lambda_C^2 + \mu_C^2 = B^2 \qquad \lambda_R^2 - \mu_R^2 = \varkappa_R^2 \tag{8.30}$$

The general solutions to Eqs. (8.29) are as follows:

$$\begin{aligned} F_C(\rho) &= A_1 J_0(\lambda_C \rho) + A_2 Y_0(\lambda_C \rho) && (a) \\ F_R(\rho) &= C_1 I_0(\lambda_R \rho) + C_2 K_0(\lambda_R \rho) && (b) \\ G_C(z) &= A_3 \cos \mu_C z + A_4 \sin \mu_C z && (c) \\ G_R(z) &= C_3 \cos \mu_R z + C_4 \sin \mu_R z && (d) \end{aligned} \tag{8.31}$$

In order to meet the boundary conditions (1) in (8.4) the coefficients A_2, A_4, and C_4 must be taken to be zero, since $Y_0(\lambda_C \rho)$ is singular at $\rho = 0$, and $\sin \mu_C z$ is not symmetric about $z = 0$. Now consider condition (2) in (8.4) which requires that the flux be continuous all along the core-reflector interface. The application of this condition to Eqs. (8.31a) and (8.31b) yields

$$\begin{aligned} \phi_C(\rho_0,z) &= \phi_R(\rho_0,z) \\ A_1 J_0(\lambda_C \rho_0) A_3 \cos \mu_C z &= C_3 \cos \mu_R z [C_1 I_0(\lambda_R \rho_0) + C_2 K_0(\lambda_R \rho_0)] \end{aligned} \tag{8.32}$$

This equation can be satisfied for all values of z if

$$\mu_C \equiv \mu_R \equiv \mu \tag{8.33}$$

If we absorb the arbitrary constants A_1, A_3, and C_3 into a single constant A, the solutions for the flux may be written in the form

$$\begin{aligned} \phi_C(\rho,z) &= A J_0(\lambda_C \rho) \cos \mu z \\ \phi_R(\rho,z) &= [C_1 I_0(\lambda_R \rho) + C_2 K_0(\lambda_R \rho)] \cos \mu z \end{aligned} \tag{8.34}$$

We now satisfy condition (4) in (8.4) that the flux vanish at every

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point of the extrapolated outer boundary of the system. Thus, over the end faces,

$$\begin{cases} \phi_C(\rho,\tilde{h}) = 0 = AJ_0(\lambda_C\rho)\cos\mu\tilde{h} \\ \phi_R(\rho,\tilde{h}) = 0 = [C_1I_0(\lambda_R\rho) + C_2K_0(\lambda_R\rho)]\cos\mu\tilde{h} \end{cases} \tag{8.35}$$

In general, the $\tilde{h}$'s in these two equations are not identical since they involve the extrapolation distance outside the core and reflector; however, for cylinders with dimensions large in comparison with the extrapolation distance in either the core or the reflector, the $\tilde{h}$'s are approximately equal. In any case, the conditions of (8.35) can be met only if

$$\cos\mu\tilde{h} \equiv 0 \tag{8.36}$$

so that the difference in extrapolation distance for the core and reflector would be difficult to take into account, since we have already required that the μ's be the same. Equation (8.36) can be satisfied only if

$$\mu = \frac{\pi}{2\tilde{h}} \tag{8.37}$$

taking, as in previous work, the lowest eigenvalue.

The requirement that the flux vanish over the extrapolated curved surface of the reflector gives a relation between C_1 and C_2; thus,

$$C_1 = -\frac{C_2K_0(\lambda_R\tilde{\rho}_1)}{I_0(\lambda_R\tilde{\rho}_1)} \tag{8.38}$$

If we call

$$L_0(\rho) \equiv I_0(\lambda_R\tilde{\rho}_1)\,K_0(\lambda_R\rho) - I_0(\lambda_R\rho)\,K_0(\lambda_R\tilde{\rho}_1) \tag{8.39}$$

the fluxes may now be written

$$\begin{aligned} \phi_C(\rho,z) &= AJ_0(\lambda_C\rho)\cos\frac{\pi z}{2\tilde{h}} \\ \phi_R(\rho,z) &= CL_0(\rho)\cos\frac{\pi z}{2\tilde{h}} \end{aligned} \tag{8.40}$$

where C and A are constants to be determined from the interface conditions on the flux and net current, (3) in (8.4); these are

$$\begin{aligned} AJ_0(\lambda_C\rho_0) &= CL_0(\rho_0) \\ D_CAJ_0'(\lambda_C\rho_0) &= D_RCL_0'(\rho_0) \end{aligned} \tag{8.41}$$

One of these equations may be used to give a relationship between A and C; however, A and C cannot both be determined from (8.41) and, as in all reactor problems, there will remain an arbitrary constant which is the flux *level* at steady-state operation. The ratio of the two equations is independent of A and C and gives the *criticality condition* for this system:

$$\frac{D_CJ_0'(\lambda_C\rho_0)}{J_0(\lambda_C\rho_0)} = \frac{D_RL_0'(\rho_0)}{L_0(\rho_0)} \tag{8.42}$$

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The results of the analysis given above might be used as follows: Suppose that a system is to be built of the type shown in Fig. 8.6 with the values of the radii ρ_0 and ρ_1 and the height $2h$ specified. Also, suppose that the reflector composition has been selected (i.e., κ_R^2 is determined). In the core, the concentration of all materials is known except that of the fuel. If the fuel concentration is low, as will be the case in many real systems, the diffusion coefficient D_C for the core may be assumed to be known, since it will be only slightly affected by small changes in the fuel concentration. Then, the criticality condition (8.42) may be used to determine λ_C, inasmuch as all the other quantities in this relation are now known. Note that from (8.30) and (8.37)

$$\lambda_R^2 = \frac{\Sigma_a^{(R)}}{D_R} + \left(\frac{\pi}{2\tilde{h}}\right)^2 \tag{8.43}$$

With the value obtained for λ_C, the buckling may be computed from

$$B^2 = \left(\frac{\pi}{2\tilde{h}}\right)^2 + \lambda_C^2 \tag{8.44}$$

The fuel concentration required to maintain this system at steady state is calculated from (8.5)

$$\frac{\nu\Sigma_f - \Sigma_a^{(C)}}{D_C} = B^2 = \left(\frac{\pi}{2\tilde{h}}\right)^2 + \lambda_C^2 \tag{8.45}$$

The results given above include, of course, the case of the very long cylinder for which the quantity $(\pi/2\tilde{h})^2$ may be neglected in comparison to the other terms.

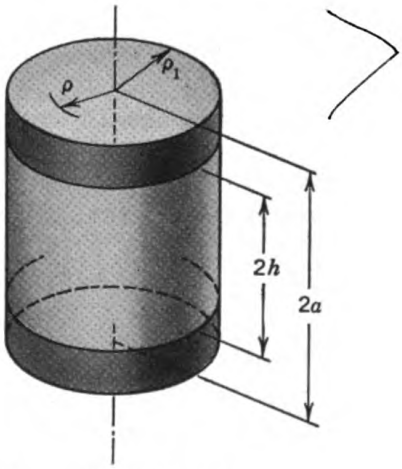

FIG. 8.7 Cylindrical reactor with end reflectors.

2. *Cylindrical Core with End Reflectors Only.* Figure 8.7 shows the geometry and dimensions for the cylindrical core with end reflectors. The work of the preceding case may be taken over here through the separation of variables since Eqs. (8.26) through (8.28) are valid for the present calculation. However, it is convenient in fitting the present boundary conditions to make a different choice for the algebraic sign of the constants. Instead of using relations (8.29), let

$$\begin{aligned} \frac{1}{\rho F_C(\rho)}\,[\rho F_C'(\rho)]' &= -\lambda_C^2 & \qquad \frac{1}{\rho F_R(\rho)}\,[\rho F_R'(\rho)]' &= -\lambda_R^2 \\ \frac{G_C''(z)}{G_C(z)} &= -\mu_C^2 & \qquad \frac{G_R''(z)}{G_R(z)} &= +\mu_R^2 \end{aligned} \tag{8.46}$$

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It follows from (8.28) that

$$\lambda_C^2 + \mu_C^2 = B^2 \qquad -\lambda_R^2 + \mu_R^2 = \varkappa_R^2 \tag{8.47}$$

The general solutions for the present case are then

$$\begin{aligned} F_C(\rho) &= A_1 J_0(\lambda_C \rho) + A_2 Y_0(\lambda_C \rho) && (a) \\ F_R(\rho) &= C_1 J_0(\lambda_R \rho) + C_2 Y_0(\lambda_R \rho) && (b) \\ G_C(z) &= A_3 \cos \mu_C z + A_4 \sin \mu_C z && (c) \\ G_R(z) &= C_3 \cosh \mu_R z + C_4 \sinh \mu_R z && (d) \end{aligned} \tag{8.48}$$

In order to meet the requirement that the flux be finite and symmetric, we must have $A_2 = C_2 = A_4 \equiv 0$. If the flux is to be continuous across the core-reflector interface, we also require

$$\lambda_C = \lambda_R \equiv \lambda \tag{8.49}$$

Consider now the reflector located on the positive z axis. (Note that the flux in the other reflector may always be obtained by symmetry.) The extrapolated boundary condition at $\tilde{a}$ yields $\phi_R(\rho,\tilde{a}) = 0$ or $G_R(\tilde{a}) = 0$. We obtain immediately from (8.48d) the relation

$$C_3 = -C_4 \tanh \mu_R \tilde{a} \tag{8.50}$$

If we use the results so far derived in the expressions for the flux, we have

$$\begin{aligned} \phi_C(\rho,z) &= A J_0(\lambda \rho) \cos \mu_C z \\ \phi_R(\rho,z) &= C J_0(\lambda \rho) \sinh \mu_R(\tilde{a} - z) \end{aligned} \tag{8.51}$$

The flux is also to vanish all along the curved surfaces; thus,

$$\phi_R(\tilde{\rho}_1,z) = \phi_C(\tilde{\rho}_1,z) = 0$$

and it follows that

$$\lambda = \frac{\nu_1}{\tilde{\rho}_1} \quad \text{where } \nu_1 = 2.405 \tag{8.52}$$

and, as before, it is assumed that the extrapolation distance is the same for the core and the reflector materials.

The continuity conditions at the interface may now be applied to obtain a relationship between the constants A and C and the criticality condition for the end-reflected cylinder. These are:

cont. flux $$C = \frac{A \cos \mu_C h}{\sinh \mu_R(\tilde{a} - h)} \tag{8.53}$$

cont. current $$D_C \mu_C \tan \mu_C h = D_R \mu_R \coth \mu_R(\tilde{a} - h) \tag{8.54}\star$$

SOLVE for μ_C

The calculation of the critical parameters of the reactor proceeds as before, with the relations obtained above and with

$$\mu_R^2 = \frac{\Sigma_a^{(R)}}{D_R} + \left(\frac{\nu_1}{\tilde{\rho}_1}\right)^2 \qquad B^2 = \left(\frac{\nu_1}{\tilde{\rho}_1}\right)^2 + \mu_C^2 \tag{8.55}$$

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d. Rectangular Geometry. We turn now to the consideration of rectangular systems; again there is no simple solution to the problem of a rectangular-block core completely reflected as shown in Fig. 8.8. However, there is one geometry which is readily solved by methods similar to those used for the reflected cylinder. Consider a block core with two

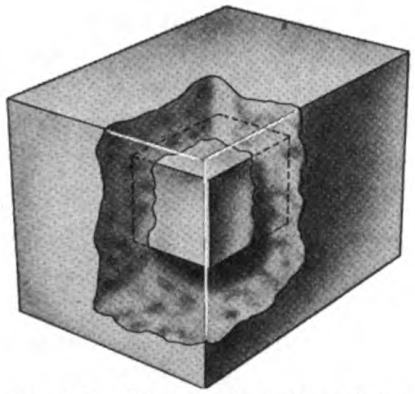

FIG. 8.8 Completely reflected rectangular-block reactor.

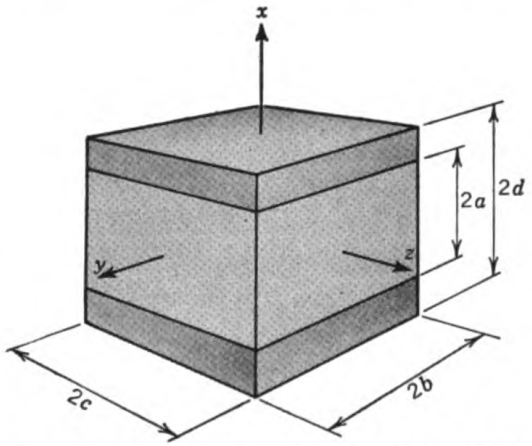

FIG. 8.9 Block reactor with slab-reflected ends.

reflecting slabs of the same thickness on two opposing faces, as shown in Fig. 8.9. As before, the differential equations for the flux in the core and reflector are given by (8.5) and the boundary conditions, by (8.4). In Cartesian coordinates, these are

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\phi_C(x,y,z) + \frac{\partial^2}{\partial y^2}\phi_C(x,y,z) + \frac{\partial^2}{\partial z^2}\phi_C(x,y,z) \\
&\qquad + B^2\phi_C(x,y,z) = 0 \qquad (a) \\
&\frac{\partial^2}{\partial x^2}\phi_R(x,y,z) + \frac{\partial^2}{\partial y^2}\phi_R(x,y,z) + \frac{\partial^2}{\partial z^2}\phi_R(x,y,z) \\
&\qquad - \kappa_R^2\phi_R(x,y,z) = 0 \qquad (b)
\end{aligned} \tag{8.56}$$

Here, as in the cylindrical case, the variables may be separated. Assume

$$\begin{aligned}
\phi_C(x,y,z) &\equiv F_C(x)\,G_C(y)\,H_C(z) \\
\phi_R(x,y,z) &\equiv F_R(x)\,G_R(y)\,H_R(z)
\end{aligned} \tag{8.57}$$

then Eqs. (8.56) take the form

$$\begin{aligned}
\frac{F_C''(x)}{F_C(x)} + \frac{G_C''(y)}{G_C(y)} + \frac{H_C''(z)}{H_C(z)} &= -B^2 \\
\frac{F_R''(x)}{F_R(x)} + \frac{G_R''(y)}{G_R(y)} + \frac{H_R''(z)}{H_R(z)} &= \kappa_R^2
\end{aligned} \tag{8.58}$$

Choose

$$\begin{aligned}
\frac{F_C''(x)}{F_C(x)} &= -\lambda_1^2 \qquad \frac{G_C''(y)}{G_C(y)} = -\lambda_2^2 \qquad \frac{H_C''(z)}{H_C(z)} = -\lambda_3^2 \\
\frac{F_R''(x)}{F_R(x)} &= \mu_1^2 \qquad \frac{G_R''(y)}{G_R(y)} = -\mu_2^2 \qquad \frac{H_R''(z)}{H_R(z)} = -\mu_3^2
\end{aligned} \tag{8.59}$$

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then Eqs. (8.58) become

$$\begin{aligned} \lambda_1^2 + \lambda_2^2 + \lambda_3^2 &= B^2 \qquad (a) \\ \mu_1^2 - \mu_2^2 - \mu_3^2 &= \varkappa_R^2 \qquad (b) \end{aligned} \tag{8.60}$$

The general solutions for the various functions in (8.59) may now be written (taking account of the symmetry requirement to eliminate the sine functions)

$$\begin{aligned} F_C(x) &= A_1 \cos \lambda_1 x & F_R(x) &= C_1 \sinh \mu_1 x + E_1 \cosh \mu_1 x \\ G_C(y) &= A_2 \cos \lambda_2 y & G_R(y) &= C_2 \cos \mu_2 y \\ H_C(z) &= A_3 \cos \lambda_3 z & H_R(z) &= C_3 \cos \mu_3 z \end{aligned} \tag{8.61}$$

In order to meet the requirement that the flux be continuous across the core-reflector interface, that is,

$$\phi_C(a,y,z) = \phi_R(a,y,z) \tag{8.62}$$

or

$$A \cos \lambda_1 a(\cos \lambda_2 y \cos \lambda_3 z) = C'(\sinh \mu_1 a + C_0 \cosh \mu_1 a) \cos \mu_2 y \cos \mu_3 z$$

where $C_0 \equiv E_1/C_1$, we set

$$\lambda_2 = \mu_2 \qquad \lambda_3 = \mu_3 \tag{8.63}$$

In order to meet the requirement that the flux vanish at all points of the extrapolated outer boundary of the system, we must have

$$C_0 = -\tanh \mu_1 \tilde{d} \qquad \lambda_2 = \mu_2 = \frac{\pi}{2\tilde{b}} \qquad \lambda_3 = \mu_3 = \frac{\pi}{2\tilde{c}} \tag{8.64}$$

(Again the assumption has been made that the extrapolation distance is the same outside the core and the reflector.) With these results we obtain for the fluxes the expressions

$$\begin{aligned} \phi_C(x,y,z) &= A \cos \lambda_1 x \cos \frac{\pi y}{2\tilde{b}} \cos \frac{\pi z}{2\tilde{c}} \\ \phi_R(x,y,z) &= C \sinh \mu_1(\tilde{d} - x) \cos \frac{\pi y}{2\tilde{b}} \cos \frac{\pi z}{2\tilde{c}} \end{aligned} \tag{8.65}$$

These solutions apply only to the region on the side of the positive x axis; the flux in the rest of the reactor is obtained by symmetry about $x = 0$.

The requirement that the net current be continuous through the core-reflector interface yields a relationship between the constants A and C and the criticality condition for this system; thus,

$$\begin{aligned} A &= \frac{C \sinh \mu_1(\tilde{d} - a)}{\cos \lambda_1 a} \qquad (a) \\ D_C \lambda_1 \tan \lambda_1 a &= D_R \mu_1 \coth \mu_1(\tilde{d} - a) \qquad (b) \end{aligned} \tag{8.66}$$

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The parameter μ_1 is obtained from Eqs. (8.60*b*) and (8.64), which yield

$$\mu_1^2 = \frac{\Sigma_a^{(R)}}{D_R} + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2 \tag{8.67}$$

The substitution of this result into (8.66*b*) gives λ_1; then B^2 may be obtained from (8.60*a*) using (8.64):

$$B^2 = \lambda_1^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2 \tag{8.68}$$

The block geometry considered here is the only rectangular system for which an analytical solution has been obtained. As indicated earlier, the completely reflected geometry is difficult to treat analytically. Unfortunately this is also true for other more simple configurations such as the block core reflected on four sides, with or without corners (Fig. 8.10). The difficulty in all these cases is that it is not possible to meet

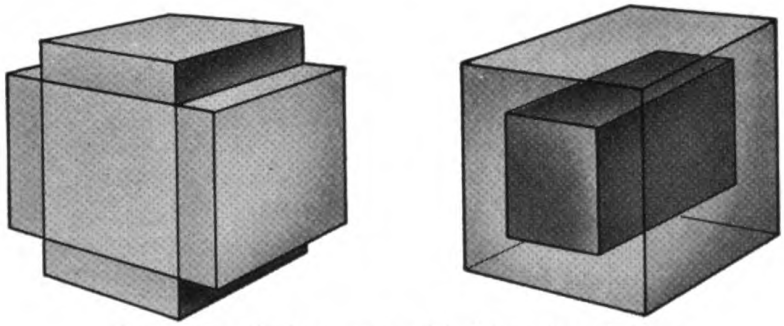

FIG. 8.10 Block reactor with slab-reflected sides.

all the boundary conditions with functions in which the variables have been separated. An approximate technique for treating these problems is discussed in Sec. 8.6.

e. Criticality Relation. The one-velocity model as used so far in this section is based upon the choice of a thermal-neutron source term which is proportional to the thermal core flux itself, that is, $\nu\Sigma_f\phi_C(\mathbf{r})$ [e.g., Eq. (8.3*a*)]. Since neutrons *are* born at high energy with the spatial distribution of the thermal flux in the core, the use of this source term neglects the diffusion of neutrons during the slowing-down process. In the case of a bare system it is true, of course, that this form of the thermal source term is correct, even when allowance is made for the migration during slowing down, since the flux preserves its spatial form at all energies. However, as we have noted, in a reflected reactor the spatial dependence of the flux will in general be different at different lethargies. In the analyses which follow, an attempt is made to account for the wanderings (and losses) of fast neutrons in reflected systems.

In order to obtain some idea of what is involved in estimating these

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effects, let us assume that the following neutron-balance relation[1] is valid for the core region:

$$-D_C\nabla^2\phi_C(\mathbf{r}) + \Sigma_a^{(C)}\phi_C(\mathbf{r}) = \nu_c\Sigma_f p_{\text{th}}^R g_{\text{th}}^R \phi_C(\mathbf{r}) \tag{8.69}$$

We define the factor p_{th}^R in this equation as the fraction of neutrons, born at high energy in the reactor, which is not absorbed in slowing down to thermal. The factor g_{th}^R is the corresponding fraction which is not lost by escapes from the reactor. In the present case of a reflected reactor these quantities could *not* be correctly calculated from the relations previously derived for the bare systems. Whatever estimate is made of these factors for a particular reactor, it must account for the absorbing and moderating properties of the reflector as well as the core.

Suppose that we accept the above equation as applicable to the reflected reactor, recognizing that the quantities p_{th}^R and g_{th}^R are to be computed by some, as yet unspecified, procedure. This relation may then be used to obtain an expression for the multiplication constant. In the usual way, we define

$$k \equiv \frac{\text{neutron production}}{\text{total neutron losses}} = \frac{\nu_c\Sigma_f p_{\text{th}}^R g_{\text{th}}^R \int_C \phi_C(\mathbf{r})\,d\mathbf{r}}{-D_C\int_C \nabla^2\phi_C(\mathbf{r})\,d\mathbf{r} + \Sigma_a^{(C)}\int_C \phi_C(\mathbf{r})\,d\mathbf{r}} \tag{8.70}$$

where $\int_C$ denotes integration over the volume of the core. The volume integral of the Laplacian term, which gives the neutron losses from the core, may be computed from the differential equation. We have from (8.69)

$$\nabla^2\phi_C + \left[\frac{\nu_c\Sigma_f p_{\text{th}}^R g_{\text{th}}^R}{D_C} - \frac{\Sigma_a^{(C)}}{D_C}\right]\phi_C = 0 \tag{8.71}$$

If we call the term in the brackets B^2

$$B^2 \equiv \frac{1}{D_C}[\nu_c\Sigma_f p_{\text{th}}^R g_{\text{th}}^R - \Sigma_a^{(C)}] \tag{8.72}$$

then $\nabla^2\phi_C = -B^2\phi_C$, and the relation for k reduces to the form

$$k = \frac{\nu_c\Sigma_f p_{\text{th}}^R g_{\text{th}}^R}{D_C B^2 + \Sigma_a^{(C)}} \tag{8.73}$$

If the system is *critical*, $k = 1$ with $\nu_c \equiv \nu$; therefore, we can compute k in the usual way; i.e.,

$$k = \frac{\nu}{\nu_c}$$

where ν_c is obtained from (8.73) by setting $k = 1$.

[1] The quantity ν_c denotes the neutron yield per fission required to maintain the system in steady state.

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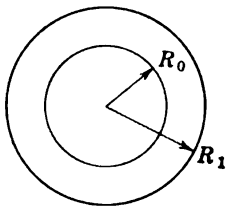
The quantity B^2 is computed from the criticality relation for the particular reflected configuration in question. We summarize these relations below for the three elementary geometries previously considered.

1. *Sphere:*

$$D_C(BR_0 \cot BR_0 - 1) = -D_R[\varkappa_R R_0 \coth \varkappa_R(\tilde{R}_1 - R_0) + 1]$$



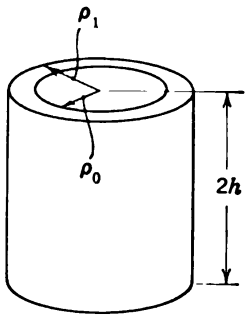
2. *Cylinder: side-reflected:*

$$\frac{D_C J_0'(\lambda_C \rho_0)}{J_0(\lambda_C \rho_0)} = \frac{D_R L_0'(\rho_0)}{L_0(\rho_0)}$$

$$L_0(\rho) \equiv I_0(\lambda_R \tilde{\rho}_1) K_0(\lambda_R \rho) - I_0(\lambda_R \rho) K_0(\lambda_R \tilde{\rho}_1)$$

$$B^2 = \lambda_C^2 + \left(\frac{\pi}{2\tilde{h}}\right)^2$$

$$\lambda_R^2 = \varkappa_R^2 + \left(\frac{\pi}{2\tilde{h}}\right)^2$$

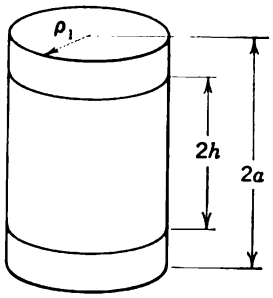


3. *Cylinder: end-reflected:*

$$D_C \mu_C \tan \mu_C h = D_R \mu_R \coth \mu_R(\tilde{a} - h)$$

$$\mu_R^2 = \varkappa_R^2 + \left(\frac{\nu_1}{\tilde{\rho}_1}\right)^2 \qquad B^2 = \mu_C^2 + \left(\frac{\nu_1}{\tilde{\rho}_1}\right)^2$$

$$\nu_1 = 2.405$$

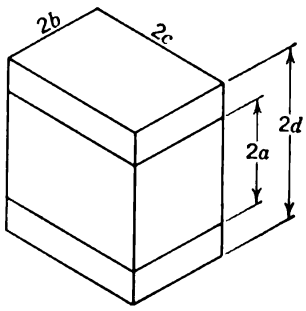


4. *Block: end-reflected:*

$$D_C \lambda_1 \tan \lambda_1 a = D_R \mu_1 \coth \mu_1(\tilde{d} - a)$$

$$\mu_1^2 = \varkappa_R^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2$$

$$B^2 = \lambda_1^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2$$



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We have noted that the probabilities p_{th}^R and g_{th}^R, in general, are dependent upon the properties of the reflector. It is possible in some instances, however, to use for these quantities the results of the bare reactor calculations. This is the case, for example, in large thermal systems of spherical geometry in which the slowing-down properties of the core and reflector are very nearly the same. Since systems of this type are of some practical importance, it is of interest to demonstrate this result.

Let us consider, therefore, the following general relation for the diffusion of thermal neutrons:

$$-D\nabla^2\phi(\mathbf{r}) + \Sigma_a\phi(\mathbf{r}) = \nu\int_C \Sigma_f\phi(\mathbf{r}')\,G(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' \tag{8.74}$$

where $\phi(\mathbf{r}) \equiv$ thermal flux at $\mathbf{r}$ in reactor, including all points in both core and reflector
$G(\mathbf{r};\mathbf{r}') \equiv$ probability that a neutron born at $\mathbf{r}'$ becomes thermal at $\mathbf{r}$ (8.75)

By our previous arguments $\nu\Sigma_f\phi(\mathbf{r}')\,d\mathbf{r}'$ is the number of neutrons appearing per unit time in $d\mathbf{r}'$ around $\mathbf{r}'$ at high (fission) energies. Thus the integral term is a generalized expression for the rate at which neutrons are entering the thermal group at $\mathbf{r}$ for any point in the reactor. Equation (8.74) is rigorously correct (within the limitations of the thermal-group diffusion model) if the proper kernel $G(\mathbf{r};\mathbf{r}')$ is chosen for the system under consideration. In general, $G(\mathbf{r};\mathbf{r}')$ depends upon the high-energy cross sections of the reactor materials and the geometry of the medium. We will not attempt here to choose an accurate kernel but will concentrate on finding a kernel which will be a good approximation in some useful cases.

Let us limit our analysis, then, to the spherical system mentioned above. As a first approximation we will assume that $G(\mathbf{r};\mathbf{r}')$ is given by the *infinite-medium slowing-down kernel* of Eq. (6.38):

$$G_\infty(\mathbf{r};\mathbf{r}') \equiv \frac{p_{th}e^{-|\mathbf{r}-\mathbf{r}'|^2/4\tau_{th}}}{(4\pi\tau_{th})^{\frac{3}{2}}} \tag{8.76}$$

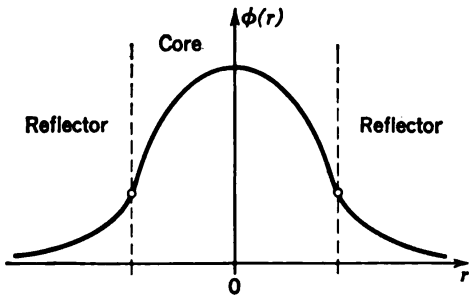

FIG. 8.11 Flux distribution in large thermal reactor.

It would be reasonable to suppose that the source function represented by the integral would be strongly peaked in the core region and that the thermal flux would be peaked also, as indicated in Fig. 8.11. This is consistent with our assumption that the slowing-down properties of the core and of the reflector are nearly the same. Thus the core is the primary source of thermal neutrons, and since the flux ϕ_C decays rapidly from the center of the reactor, little error

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will be made in extending the integral of Eq. (8.74) over all space when describing the flux distribution in the core; therefore, take

$$-D_C\nabla^2\phi_C(\mathbf{r}) + \Sigma_a^{(C)}\phi_C(\mathbf{r}) = \int_\infty \nu\Sigma_f\phi(\mathbf{r}')\,G_\infty(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' \tag{8.77}$$

Moreover, because of the relatively small thermal flux in the reflector and the rapid decay of the kernel G_∞ for large values of $|\mathbf{r} - \mathbf{r}'|$, we would expect that in the reflector region the integral term will be negligibly small. Thus we assume that the neutron balance in the reflector is given by

$$-D_R\nabla^2\phi_R(\mathbf{r}) + \Sigma_a^{(R)}\phi_R(\mathbf{r}) = 0$$

which is identical to Eq. (8.3*b*).

The integral of Eq. (8.77) may be evaluated by expanding the flux $\phi(\mathbf{r}')$ in a Taylor series about point $\mathbf{r}$; the first three terms are

$$\phi(\mathbf{r}') = \phi(\mathbf{r}) + (\mathbf{r} - \mathbf{r}')\cdot\nabla\phi(\mathbf{r}) + \tfrac{1}{2}[(\mathbf{r} - \mathbf{r}')\cdot\nabla]^2\phi(\mathbf{r}) + \cdots \tag{8.78}$$

The substitution of this expression into (8.77), along with (8.76) for G_∞, yields the following results for the three terms:

$$\begin{aligned}
\nu\Sigma_f\int_\infty \phi(\mathbf{r})\,G_\infty(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' &= \nu\Sigma_f p_{\text{th}}\phi(\mathbf{r}) \\
\nu\Sigma_f\int_\infty (\mathbf{r} - \mathbf{r}')\cdot\nabla\phi(\mathbf{r})\,G_\infty(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' &= 0 \\
\tfrac{1}{2}\nu\Sigma_f\int_\infty [(\mathbf{r} - \mathbf{r}')\cdot\nabla]^2\phi(\mathbf{r})\,G_\infty(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' &= \nu\Sigma_f\tau_{\text{th}}p_{\text{th}}\nabla^2\phi(\mathbf{r})
\end{aligned} \tag{8.79}$$

Since the point $\mathbf{r}$ is in the core, the fluxes on the right of these equations are core fluxes. The application of these results to the core equation yields

$$(D_C + \nu\Sigma_f p_{\text{th}}\tau_{\text{th}})\nabla^2\phi_C(\mathbf{r}) + [\nu\Sigma_f p_{\text{th}} - \Sigma_a^{(C)}]\phi_C(\mathbf{r}) = 0 \tag{8.80}$$

If we define

$$B^2 \equiv \frac{\nu\Sigma_f p_{\text{th}} - \Sigma_a^{(C)}}{D_C + \nu\Sigma_f p_{\text{th}}\tau_{\text{th}}} \equiv \frac{k_\infty - 1}{L_C^2 + k_\infty\tau_{\text{th}}} \tag{8.81}$$

where

$$k_\infty \equiv \frac{\nu\Sigma_f p_{\text{th}}}{\Sigma_a^{(C)}} \qquad L_C^2 \equiv \frac{D_C}{\Sigma_a^{(C)}} \tag{8.82}$$

then (8.80) reduces to the form (8.5*a*). Since the differential equations for the flux in both the core and reflector are now identical to the set (8.5), it follows that the multiplication constant is given by the expression

$$k = \frac{\nu\Sigma_f p_{\text{th}}}{\Sigma_a^{(C)}[1 + B^2(L_C^2 + k_\infty\tau_{\text{th}})]} \tag{8.83}$$

This equation would be expected to give good results in the case of a large thermal reactor with slowing-down properties much the same in the core as in the reflector. If this result is compared to (8.73), it may be

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seen that the analysis above has amounted to obtaining an estimate for g_{th}^R. That this is indeed a reasonable estimate may be established by reviewing the results obtained for the bare reactor, using the Fermi model (6.81). In that case

$$g_{\text{th}} \equiv e^{-B^2\tau_{\text{th}}} \simeq \frac{1}{1 + B^2\tau_{\text{th}}} \tag{8.84}$$

for $B^2\tau_{\text{th}} \ll 1$ (i.e., large reactors). If this form had been used directly in (8.73) for g_{th}^R, then

$$k \simeq \frac{\nu\Sigma_f p_{\text{th}}^R}{\Sigma_a^{(C)}(1 + L_C^2B^2)(1 + B^2\tau_{\text{th}})} \simeq \frac{\nu\Sigma_f p_{\text{th}}^R}{\Sigma_a^{(C)}[1 + B^2(L_C^2 + \tau_{\text{th}})]} \tag{8.85}$$

But this is the relation (8.83) when $k_\infty \simeq 1$ and p_{th}^R is taken as p_{th} for the core material; thus, the two results (8.83) and (8.85) are comparable. It is of interest to note also that (8.85) is in the form (6.113) if we identify $L_C^2 + \tau_{\text{th}}$ as the migration area M^2.

8.3 Reactors with Spherical Symmetry: Serber-Wilson Condition

a. Introduction. The accuracy of the methods used in the one-velocity model developed in the preceding section is limited by the interface conditions selected for the solution of the differential equations. The two conditions imposed in these calculations were the continuity of the neutron net current and the neutron flux. The purpose of the present section is to select a new set of boundary conditions which will give a more accurate representation of the neutron flux in the regions of importance, particularly in regions wherein the selected analytical expressions for the flux are known to be valid.

In this calculation, the continuity of net-current condition will be retained. This choice is made because the net current out of any region is proportional to the volume integral of the neutron density (or flux) over that region. Thus in computing the neutron leakage from a volume V having surface area A, one might use a relation of the type

$$\text{Leakage} = -\int_A D\nabla\phi \cdot d\mathbf{A} = -D\int_V \nabla^2\phi \, d\mathbf{r} \tag{8.86}$$

It is evident from this expression that so long as the computed flux ϕ gives a reasonably good picture of the spatial distribution of neutrons over most of the region, fairly large errors can be tolerated over small portions of the region, for example, near boundaries (see Fig. 8.12). The present argument, then, is based on the fact that the net-current condition is an integral-type relation and requires simply that a certain volume integral of the estimated flux approximate closely the integral for the exact calculation. The continuity-of-flux condition, on the other hand, although physically acceptable, is not always the best condition

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to use. This is especially true when using the diffusion-theory approximation for the flux distribution, namely, that the flux is well represented by an isotropic term and a term in the first derivative. Of course, this is not the case near outer boundaries and discontinuities in the material composition of the system. Thus the requirement that the diffusion-theory expressions for the flux be continuous in these regions can introduce large errors in estimating the current at an interface.

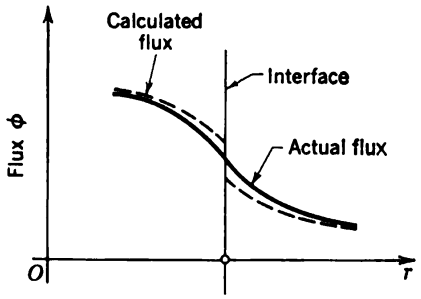

FIG. 8.12 Flux distribution around interface.

In the present treatment we discard the condition that the flux be continuous at an interface, and select instead a relation of the integral type (as in the case of the net-current condition). The boundary conditions to be applied here in place of the continuity conditions (1) and (2) of (8.4) are as follows:

(1) Continuity of net current at interface
(2) Expression for neutron flux must satisfy the transport equation at the center of the reactor (8.87)

The net-current condition (1) of (8.87) is the same as that previously used. Condition (2) of (8.87) replaces the continuity-of-flux condition and requires that the general solutions which are selected satisfy the accurate equations for the flux distribution in the region of greatest importance, namely, in the central region of the core. This is known as the *Serber-Wilson* condition and is an integral relation which is applied at the center of the reactor. Thus, discontinuities in the expression for the flux may very likely result at the interface; however, it is recognized that this is of no great consequence since the solutions obtained will not be valid in these regions, in any event (Fig. 8.12).

The method based on the Serber-Wilson condition is especially suited to the treatment of reactor systems which are spherically symmetric. We will develop the basic ideas involved in this method by analyzing, on the basis of the one-velocity model, the simple case of a spherical core surrounded by an infinite reflector. However, the method is generally applicable to finite systems,[1] to multiregion configurations, and to multigroup (energy) calculations.[2] These problems will not be discussed here. In order to demonstrate the effect on the criticality requirements of the

[1] A. H. Wilson, The General Properties of the Transport Equation and Its Uses in Determining the Multiplication in Bodies Having Spherical Symmetry, AERE-M/S-105A, Dec. 7, 1944.

[2] Bengt Carlson, "Multi-velocity Serber-Wilson Neutron Diffusion Calculations," Los Alamos Scientific Laboratory, Report LA-1391, Mar. 24, 1952, declassified.

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Serber-Wilson condition, the critical radius of a spherical reactor with an infinite reflector will be computed for various core and reflector compositions, and the results will be compared with the computations based on the method of Sec. 8.2b (using the continuity-of-flux condition).

b. One-velocity Transport Equation and Asymptotic Solutions. The development of the present method is based on the general one-velocity transport equation for nonstationary systems. For this calculation we introduce the following assumptions: (1) the core is a homogeneous multiplying medium with no external (nonfission) sources; (2) the reflector is a source-free homogeneous medium; and (3) scattering is isotropic in the laboratory system of coordinates. It was shown previously in Sec. 7.4c that the appropriate integrodifferential equation for a system of this type could be written in the form

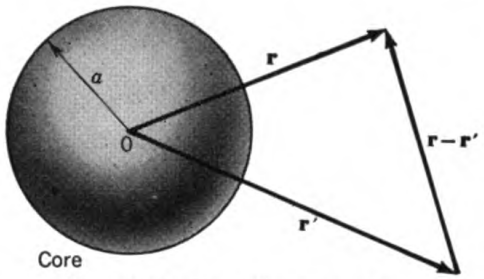

FIG. 8.13 Coordinate system.

$$\boldsymbol{\Omega} \cdot \nabla \phi(\mathbf{r},\boldsymbol{\Omega}) + \bar{\Sigma}(\mathbf{r})\, \phi(\mathbf{r},\boldsymbol{\Omega}) = \frac{1}{4\pi} \Sigma_t(\mathbf{r})\, c(\mathbf{r})\, \phi(\mathbf{r}) \qquad (7.245)$$

where

$$\phi(\mathbf{r}) = \int_V c(\mathbf{r}')\, \Sigma_t(\mathbf{r}')\, \phi(\mathbf{r}')\, G(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \qquad (8.88)$$

and the volume integral is to extend over all space. The quantity c is given by

$$c(\mathbf{r}) \equiv \frac{\nu\Sigma_f(\mathbf{r}) + \Sigma_s(\mathbf{r})}{\Sigma_t(\mathbf{r})} \qquad (7.236)$$

and the effective absorption cross section by

$$\bar{\Sigma}(\mathbf{r}) \equiv \Sigma_t(\mathbf{r}) + \frac{\alpha}{v} \qquad (7.246)$$

The kernel G which appears in the expression for $\phi(\mathbf{r})$ is defined

$$G(\mathbf{r};\mathbf{r}') \equiv \frac{e^{-\beta(\mathbf{r}';\mathbf{r})}}{4\pi|\mathbf{r} - \mathbf{r}'|^2} \qquad (8.89)$$

and β is given by (7.196) with Σ_a replaced by $\bar{\Sigma}$.

In the present analysis our interest is in the total flux distribution $\phi(\mathbf{r})$ throughout a system consisting of a spherical core of radius a and an infinite reflector (Fig. 8.13). For convenience in analysis, we place the origin of coordinates at the center of the core. Since this is a simple two-region problem, the integral of (8.88) is conveniently separated into the form:

$$\phi(\mathbf{r}) = c_C\Sigma_t^{(C)} \int_C \phi_C(\mathbf{r}')\, G(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' + c_R\Sigma_t^{(R)} \int_R \phi_R(\mathbf{r}')\, G(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \qquad (8.90)$$

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This relation gives the flux at *any* general point $\mathbf{r}$ in the system in terms of integrals involving the flux in both the core and the reflector. The notation $(\ )_C$ refers to properties of the core and $(\ )_R$ to properties of the reflector; $\int_C$ denotes integration over the core, and $\int_R$ denotes integration over the reflector volume. Also, from our previous definition of c, we have

$$c_C \equiv \frac{\nu\Sigma_f + \Sigma_s^{(C)}}{\Sigma_t^{(C)}} \qquad c_R \equiv \frac{\Sigma_s^{(R)}}{\Sigma_t^{(R)}} \tag{8.91}$$

The kernel $G(\mathbf{r};\mathbf{r}')$ is to take on the appropriate form, depending on the relative position of the field point $\mathbf{r}$ and the position of the integration variable $\mathbf{r}'$. Since we have assumed that the two regions of the system are homogeneous, the $\bar{\Sigma}$ which appears in the expression for β will be constant over each region, and β reduces to relations of the type

$$\beta(\mathbf{r}';\mathbf{r}) = \bar{\Sigma}|\mathbf{r} - \mathbf{r}'| \tag{8.92}$$

c. The Serber-Wilson Condition. The Serber-Wilson condition is established by requiring that the solutions selected for the flux distributions throughout the core and reflector satisfy rigorously the integral equation (8.90) at the origin. At this point in the system, (8.90) takes on an especially simple form by virtue of the fact that all integration paths extend along radial lines. Thus, using (8.92),

$$\phi_C(0) = c_C\Sigma_t^{(C)} \int_C \frac{\phi_C(\mathbf{r}')e^{-\bar{\Sigma}_C r'}\,d\mathbf{r}'}{4\pi r'^2} + c_R\Sigma_t^{(R)} \int_R \frac{\phi_R(\mathbf{r}')e^{-\bar{\Sigma}_C a - \bar{\Sigma}_R(r'-a)}\,d\mathbf{r}'}{4\pi r'^2} \tag{8.93}$$

We are dealing here with a spherically symmetric system, so that $\phi(\mathbf{r}) \equiv \phi(r)$, and $d\mathbf{r} = 4\pi r^2\,dr$. With these substitutions, (8.93) reduces to

$$\phi_C(0) = c_C\Sigma_t^{(C)} \int_0^a \phi_C(r)e^{-\bar{\Sigma}_C r}\,dr + c_R\Sigma_t^{(R)} \int_a^\infty \phi_R(r)e^{-\bar{\Sigma}_C a - \bar{\Sigma}_R(r-a)}\,dr \tag{8.94}$$

It will be convenient to separate the integration over the core into the form $\int_0^a = \int_0^\infty - \int_a^\infty$. Thus

$$\phi_C(0) = c_C\Sigma_t^{(C)} \left[\int_0^\infty \phi_C(r)e^{-\bar{\Sigma}_C r}\,dr - \int_a^\infty \phi_C(r)e^{-\bar{\Sigma}_C r}\,dr\right] + c_R\Sigma_t^{(R)} \int_a^\infty \phi_R(r)e^{-\bar{\Sigma}_C a - \bar{\Sigma}_R(r-a)}\,dr \tag{8.95}$$

For the functions $\phi_C(r)$ and $\phi_R(r)$ which appear in (8.95) we use first-order approximations of the solutions to the transport equation. The basis for this choice is derived from the following argument. It was shown previously in Sec. 7.4d that the elementary solution to the one-velocity transport equation for the source-free infinite multiplying

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medium was the function $e^{iB\cdot r}$, where the eigenvalue B is given by the transcendental equation (7.254). In the present case, it is seen that the substitution of this form into (8.88), with G given by (8.89) and β in terms of $\bar{\Sigma}$ in place of Σ_a, gives

$$\left(\frac{\Sigma_t}{B}\right)\tan^{-1}\left(\frac{B}{\bar{\Sigma}}\right) = \frac{1}{c} \tag{8.96}$$

We observe, however, that $e^{iB\cdot r}$ is also a solution of the diffusion equation for the multiplying medium

$$\nabla^2\phi + B^2\phi = 0 \qquad B \text{ real} \tag{5.161}$$

as demonstrated in Eq. (7.256) and the subsequent discussion. Thus the general solution of the integral equation (8.88), which can be constructed from the elementary solution [see, for example, Eq. (7.255)], will also satisfy the diffusion equation.[1] Now the only solution of the diffusion equation which is finite everywhere and which is spherically symmetric is

$$\phi(r) = \frac{\sin Br}{r} \tag{8.97}$$

This function should be a satisfactory solution to Eq. (8.88) at points far removed from the boundaries of the medium because in these regions the angular distribution of the neutron flux is nearly isotropic, and the diffusion-theory results will give an adequate representation of the spatial distribution of the neutrons. Thus $\sin Br/r$ is sometimes called the asymptotic solution to the transport equation.

The transcendental equation (8.96) may be written more symmetrically

$$\left(\frac{\bar{\Sigma}}{B}\right)\tan^{-1}\left(\frac{B}{\bar{\Sigma}}\right) = \frac{1}{\bar{c}} \tag{8.98}$$

wherein we have introduced the quantity $\bar{c}$ defined by

$$\begin{aligned}\bar{c} &\equiv \text{average number of neutrons produced per collision in the system described by (7.245)} \\ &= \frac{\nu\Sigma_f + \Sigma_s}{\bar{\Sigma}}\end{aligned} \tag{8.99}$$

Note that

$$\bar{c}\bar{\Sigma} = c\Sigma_t \tag{8.100}$$

It is seen from Eq. (8.98) that the requirement that B be real implies that $\bar{c} \geq 1$, which corresponds to a multiplying medium [see (7.236)]. (In the present context, multiplying means capable of supporting a self-sustaining chain reaction.) In the event that the medium is nonmultiplying, $\bar{c} < 1$ and B is imaginary. Then, it is convenient to define

[1] Wilson, *op. cit.*

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$B \equiv i\varkappa$, and Eq. (8.98) takes the form

$$\frac{\tanh^{-1}\kappa}{\kappa} = \frac{1}{\bar{c}} \qquad \text{with } \kappa \equiv \frac{\varkappa}{\bar{\Sigma}} \tag{8.101}$$

The corresponding expression for the flux is obtained from (5.161) and is seen to be

$$\nabla^2\phi - \varkappa^2\phi = 0 \qquad \varkappa \text{ real} \tag{8.102}$$

In the analysis which follows we use the solutions to Eqs. (5.161) and (8.102) to represent ϕ_C and ϕ_R, with B and $\varkappa$ defined by Eqs. (8.98) and (8.101).

It will be convenient in developing the present analysis to introduce another new quantity φ, which is related to c through the equation

$$\varphi(\mathbf{r}) \equiv c(\mathbf{r}) - 1 \tag{8.103}$$

Thus, physically, φ is the excess of neutrons, over the incident particle, which is released per collision. If we use the usual definition of Σ_t

$$\Sigma_t \equiv \Sigma_s + \Sigma_f + \Sigma_\gamma \tag{8.104}$$

then

$$\varphi = \frac{(\nu - 1)\Sigma_f - \Sigma_\gamma}{\Sigma_t} \tag{8.105}$$

Some limiting values of c and φ are given below:

Material	c	φ
Pure scatterer	1	0
Pure absorber (no fissions)	0	-1
Pure absorber (with fissions)	η	$\eta - 1$

where η has the customary definition. Note that $c > 1$, and $\varphi > 0$ only if the average number of neutrons produced per absorption (η) is greater than unity. These parameters may also be defined in terms of the effective absorption cross section $\bar{\Sigma}$. The appropriate definition for $\bar{c}$ was given previously in (8.99). The corresponding expression for the net neutron production per collision $\bar{\varphi}$ is then

$$\bar{\varphi} \equiv \bar{c} - 1 = \frac{\nu\Sigma_f + \Sigma_s - \bar{\Sigma}}{\bar{\Sigma}} \tag{8.106}$$

It follows from (8.105), (8.99), and (8.106) that

$$\bar{\varphi}\bar{\Sigma} = \varphi\Sigma_t - \frac{\alpha}{v} \tag{8.107}$$

The asymptotic form to be used in the core region is the solution (8.97) of Eq. (5.161). The corresponding solution for the reflector region

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is given by Eq. (8.102); the only solution to this equation which is spherically symmetric is the negative exponential $e^{-\varkappa r}/r$. Thus for the present analysis we select the following asymptotic expressions, denoted by the symbol ϕ^∞:

$$\phi_C^\infty \equiv \frac{\sin Br}{r} \qquad \phi_R^\infty \equiv \frac{Ae^{-\varkappa r}}{r} \tag{8.108}$$

where B is to be computed from (8.98) and $\varkappa$ from (8.101). These are conveniently summarized in the forms

$$\frac{\tan \omega}{\omega} = 1 + \bar{\varphi}_C \qquad \text{with } \tan \omega \equiv \frac{B}{\Sigma_C} \tag{8.109}$$

$$\frac{\kappa}{\tanh^{-1} \kappa} = 1 + \bar{\varphi}_R \qquad \text{with } \kappa \equiv \frac{\varkappa}{\Sigma_R} \tag{8.110}$$

Note that, if the material composition of the core and reflector is specified, the parameters B and $\varkappa$ may be obtained directly from the above transcendental relations. The quantities ω and κ have been extensively tabulated[1] as functions of φ_C and φ_R.

The expressions (8.108) are now applied to Eq. (8.95). First we note that ϕ_C^∞ must satisfy the integral equation (8.88); therefore

$$\phi_C^\infty(0) = c_C \Sigma_i^{(C)} \int_0^\infty \phi_C^\infty(r) e^{-\bar{\Sigma}_C r}\, dr \tag{8.111}$$

The substitution of the asymptotic solutions ϕ_C^∞ and ϕ_R^∞ into (8.95), along with the application of (8.111), yields the relation

$$c_C \Sigma_i^{(C)} e^{\bar{\Sigma}_C a} \int_a^\infty \phi_C^\infty(r) e^{-\bar{\Sigma}_C r}\, dr = c_R \Sigma_i^{(R)} e^{\bar{\Sigma}_R a} \int_a^\infty \phi_R^\infty(r) e^{-\bar{\Sigma}_R r}\, dr \tag{8.112}$$

[Note that the use of (8.111) allows us to cancel the term at the left of (8.95) with the first term at the right.] The equation (8.112) is the Serber-Wilson condition. It is based on the requirement that the solution for the flux satisfy the transport equation at the center of the reactor, and it replaces the condition of continuity of flux at the core-reflector interface.

The second boundary condition to be used is the requirement that the net current be continuous at the interface. For the present calculation, this condition is more conveniently stated in integral form; thus,

Net number of neutrons produced throughout the reactor = 0 (8.113)

This condition must be satisfied if the system is to be at steady state.

[1] See, for example, D. Sweeney, M. Goldstein, and B. Carlson, "Tables of Functions Associated with the Serber-Wilson Method for Neutron Diffusion Calculations in a Spherical Geometry," vol. I, LA-1364 (May 19, 1952), vol. II, LA-1365 (May 5, 1952), vol. III, LA-1366 (Apr. 30, 1952); also, K. M. Case, F. de Hoffmann, and G. Placzek, "Introduction to the Theory of Neutron Diffusion," pp. 56–57, Los Alamos Scientific Laboratory, 1953.

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However, it applies with equal validity to noncritical systems (nonstationary systems) since these systems can always be described by means of an equivalent steady-state problem through the use of the "effective-absorption" cross section $\bar{\Sigma}$ [cf. Eq. (7.245)]. If we use the parameters appropriate to a system with "time absorption," as implied in $\bar{\Sigma}$, Eq. (8.113) may be written

$$\int_V \bar{\varphi}(\mathbf{r})\, \bar{\Sigma}(\mathbf{r})\, \phi(\mathbf{r})\, d\mathbf{r} = 0 \tag{8.114}$$

This integral may be separated into the core and reflector terms; thus in terms of the solutions (8.108),

$$\bar{\varphi}_C \bar{\Sigma}_C \int_C \phi_C^\infty(\mathbf{r})\, d\mathbf{r} = -\bar{\varphi}_R \bar{\Sigma}_R \int_R \phi_R^\infty(\mathbf{r})\, d\mathbf{r} \tag{8.115}\star$$

This integral equation along with (8.112) provides the boundary conditions to be satisfied by the solutions ϕ^∞.

d. Criticality Equation. The criticality equation for the spherical reactor with infinite reflector is obtained by dividing the flux condition (8.112) by the current condition (8.115). Before performing this operation, we first evaluate the integrals involved in these expressions. In every case, the asymptotic solutions (8.108) for ϕ_C and ϕ_R will be used.

The first integral to be computed is that from the left-hand side of (8.115):

$$\int_C \phi_C^\infty(\mathbf{r})\, d\mathbf{r} = 4\pi \int_0^a \frac{r^2 \sin Br\, dr}{r} = \frac{4\pi}{B^2}(\sin aB - aB \cos aB) \tag{8.116}$$

From the right-hand side of (8.115) we obtain

$$\int_R \phi_R^\infty(\mathbf{r})\, d\mathbf{r} = 4\pi A \int_a^\infty re^{-\varkappa r}\, dr = \frac{4\pi A}{\varkappa^2}(a\varkappa + 1)e^{-\varkappa a} \tag{8.117}$$

and from the right-hand side of (8.112) we have

$$A \int_a^\infty \frac{e^{-\bar{\Sigma}_R r - \varkappa r}\, dr}{r} = A \int_{a(\varkappa + \bar{\Sigma}_R)}^\infty e^{-s} \frac{ds}{s} = AE_1(a\bar{\Sigma}_R + a\varkappa) \tag{8.118}$$

where E_1 is the exponential integral.[1] The term from the left-hand side of Eq. (8.112) gives [cf. (7.268)]

$$\int_a^\infty \frac{e^{-\bar{\Sigma}_C r} \sin Br\, dr}{r} = -\operatorname{Im} E_1(\bar{\Sigma}_C a + iBa) \tag{8.119}$$

where Im $E_1(z)$ denotes the imaginary part of $E_1(z)$.

The substitution of the above results into the relations (8.112) and (8.115), followed by the division of (8.112) by (8.115), yields the criti-

[1] Case, de Hoffmann, and Placzek, *op. cit.*

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cality condition

$$-\frac{c_C\Sigma_t^{(C)}e^{\bar{\Sigma}_C a}\ \mathrm{Im}\ E_1(\bar{\Sigma}_C a + iaB)}{\bar{\varphi}_C\bar{\Sigma}_C\dfrac{1}{a^2B^2}(\sin aB - aB\cos aB)} = -\frac{c_R\Sigma_t^{(R)}(a\varkappa)^2E_1(a\bar{\Sigma}_R + a\varkappa)e^{a(\bar{\Sigma}_R+\varkappa)}}{\bar{\varphi}_R\bar{\Sigma}_R(a\varkappa+1)} \tag{8.120}$$

This result was obtained by both Wilson and by R. Serber and gives the critical radius of the core for a specified core and reflector composition (or vice versa). We will discuss this relation in more detail shortly.

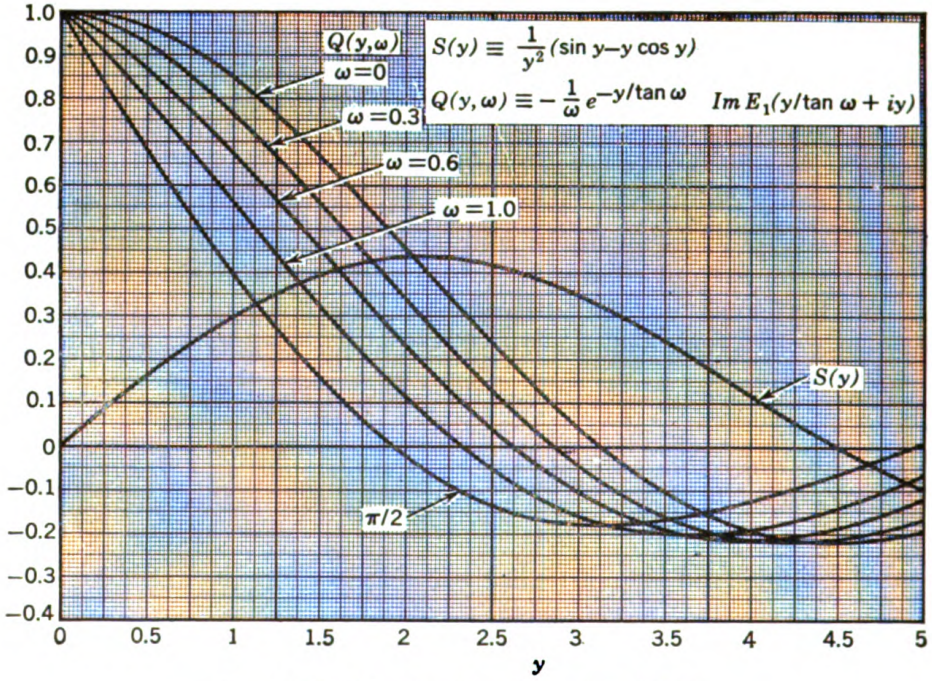

FIG. 8.14 The functions $Q(y,\omega)$ and $S(y)$.

In order to simplify the notation, we introduce the quantities

$$\bar{x} \equiv \bar{\Sigma}_C a \qquad \bar{\zeta} \equiv \bar{\Sigma}_R a$$
$$\text{and} \qquad aB \equiv a\bar{\Sigma}_C\tan\omega = \bar{x}\tan\omega \equiv y \tag{8.121}$$

Some of the functions on the left side of (8.120) have been tabulated.[1] It will be convenient, therefore, to rewrite the criticality equation in terms of the tabulated functions (see also Fig. 8.14):

$$S(y) \equiv \frac{1}{y^2}(\sin y - y\cos y)$$
$$Q(y,\omega) = -\frac{1}{\omega}e^{-y/\tan\omega}\ \mathrm{Im}\ E_1\left(\frac{y}{\tan\omega} + iy\right) \tag{8.122}$$

[1] *Ibid.*

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If these relations are used in (8.120), we obtain

$$\left[\frac{c_C\Sigma_t^{(C)}}{\bar{\varphi}_C\bar{\Sigma}_C}\right]\frac{\omega Q(y,\omega)}{S(y)} = -\left[\frac{c_R\Sigma_t^{(R)}}{\bar{\varphi}_R\bar{\Sigma}_R}\right]\frac{(\kappa\bar{\zeta})^2E_1[\bar{\zeta}(1+\kappa)]e^{\bar{\zeta}(1+\kappa)}}{\kappa\bar{\zeta}+1} \quad (8.123)\star$$

We have used here the definition for κ from (8.110), along with (8.121) to obtain

$$a\varkappa = \kappa\bar{\Sigma}_R a = \kappa\bar{\zeta} \quad (8.124)$$

Equation (8.123) is the criticality equation for the equivalent stationary reactor. The solution of this equation for a system of given size and composition yields the multiplication constant of the actual reactor. Before dealing with time-dependent systems, we first examine the problem of the stationary, or critical, reactor.

e. Critical Systems. The steady-state system may be described in terms of our present notation as the special case when the reactor period $1/\alpha$ is infinite. This corresponds to $\alpha \to 0$ and the effective absorption cross section $\bar{\Sigma} \to \Sigma_t$ [see Eq. (7.246)]. Formally, this change affects only the $\bar{\Sigma}$ terms which appear in the preceding analysis. In every instant $\bar{\Sigma}$ becomes Σ_t, and the results derived above take the forms summarized below. The criticality equation (8.123) becomes

$$\left(\frac{1+\varphi_C}{\varphi_C}\right)\frac{\omega Q(y,\omega)}{S(y)} = -\left(\frac{1+\varphi_R}{\varphi_R}\right)\frac{(\kappa\zeta)^2E_1[\zeta(1+\kappa)]e^{\zeta(1+\kappa)}}{\kappa\zeta+1} \quad (8.125)\star$$

The characteristic equations (8.109) and (8.110) are now

$$\frac{\tan\omega}{\omega} = 1+\varphi_C \qquad \text{with } \tan\omega \equiv \frac{B}{\Sigma_t^{(C)}} \quad (8.126)$$

and

$$\frac{\kappa}{\tanh^{-1}\kappa} = 1+\varphi_R \qquad \text{with} \kappa \equiv \frac{\varkappa}{\Sigma_t^{(R)}} \quad (8.127)$$

also, $x = \Sigma_t^{(C)}a \qquad \zeta = \Sigma_t^{(R)}a \qquad \text{and} \qquad y = x\tan\omega = aB \quad (8.128)$

Equation (8.125) is the criticality equation for a reactor operating at steady state. In its present form, this equation relates the properties of the core (left side of the equation) to the properties of the reflector (right side). Thus, given the properties of the reflector, (8.125) determines the critical fuel concentration (or size) of the core. In making computations with this relation, it will be convenient to call the core term $T_C(x,\varphi_C)$ and the reflector term $T_R(\zeta,\varphi_R)$. Note that ω is determined by φ_C [according to (8.126)] and that κ is determined by φ_R (8.127), so that the only variables which remain are x and ζ, once the φ's (i.e., cross sections) have been specified. The criticality condition is that

$$T_C(x,\varphi_C) = T_R(\zeta,\varphi_R) \quad (8.129)$$

Thus, given a value of ζ, T_R is known. The problem, then, is to find x such that $T_C = T_R$. If the core composition $\Sigma_t^{(C)}$ is specified, this value

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of x gives the critical radius a from relation (8.128); if the magnitude of a is specified, (8.128) gives the composition $\Sigma_t^{(C)}$ of the core.

The solution of Eq. (8.125) has been carried out in parametric form, and the results are presented in graphs suitable for criticality computations for a range of reactor compositions of practical interest. Figure 8.15 is a plot of $T_C(x)$ for various values of φ_C, and Fig. 8.16 is a plot of $T_R(\zeta)$ for various values of φ_R. It should be noted that x and ζ are related by

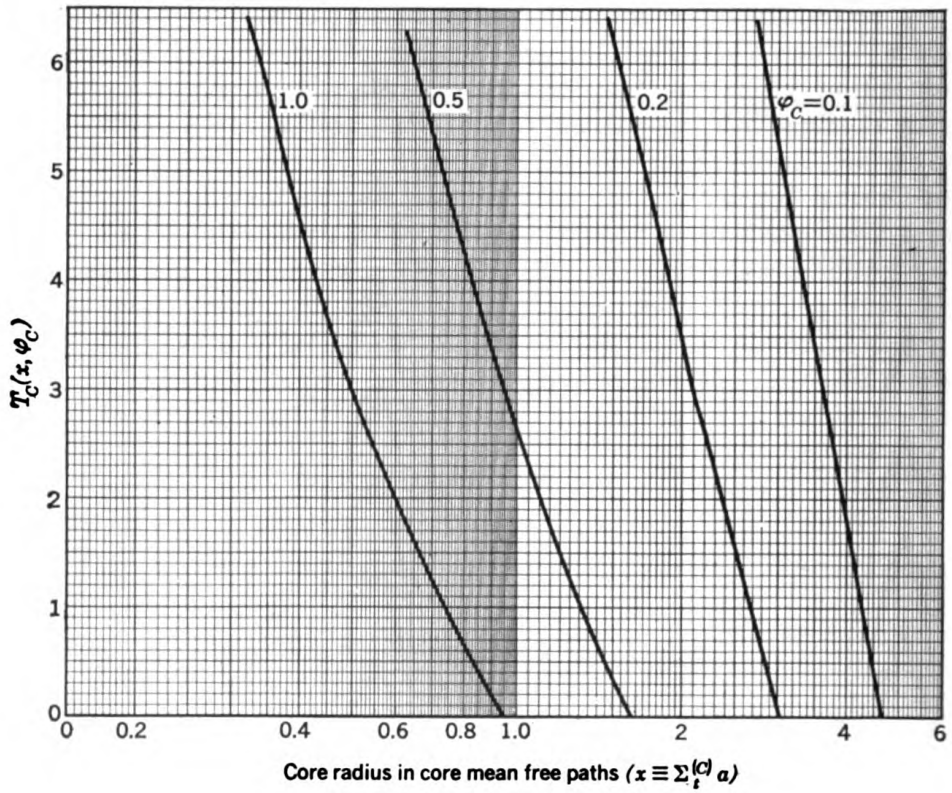

FIG. 8.15 Core properties.

the ratio of the total cross section in the core to the total cross section in the reflector; thus

$$x \equiv \theta\zeta = \frac{\Sigma_t^{(C)}}{\Sigma_t^{(R)}}\zeta \tag{8.130}$$

The application of these figures to an actual reactor computation proceeds in the following manner: (1) Compute φ_C, φ_R, and θ from the properties of the reactor. (2) Select a value of ζ and from Fig. 8.16 find $T_R(\zeta,\varphi_R)$. (3) By using this value for T_C, select the value of x from the appropriate curve (for the given φ_C) in Fig. 8.15. (4) See if this value of x and the assumed value of ζ satisfy Eq. (8.130); if they do not, repeat the process until the selected value of ζ and the corresponding value of x do so.

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The critical core radius x (in mean free paths) has been computed for some combinations of φ_C and φ_R which may be encountered in reactor design. These data are presented as a function of θ, the core-reflector cross section ratio, in Fig. 8.17. For these cases, the critical size can be obtained directly from this figure if the core and reflector compositions are given. For other combinations of φ_C and φ_R, the procedure outlined above can always be applied.

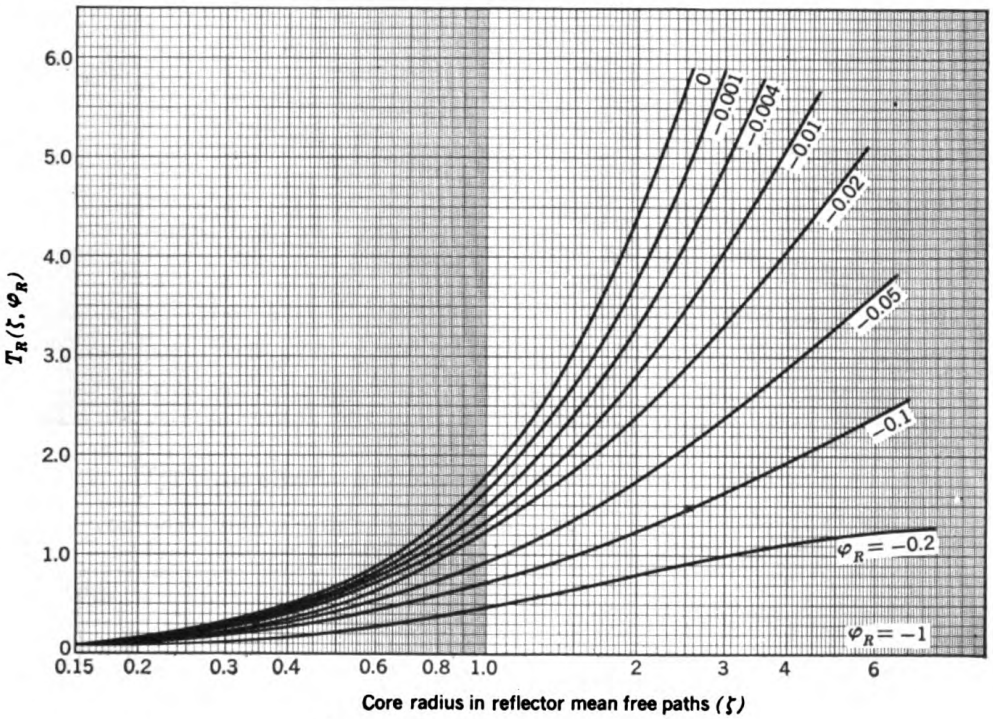

FIG. 8.16 Reflector properties.

The results based on the Serber-Wilson method have been compared with those obtained from the continuity-of-flux condition. The critical radii obtained by using the two methods are plotted in Fig. 8.18 as a function of θ for the case $\varphi_C = 0.2$, $\varphi_R = 0$. The Serber-Wilson calculations were based on Eq. (8.125), and the continuity-of-flux condition was based on (8.11). The appropriate form of (8.11) in the present notation is

$$\beta_C x \cot \beta_C x = 1 - \theta - \beta_R x \tag{8.131}$$

where $$\beta_C^2 \equiv 3\varphi_C \qquad \text{and} \qquad \beta_R^2 \equiv -3\varphi_R \tag{8.132}$$

The flux distributions for the case $\varphi_C = 0.2$, $\varphi_R = 0$, and $\theta = 0.8$ are shown in Fig. 8.19. It is of interest to note the discontinuity in the flux due to the application of the Serber-Wilson condition. Also, note that the critical mass computed by the continuity-of-flux condition exceeds the Serber-Wilson estimate by as much as 40 per cent for small reactors.

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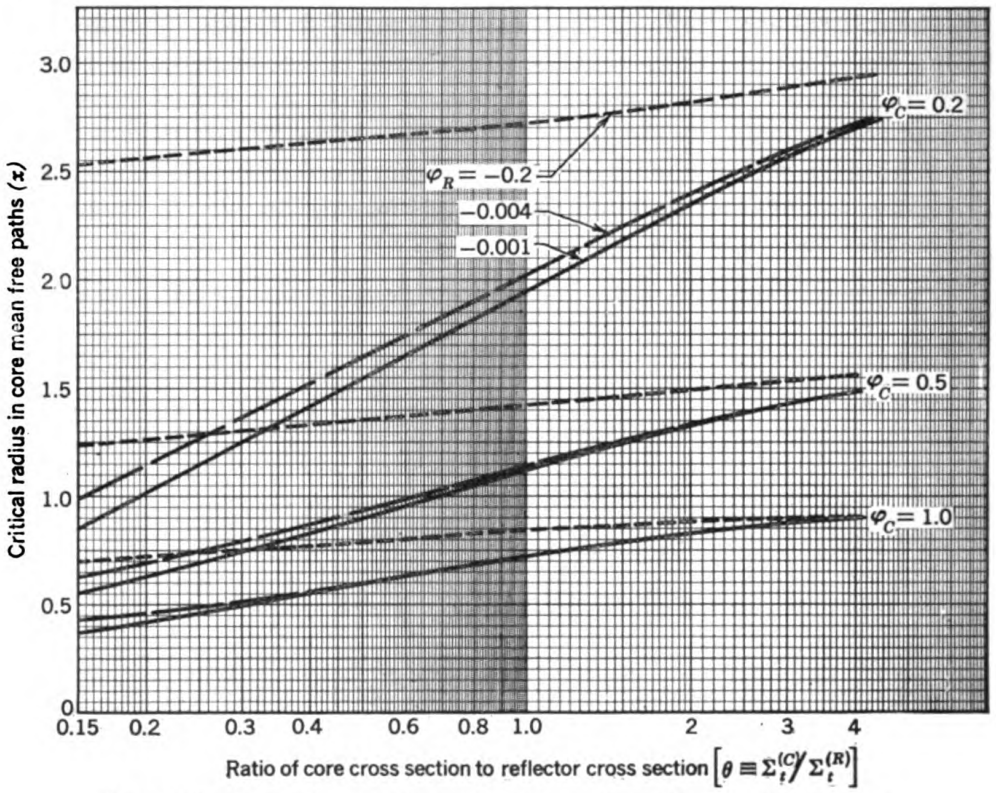

FIG. 8.17 Critical radius for spherical reactor with infinite reflector.

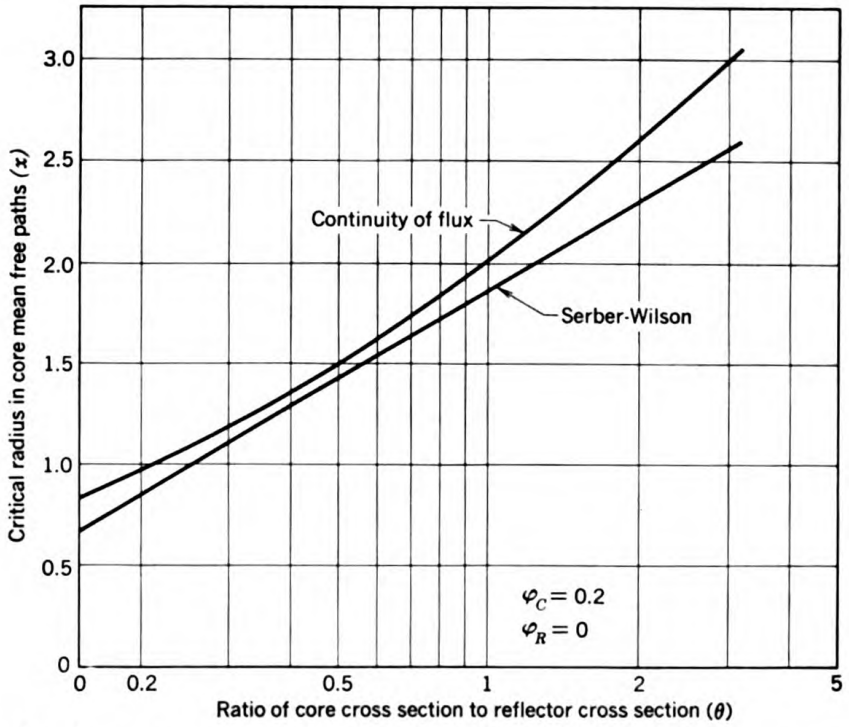

FIG. 8.18 Comparison of results from Serber-Wilson condition and from continuity-of-flux condition.

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f. Noncritical Systems. We return now to examine the analysis of systems which are not at steady state. The neutron distribution in these cases is described by the integrodifferential equation (7.243). It was noted in Sec. 7.4c that a solution to this equation could be written in the

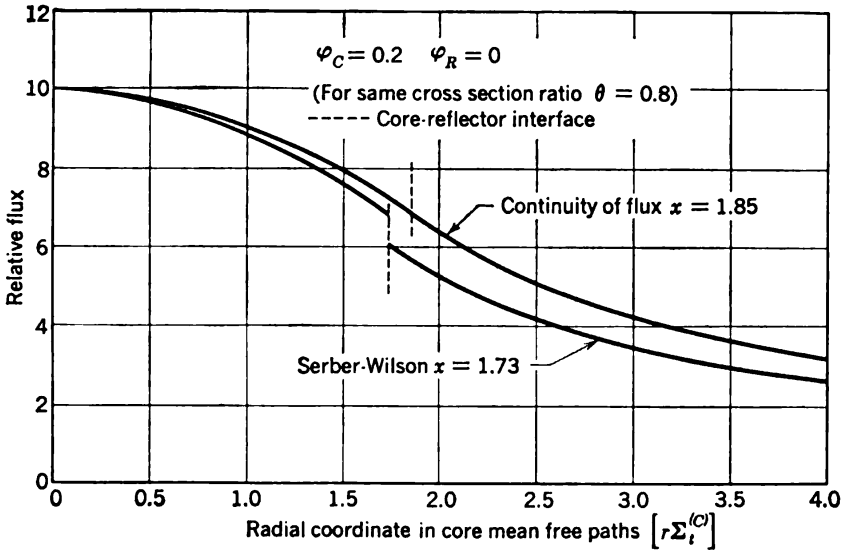

FIG. 8.19 Flux distributions.

form (7.244). The corresponding expression for the total flux is obtained by integrating (7.244) over all Ω; thus

$$\phi(\mathbf{r},t) = \phi(\mathbf{r})e^{\alpha t} \tag{8.133}$$

where the period $1/\alpha$ is yet to be determined. In dealing with the nonstationary system we found it convenient to define an equivalent steady-state system by introducing the effective cross section $\tilde{\Sigma}$. The relation which determines this cross section, and consequently the reactor period, is Eq. (8.123). Our purpose in the present calculation is to obtain a procedure for computing α from (8.123) and, subsequently, the multiplication constant k of the reactor.

It will be convenient at this point to work, not with α, but with another parameter γ which is related to α through the equation

$$\gamma \equiv \frac{\alpha}{v\Sigma_t} \tag{8.134}$$

The effective absorption cross section can be written in terms of γ by substituting the above expression into (7.246). Thus, for the core

$$\tilde{\Sigma}_C = \Sigma_t^{(C)}(1 + \gamma) \qquad \text{or} \qquad \frac{\tilde{\Sigma}_C}{\Sigma_t^{(C)}} = \frac{\tilde{x}}{x} = 1 + \gamma \tag{8.135}$$

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For the reflector, using (8.130) for θ,

$$\begin{aligned} \bar{\Sigma}_R &= \Sigma_i^{(R)}\left[1 + \frac{\gamma\Sigma_i^{(C)}}{\Sigma_i^{(R)}}\right] = \Sigma_i^{(R)}(1 + \gamma\theta) \\ \bar{\zeta} &= \zeta(1 + \gamma\theta) \end{aligned} \tag{8.136}$$

We can now write the cross-section ratios which appear in (8.123) in terms of γ. Observe first that, from (8.107),

$$\begin{aligned} \bar{\varphi}_C\bar{\Sigma}_C &= \Sigma_i^{(C)}(\varphi_C - \gamma) \\ \bar{\varphi}_R\bar{\Sigma}_R &= \Sigma_i^{(R)}(\varphi_R - \gamma\theta) \end{aligned} \tag{8.137}$$

Thus the cross-section ratios in (8.123) take the form

$$\frac{c_C\Sigma_i^{(C)}}{\bar{\varphi}_C\bar{\Sigma}_C} = \frac{1 + \varphi_C}{\varphi_C - \gamma} \quad \text{and} \quad \frac{c_R\Sigma_i^{(R)}}{\bar{\varphi}_R\bar{\Sigma}_R} = \frac{1 + \varphi_R}{\varphi_R - \gamma\theta} \tag{8.138}$$

The parameters ω and κ may also be written in terms of γ; the appropriate relations are easily found to be

$$\frac{\tan\omega}{\omega} = \frac{1 + \varphi_C}{1 + \gamma} \quad \text{and} \quad \frac{\kappa}{\tanh^{-1}\kappa} = \frac{1 + \varphi_R}{1 + \gamma\theta} \tag{8.139}$$

With these results (8.123) may be written

$$\left(\frac{1 + \varphi_C}{\varphi_C - \gamma}\right)\frac{\omega Q(y,\omega)}{S(y)} = \left(\frac{1 + \varphi_R}{\gamma\theta - \varphi_R}\right)\frac{(\kappa\bar{\zeta})^2 E_1[\bar{\zeta}(1 + \kappa)]e^{\bar{\zeta}(1+\kappa)}}{\kappa\bar{\zeta} + 1} \tag{8.140}\star$$

It should be noted that, for the steady-state condition, $\gamma \to 0$, and this equation reduces to the form (8.125).

The problem of computing the multiplication constant k of a noncritical reactor entails the solution of (8.140) for the value of γ which satisfies the specified geometric and material properties of the system in question. The multiplication constant can then be computed from the number γ by a simple procedure, as will be demonstrated shortly. In the usual problem of interest the size and composition of the reactor are specified; thus $x = \Sigma_i^{(C)}a$, $\zeta = \Sigma_i^{(R)}a$, φ_C and φ_R are known, and from these θ can be computed by means of the relation $\theta = x/\zeta$. The determination of γ involves a straightforward trial-and-error procedure for the solution of (8.140). The procedure is first to select a γ and compute ω and κ from Eqs. (8.139). Next, the values of y and $\bar{\zeta}$ are found from Eqs. (8.135) and (8.136); note that

$$\bar{x} = x(1 + \gamma) = \frac{y}{\tan\omega} \tag{8.141}$$

These values of y and $\bar{\zeta}$ are then inserted in (8.140), and, if they satisfy the equation, the selected value of γ is the solution for the system in

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question. A sample calculation based on this scheme is included at the end of the present section.

An expression for the usual multiplication constant k may be derived in terms of the parameter γ. We start with the core equation of the equivalent steady-state system [cf. (7.245)].

$$\Omega \cdot \nabla \phi_C(\mathbf{r},\Omega) + \Sigma_C \phi_C(\mathbf{r},\Omega) = \frac{1}{4\pi} \Sigma_i^{(C)} c_C \phi_C(\mathbf{r}) \tag{8.142}$$

and integrate over all directions Ω and over the core volume. The result is

$$\int_C d\mathbf{r} \int_\Omega \Omega \cdot \nabla \phi_C(\mathbf{r},\Omega)\, d\Omega + \Sigma_C \Phi_C = c_C \Sigma_i^{(C)} \Phi_C \tag{8.143}$$

where

$$\Phi_C \equiv \int_C \phi_C(\mathbf{r})\, d\mathbf{r} \tag{8.144}$$

Now the multiplication constant is defined by

$$k \equiv \frac{\text{total production } (n + 1\text{st generation})}{\text{total population } (n\text{th generation})} = \frac{\text{production}}{\text{leakage} + \text{absorption}} \tag{8.145}$$

For the actual reactor *not* at steady state, $k \neq 1$, and (8.145) takes the form

$$k = \frac{c_C \Sigma_i^{(C)} \Phi_C}{\text{leakage} + \Sigma_i^{(C)} \Phi_C}$$

If we write the total absorptions in terms of Σ_C by using (7.246), then

$$k = \frac{c_C \Sigma_i^{(C)} \Phi_C}{\text{leakage} + \Sigma_C \Phi_C - \gamma \Sigma_i^{(C)} \Phi_C}$$

The sum of the first two terms in the denominator of this expression is identically $c_C \Sigma_i^{(C)} \Phi_C$ [cf. (8.143)]; therefore,

$$k = \frac{c_C}{c_C - \gamma} \tag{8.146}\star$$

A sample calculation in which this method was used is given below. The computations are for a spherical reactor with infinite reflector having the following specifications:

$$\varphi_C = 0.3 \qquad \varphi_R = -0.03$$

$$x \equiv \Sigma_i^{(C)} a = 3.0 \qquad \zeta \equiv \Sigma_i^{(R)} a = 3.3 \qquad \theta \equiv \frac{x}{\zeta} = 0.91$$

The problem was to determine the multiplication constant of this system. The values of the various functions appearing in (8.140) have been computed for three selected values of γ, and the results are listed in

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Table 8.2. By interpolation it was found that $\gamma = 0.1455$. The corresponding value of k is found from (8.146); it is $k = 1.127$.

TABLE 8.2 COMPUTATION OF THE MULTIPLICATION CONSTANT

Function	$\gamma = 0.130$	$\gamma = 0.142$	$\gamma = 0.150$
ω	0.617	0.596	0.579
κ	0.599	0.615	0.620
$\bar{x}$	3.39	3.43	3.45
y	2.41	2.32	2.25
$Q(y,\omega)$	0.0674	0.1066	0.1375
$S(y)$	0.424	0.425	0.433
ξ	3.69	3.73	3.75
$E_1[\xi(1+\kappa)]$	4.04×10^{-4}	3.5×10^{-4}	3.29×10^{-4}
$T_R(\zeta,\varphi_R)$	1.468	1.405	1.360
$T_C(\bar{x},\varphi_C)$	0.748	1.230	1.590
T_C/T_R	0.510	0.895	1.168

8.4 The Two-group Model

a. Reactor Equations. The analytical model of a reactor which describes the neutron population as two separate groups of diffusing neutrons, one fast and one thermal, has a great deal of usefulness in reactor calculations. To begin with, it is sufficiently simple from an analytical and computational standpoint to permit calculations to be made rapidly. Moreover, the relative simplicity of the method allows one to trace rather easily the various factors which influence the physical features of the system as predicted by the model. Also, for many reactors, particularly large thermal reactors, the method gives acceptable first estimates of the important characteristics. Finally, there is sufficient latitude allowed in the choice of certain parameters which appear in the calculation that any given "family" of reactor types can be analyzed with increasing accuracy as experience is gained in the selection of the best values for these parameters. This feature becomes increasingly important as the engineering practicality of certain reactor types is established and experimental information becomes available for comparison with the predictions of the model.

Physically, the two-group model assumes that the behavior of the fast neutrons in a reflected reactor can be represented by a single diffusion equation (in each region) in terms of suitably selected fast-neutron cross sections. The thermal neutrons are represented by a second group in the usual way. Thus in the application of the model to the general multiregion reactor, two differential equations are introduced for each region, one to describe the thermal group and one to describe the fast group. The solutions to these equations in each region are coupled to

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the solutions in adjacent regions by appropriate boundary conditions for each group, along with requirements on the solutions at the center of the reactor and at the outer boundary. The sources for the thermal group are in each region proportional to the fast flux, and in those regions containing fissionable material the sources for the fast group are proportional to the thermal flux. The following treatment is focused on the analytical statement and solution of a particular case, the two-region reactor with outer region nonmultiplying. The methods developed for this case are easily extended (in principle) to more general situations.

The analytical statement of the two-region system is as follows. Consider first the diffusion equations for the fast and thermal neutrons in the core:

$$\begin{aligned} &\textit{Fast group:} \quad -D_1\nabla^2\phi_1(\mathbf{r}) + \Sigma_R^{(1)}\phi_1(\mathbf{r}) = \nu\Sigma_f\phi_2(\mathbf{r}) \qquad (a) \\ &\textit{Thermal group:} \quad -D_2\nabla^2\phi_2(\mathbf{r}) + \Sigma_a^{(2)}\phi_2(\mathbf{r}) = p_C\Sigma_R^{(1)}\phi_1(\mathbf{r}) \qquad (b) \end{aligned} \tag{8.147}\star$$

The quantity $\Sigma_R^{(1)}$ which appears in (a) is to be chosen so that $\Sigma_R^{(1)}\phi_1(\mathbf{r})$ is a good estimate of the slowing-down density of fast neutrons out of the fast group at $\mathbf{r}$. The source of fast neutrons is taken to be the neutrons produced by fissions caused by thermal neutrons $\nu\Sigma_f\phi_2(\mathbf{r})$. The parameters D_2 and $\Sigma_a^{(2)}$ are the thermal-diffusion coefficient and thermal-absorption cross section in the core, respectively, computed in the usual way. The source term for the thermal neutron equation (b) is taken to be the number of neutrons slowing down out of the fast group, reduced by the number lost in actual absorption during slowing down. We define

$$p_C \equiv \frac{\text{fraction of neutrons leaving fast group which reach thermal (in core)}}{} \tag{8.148}$$

and assume that p_C is given by the resonance-escape probability in the core. It should be noted that this selection is, to a certain extent, arbitrary, since there is still the choice of $\Sigma_R^{(1)}$ to be made. Thus, we have taken (8.148) as a convenient definition and reserved all necessary adjustments in specifying the thermal-source term $p_C\Sigma_R^{(1)}\phi_1$ for the quantity $\Sigma_R^{(1)}$.

For the reflector we assume the following set of equations:

$$\begin{aligned} &\textit{Fast group:} \quad -D_3\nabla^2\phi_3(\mathbf{r}) + \Sigma_R^{(3)}\phi_3(\mathbf{r}) = 0 \qquad (c) \\ &\textit{Thermal group:} \quad -D_4\nabla^2\phi_4(\mathbf{r}) + \Sigma_a^{(4)}\phi_4(\mathbf{r}) = p_R\Sigma_R^{(3)}\phi_3(\mathbf{r}) \qquad (d) \end{aligned} \tag{8.149}\star$$

The parameter $\Sigma_R^{(3)}$ corresponds to $\Sigma_R^{(1)}$ for the core. Both of these quantities are called the fast *removal cross sections.* Note that the form (c) explicitly assumes a nonmultiplying reflector material, since a zero source of fast neutrons has been taken in the reflector. As in the core, D_4 and $\Sigma_a^{(4)}$ are thermal properties of the reflector calculated in the usual way;

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the source term $p_R\Sigma_R^{(3)}\phi_3$ is entirely analogous to the corresponding term discussed above for the core. The quantities D_1 and D_3 represent the fast-diffusion coefficients in the core and reflector, respectively; these must also be chosen in some way to represent the entire fast range.

b. Group Constants. We can obtain some idea of how one might go about identifying and computing the fast-group constants from a study of two simpler multiplying systems. Consider first an infinite multiplying medium. In this case there is no spatial variation in the neutron flux, either fast or thermal, and the set of Eqs. (8.147) is reduced to the form

$$\Sigma_R^{(1)}\phi_1 = \nu\Sigma_f\phi_2 \qquad \Sigma_a^{(2)}\phi_2 = p_C\Sigma_R^{(1)}\phi_1 \tag{8.150}$$

If we take the ratio of these two equations, the constants ϕ_1 and ϕ_2 cancel out, and the resulting expression may be written

$$\frac{\nu\Sigma_f p_C}{\Sigma_a^{(2)}} = 1 \tag{8.151}$$

A comparison of this relation and the appropriate criticality equation for this system which was obtained in Chap. 4, Eq. (4.274), wherein $\epsilon \equiv 1$, reveals that the quantity p_C may be correctly identified as the resonance-escape probability to thermal. This was previously indicated.

The removal cross sections may be identified from a similar analysis for the bare multiplying system. It follows from our study of the Fermi age model that the spatial variation in the neutron flux is of the same functional form at *all* energies. Thus we may take

$$\phi_1(\mathbf{r}) \equiv \phi_1 F(\mathbf{r}) \qquad \text{and} \qquad \phi_2(\mathbf{r}) = \phi_2 F(\mathbf{r}) \tag{8.152}$$

where $F(\mathbf{r})$ satisfies the differential equation $\nabla^2 F + B^2 F = 0$ and B^2 is given by the various relations listed in Table 5.3. In the usual way, we have required that both the fast and thermal fluxes vanish on the extrapolated boundary surfaces of the reactor; the extrapolation distance is assumed to be the same for the fast and thermal groups.

The application of the above relations to Eqs. (8.147) yields the set

$$D_1B^2\phi_1 + \Sigma_R^{(1)}\phi_1 = \nu\Sigma_f\phi_2 \qquad D_2B^2\phi_2 + \Sigma_a^{(2)}\phi_2 = p_C\Sigma_R^{(1)}\phi_1 \tag{8.153}$$

As before we take the ratio of these equations and upon canceling out the constants ϕ_1 and ϕ_2 obtain

$$\frac{\nu\Sigma_f}{D_2B^2 + \Sigma_a^{(2)}} = \frac{D_1B^2 + \Sigma_R^{(1)}}{p_C\Sigma_R^{(1)}} \tag{8.154}$$

This may be written in the form

$$\frac{\nu\Sigma_f p_C}{\Sigma_a^{(2)}(1 + L_2^2B^2)(1 + L_1^2B^2)} = 1 \qquad \text{with } L_1^2 \equiv \frac{D_1}{\Sigma_R^{(1)}} \tag{8.155}$$

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If we compare this result with the Fermi age analysis of the bare reactor (6.78) for the case wherein there are no fast fissions ($\epsilon \equiv 1$), then we can immediately identify the factor $(1 + L_1^2B^2)^{-1}$ as the fast nonleakage probability; thus,

$$(1 + L_1^2B^2)^{-1} \equiv e^{-B^2\tau_{th}} \tag{8.156}$$

For a large reactor $e^{-B^2\tau_{th}} \simeq (1 + B^2\tau_{th})^{-1}$, and we note, then, that

$$L_1^2 \simeq \tau_{th} \qquad \text{and} \qquad \Sigma_R^{(1)} = \frac{D_1}{\tau_{th}} \tag{8.157}$$

This is one way in which one could define the removal cross sections Σ_R. It would certainly be an adequate first approximation. As experience is gained from the comparison of theory and experiment, better estimates can be made for this parameter.

The fast-diffusion coefficients may be computed from a suitable expression for the neutron leakage. Thus we take

$$\bar{D} \equiv \frac{\int D(u)\, \nabla\phi(\mathbf{r},u)\, du}{\int \nabla\phi(\mathbf{r},u)\, du} \tag{8.158}$$

If we assume that, in first approximation, the neutron-energy spectrum is the same for all $\mathbf{r}$, and in fact varies as $1/E$, then it follows from the above expression that an acceptable definition of D_1 would be

$$D_1 = \frac{1}{u_{th}} \int_0^{u_{th}} D(u)\, du \tag{8.159}$$

Finally, from (8.157),

$$\Sigma_R^{(1)} = \frac{D_1}{\tau_{th}} = \frac{1}{u_{th}\tau_{th}} \int_0^{u_{th}} D(u)\, du \tag{8.160}\star$$

In a medium in which the scattering cross section varies only slightly with lethargy and in which the fast-absorption cross section is negligible in comparison with the fast-scattering cross section, (8.160) may be approximated as

$$\Sigma_R^{(1)} \simeq \frac{\xi}{\lambda_s u_{th}} \tag{8.161}$$

Thus the "mean free path to removal" from the fast group is seen to be

$$\lambda_R^{(1)} \equiv \frac{1}{\Sigma_R^{(1)}} = \frac{\lambda_s u_{th}}{\xi} = \begin{pmatrix}\text{mean free path}\\ \text{for scattering}\end{pmatrix}\begin{pmatrix}\text{average number of}\\ \text{collisions to thermal}\end{pmatrix} \tag{8.162}$$

[see Eq. (4.47)]. The parameter $\Sigma_R^{(1)}$ is usually calculated by choosing a fast-diffusion coefficient for the entire group and dividing by the age to thermal as in (8.160). Note that (8.147*a*) may be written directly in terms of the slowing-down density $q \equiv \Sigma_R^{(1)}\phi_1$; thus,

$$-\tau_{th}\nabla^2 q + q = \nu\Sigma_f\phi_2 \tag{8.163}$$

where we have used (8.160).

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c. General Solution. Consider now the analysis of the reflected reactor as described by the differential equations (8.147) and (8.149); it is understood that the parameters which appear in these equations are to be chosen by some means such as that outlined above. The general solutions to these equations, for each region of the reactor, will be given as a linear combination of two known functions. In selecting these functions we will be guided by our previous experience with the one-velocity model. The proper weighting factors (coefficients) for the two functions in each combination will be determined by the requirement that each general solution satisfy the appropriate differential equation from the set (8.147) and (8.149) as well as the various boundary and symmetry conditions imposed on the system.

We begin with the differential equations for the core. Let $Z(\mathbf{r})$ and $W(\mathbf{r})$ be two functions such that the following linear combinations give the fast and thermal fluxes in the core:

$$\begin{aligned} \phi_1(\mathbf{r}) &\equiv A_1 Z(\mathbf{r}) + C_1 W(\mathbf{r}) \\ \phi_2(\mathbf{r}) &\equiv A_2 Z(\mathbf{r}) + C_2 W(\mathbf{r}) \end{aligned} \tag{8.164}$$

We define the functions Z and W as the solutions to the equations

$$\nabla^2 Z(\mathbf{r}) + B_1^2 Z(\mathbf{r}) = 0 \qquad \nabla^2 W(\mathbf{r}) + B_2^2 W(\mathbf{r}) = 0 \tag{8.165}$$

The parameters B_1 and B_2 are to be chosen so that the expressions (8.164) for ϕ_1 and ϕ_2 are acceptable solutions of Eqs. (8.147). Thus we substitute Eqs. (8.164) into (8.147) and apply (8.165). The result may be written in the form:

$$\begin{aligned} &Z(\mathbf{r})[A_1 D_1 B_1^2 + \Sigma_R^{(1)} A_1 - \nu\Sigma_f A_2] \\ &\qquad + W(\mathbf{r})[C_1 D_1 B_2^2 + C_1 \Sigma_R^{(1)} - C_2 \nu\Sigma_f] = 0 \\ &Z(\mathbf{r})[A_2 D_2 B_1^2 + \Sigma_a^{(2)} A_2 - p_C \Sigma_R^{(1)} A_1] \\ &\qquad + W(\mathbf{r})[C_2 D_2 B_2^2 + C_2 \Sigma_a^{(2)} - p_C C_1 \Sigma_R^{(1)}] = 0 \end{aligned} \tag{8.166}$$

Since in general $B_1 \neq B_2$, $Z(\mathbf{r})$ and $W(\mathbf{r})$ are independent functions. It follows, then, that the above equations can be satisfied only if each coefficient is identically zero. This requirement yields the relations

$$\begin{aligned} A_1 &= A_2 \left[\frac{\nu\Sigma_f}{D_1 B_1^2 + \Sigma_R^{(1)}} \right] = A_2 \left[\frac{D_2 B_1^2 + \Sigma_a^{(2)}}{p_C \Sigma_R^{(1)}} \right] \\ C_2 &= C_1 \left[\frac{D_1 B_2^2 + \Sigma_R^{(1)}}{\nu\Sigma_f} \right] = C_1 \left[\frac{p_C \Sigma_R^{(1)}}{D_2 B_2^2 + \Sigma_a^{(2)}} \right] \end{aligned} \tag{8.167}$$

Or, if we cancel out the factors C_1 and A_2,

$$\begin{cases} [D_1 B_1^2 + \Sigma_R^{(1)}][D_2 B_1^2 + \Sigma_a^{(2)}] = \nu p_C \Sigma_f \Sigma_R^{(1)} \\ [D_1 B_2^2 + \Sigma_R^{(1)}][D_2 B_2^2 + \Sigma_a^{(2)}] = \nu p_C \Sigma_f \Sigma_R^{(1)} \end{cases} \tag{8.168}$$

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Thus B_1^2 and B_2^2 are solutions of the quadratic equation

$$(B^2 + \kappa_2^2)(B^2 + \kappa_1^2) = \frac{\nu p_C \Sigma_f \Sigma_R^{(1)}}{D_1 D_2} \equiv k_\infty \kappa_1^2 \kappa_2^2 \qquad (8.169)\star$$

where we have defined

$$\kappa_1^2 \equiv \frac{\Sigma_R^{(1)}}{D_1} \qquad \kappa_2^2 \equiv \frac{\Sigma_a^{(2)}}{D_2} \qquad k_\infty \equiv \frac{\nu \Sigma_f p_C}{\Sigma_a^{(2)}} \qquad (8.170)$$

We find, then, that if the expressions (8.164) are to be solutions of the core equations (8.147), with Z and W given by (8.165), the quantities B_1 and B_2 must be taken as the roots of the B^2 equation (8.169). These are seen to be

$$B^2 = -\frac{\kappa_1^2 + \kappa_2^2}{2} \pm \frac{1}{2}[(\kappa_1^2 + \kappa_2^2)^2 + 4\kappa_1^2\kappa_2^2(k_\infty - 1)]^{\frac{1}{2}} \qquad (8.171)$$

Since our primary interest in this model is in the application to reactors, it is clear that $k_\infty > 1$. In this case

$$\frac{[(\kappa_1^2 + \kappa_2^2)^2 + 4\kappa_1^2\kappa_2^2(k_\infty - 1)]^{\frac{1}{2}}}{\kappa_1^2 + \kappa_2^2} > 1 \qquad (8.172)$$

It follows, then, that if μ^2 and λ^2 are defined as positive and real then the solutions to (8.171) are

$$\begin{aligned} B_1^2 &\equiv \mu^2 && \text{small and positive} \\ B_2^2 &\equiv -\lambda^2 && \text{large and negative} \end{aligned} \qquad (8.173)$$

With these choices, Eqs. (8.165) take the form

$$\nabla^2 Z(\mathbf{r}) + \mu^2 Z(\mathbf{r}) = 0 \qquad \nabla^2 W(\mathbf{r}) - \lambda^2 W(\mathbf{r}) = 0 \qquad (8.174)$$

The solutions to these equations have already been discussed in much detail. The μ equation yields for Z the trigonometric and ordinary Bessel functions, and the λ equation, the hyperbolic and modified Bessel functions.

We summarize the general solutions for the flux distributions in the core with the aid of the results (8.167). If we define

$$\begin{aligned} a_1 &\equiv \frac{p_C \Sigma_R^{(1)}}{D_2\mu^2 + \Sigma_a^{(2)}} = \frac{D_1\mu^2 + \Sigma_R^{(1)}}{\nu\Sigma_f} \\ a_2 &\equiv \frac{p_C \Sigma_R^{(1)}}{-D_2\lambda^2 + \Sigma_a^{(2)}} = \frac{-D_1\lambda^2 + \Sigma_R^{(1)}}{\nu\Sigma_f} \end{aligned} \qquad (8.175)\star$$

then (8.164) may be written

$$\begin{aligned} \phi_1(\mathbf{r}) &= A_1 Z(\mathbf{r}) + C_1 W(\mathbf{r}) \\ \phi_2(\mathbf{r}) &= a_1 A_1 Z(\mathbf{r}) + a_2 C_1 W(\mathbf{r}) \end{aligned} \qquad (8.176)\star$$

Note that, since in all geometries there will be conditions of symmetry or nonsingularity on ϕ_1 and ϕ_2, the functions Z and W must be chosen

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to meet those conditions; thus Z and W do not involve any undetermined constants.

We consider next the reflector equations (8.149), which may be written

$$\nabla^2\phi_3(\mathbf{r}) - \kappa_3^2\phi_3(\mathbf{r}) = 0$$
$$\nabla^2\phi_4(\mathbf{r}) - \kappa_4^2\phi_4(\mathbf{r}) = -\frac{p_R\Sigma_R^{(3)}\phi_3(\mathbf{r})}{D_4} \tag{8.177}$$

with
$$\kappa_3^2 \equiv \frac{\Sigma_R^{(3)}}{D_3} \qquad \kappa_4^2 \equiv \frac{\Sigma_a^{(4)}}{D_4} \tag{8.178}$$

The solutions to the above equations are to vanish at the extrapolated outer surface of the reflector. By the usual arguments [refer to discussion following Eqs. (6.56)] we specify that the extrapolation distance be the same for the fast and thermal groups. Thus, if $\tilde{\mathbf{R}}'$ denotes the extrapolated outer surface of the reflector,

$$\phi_3(\tilde{\mathbf{R}}') = \phi_4(\tilde{\mathbf{R}}') = 0 \tag{8.179}$$

We look for solutions to Eqs. (8.177) which may be given as a linear combination of the functions U and V, where these functions satisfy the equations

$$\nabla^2 U(\mathbf{r}) - \kappa_3^2 U(\mathbf{r}) = 0 \qquad \nabla^2 V(\mathbf{r}) - \kappa_4^2 V(\mathbf{r}) = 0 \tag{8.180}$$

and
$$U(\tilde{\mathbf{R}}') = V(\tilde{\mathbf{R}}') = 0 \tag{8.181}$$

Now, the form of the ϕ_3 equation is seen to be the same as the reflector relation used in the one-velocity model [cf. Eq. (8.5*b*)], and it was found that the solution to that equation could be given in terms of a single function. It follows, then, that

$$\phi_3(\mathbf{r}) \equiv S_1 U(\mathbf{r}) \tag{8.182}$$

where S_1 is some constant to be determined. The function ϕ_4, on the other hand, must be expressed in the more general form:

$$\phi_4(\mathbf{r}) \equiv T_1 V(\mathbf{r}) + T_2 U(\mathbf{r}) \tag{8.183}$$

The substitution of these relations for ϕ_3 and ϕ_4 into the ϕ_4 equation of (8.177) yields the requirement

$$T_2 = -a_3 S_1 \qquad \text{with } a_3 \equiv \frac{p_R\Sigma_R^{(3)}}{D_4(\kappa_3^2 - \kappa_4^2)} \tag{8.184}$$

Thus we have for the reflector fluxes

$$\phi_3(\mathbf{r}) = S_1 U(\mathbf{r}) \qquad \phi_4(\mathbf{r}) = T_1 V(\mathbf{r}) - a_3 S_1 U(\mathbf{r}) \tag{8.185}$$

d. Criticality Condition. The general solutions (8.176) and (8.185) give the thermal and fast flux distributions in both core and reflector in terms of the functions $Z(\mathbf{r})$, $W(\mathbf{r})$, $U(\mathbf{r})$, and $V(\mathbf{r})$. These functions, in turn, are defined by Eqs. (8.174) and (8.180) and by the symmetry and

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extrapolated boundary requirements. The only quantities still undetermined are the constants A_1, C_1, T_1, and S_1. These may be computed from the core-reflector interface conditions which have yet to be applied. These conditions are, physically, the continuity of flux and net current in both the fast and thermal groups. If we denote by $\mathbf{R}$ the core-reflector interface surface, then the appropriate analytical statements are

$$\begin{aligned}
&(1)\ \phi_1(\mathbf{R}) = \phi_3(\mathbf{R}) \\
&(2)\ -D_1\nabla\phi_1(\mathbf{R})\cdot d\mathbf{A}(\mathbf{R}) = -D_3\nabla\phi_3(\mathbf{R})\cdot d\mathbf{A}(\mathbf{R}) \\
&(3)\ \phi_2(\mathbf{R}) = \phi_4(\mathbf{R}) \\
&(4)\ -D_2\nabla\phi_2(\mathbf{R})\cdot d\mathbf{A}(\mathbf{R}) = -D_4\nabla\phi_4(\mathbf{R})\cdot d\mathbf{A}(\mathbf{R})
\end{aligned} \tag{8.186}\star$$

The application of these conditions to the general solutions (8.176) and (8.185) yields the following set of equations in the four unknowns A_1, C_1, T_1, and S_1:

$$\begin{aligned}
(Z)A_1 + (W)C_1 + (-U)S_1 &= 0 \qquad (a) \\
(-D_1Z')A_1 + (-D_1W')C_1 + (D_3U')S_1 &= 0 \qquad (b) \\
(a_1Z)A_1 + (a_2W)C_1 + (a_3U)S_1 + (-V)T_1 &= 0 \qquad (c) \\
(-a_1D_2Z')A_1 + (-a_2D_2W')C_1 + (-a_3D_4U')S_1 & \\
+ (D_4V')T_1 &= 0 \qquad (d)
\end{aligned} \tag{8.187}$$

The symbols Z, W, U, V and Z', W', U', V' denote the functions and their normal derivatives evaluated at $\mathbf{r} = \mathbf{R}$. N.B. argument will be $W(\mu r)$ e.g.

The above set consists of four simultaneous algebraic equations in the four unknown constants. Since these equations are homogeneous, it is not possible to obtain explicit expressions for all four quantities, and, as in all steady-state reactor problems, there will remain one undetermined constant (the flux or power level in the reactor). Thus three of the unknowns will be determined in terms of the fourth. The only nontrivial solution of this system of equations requires that the determinant Δ of the coefficients of A_1, C_1, T_1, and S_1 vanish; that is,

$$\Delta \equiv \begin{vmatrix} Z & W & -U & 0 \\ -D_1Z' & -D_1W' & D_3U' & 0 \\ a_1Z & a_2W & a_3U & -V \\ -a_1D_2Z' & -a_2D_2W' & -a_3D_4U' & D_4V' \end{vmatrix} \equiv 0 \tag{8.188}\star$$

This expression may be regarded as the *criticality condition* for the reflected reactor in the two-group approximation, since it involves both the group constants and the dimensions of the reactor. In the usual way, the specification of the fuel concentration or the core size allows the determination of the other through the use of the above relation. Unlike the simpler models and applications previously considered, however, the computational procedure is somewhat more involved. The details of an appropriate procedure will be discussed later in connection with a sample calculation, and some computational approximations which have been

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found to be useful will be demonstrated. For the present we confine our remarks to a general outline of the method and the necessary steps involved in a computation.

Suppose, for example, that the shape and dimensions of the reactor have been specified, and that the composition of the nonfuel components is known. The problem, then, is to determine the critical fuel concentration. The calculation may be carried out by assuming various fuel loadings and finally selecting that loading which satisfies all the system requirements determined above. It should be recognized that, in this procedure, it is necessary to recompute several of the nuclear constants $\kappa_1, \kappa_2, \ldots$ with each assumed value of the fuel concentration. As a rule, only a few of these constants are markedly affected by variations in the fuel concentration, so that as a first approximation one need make adjustments only for these quantities. When a fairly reliable estimate has been obtained for the critical concentration, the entire computation may be repeated, using the corrected values of the less sensitive parameters.

Once a value has been assumed for the fuel concentration, and all the nuclear parameters computed, the corresponding values of λ^2 and μ^2 may be obtained from the general relation (8.171). These values, along with the appropriate data for the various functions Z, W, . . . , may then be inserted in the determinant (8.188). If the value of the determinant is found to be identically zero, the assumed fuel concentration is in fact the critical concentration, if not, the procedure is repeated until this condition is satisfied.

In the event that the problem of interest is the determination of the multiplication constant of a given reactor system, the procedure outlined above is slightly modified. For this case, all the dimensions and nuclear constants are fixed and the immediate objective is to determine a ν_c such that the determinant (8.188) vanishes. This is best accomplished by selecting various values of k_∞ (which is to say ν_c), computing the corresponding values of λ^2 and μ^2, and finally obtaining the value of Δ. That value of k_∞, call it k_∞^c, which satisfies (8.188) gives the actual multiplication constant k. Thus if k_∞ corresponds to the experimental value of ν, then the multiplication of a noncritical system is given by

$$k = \frac{k_\infty}{k_\infty^c} \tag{8.189}$$

It is of interest to note that the general method outlined here is entirely analogous to the one used for the bare-reactor calculations, using either the one-velocity or Fermi age models. For example, in the case of the Fermi age model, the problem was to solve the equation

$$\left(\frac{\nu\Sigma_f}{\Sigma_a}\right) p_{\text{th}} \frac{e^{-B^2\tau_{\text{th}}}}{1 + L^2B^2} = 1$$

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for B^2 and then compute the critical dimensions from an equation such as $B = \pi/\tilde{R}$ (for a sphere). In the two-group model, the equivalent relation is obtained from (8.169), which may be written

$$\left[\frac{\nu\Sigma_f}{\Sigma_a^{(2)}}\right]\frac{pc}{(1 + L_1^2\mu^2)(1 + L_2^2\mu^2)} = 1 \tag{8.190}$$

where

$$L_1^2 \equiv \frac{1}{\kappa_1^2} \quad \text{and} \quad L_2^2 \equiv \frac{1}{\kappa_2^2} \tag{8.191}$$

Equation (8.190), along with the determinant (8.188), may be regarded as the criticality requirements for this model. [Note that the last two factors on the left side of (8.190) give the fast and thermal nonleakage probabilities, respectively.] As in the bare-reactor calculation, (8.190) is solved for μ^2 (and λ^2) and the critical size (or concentration) obtained from the determinant.

e. Application to Spherical, Cylindrical, and Rectangular Block Reactors. The functions appearing in the determinant (8.188) are summarized for a few principal geometries in Table 8.3. Note that in each geometry, such as that of the block core, the boundary conditions apply for all y and z along the core-reflector interfaces. As used in (8.188), however, the function $\cos n_2 y \cos n_3 z$ appears as a factor in every term, and since this factor is not identically zero, it may be factored out of the determinant, which, then, becomes independent of y and z.

f. Comparison with One-group Method. Some idea of the errors in the criticality requirements which can be caused by the omission of the effects of the fast neutrons in a reactor can be obtained from the comparison of the results based on the two-group model with those of the one-group (velocity) model. As a representative case, we consider a spheri-

TABLE 8.3. SUMMARY OF FUNCTIONS FOR PRINCIPAL REACTOR GEOMETRIES

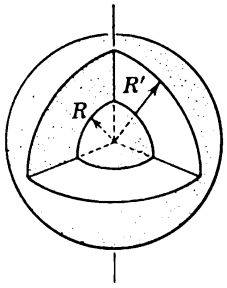
Spherical Geometry

$$Z(\mathbf{R}) = \frac{\sin \mu R}{R}$$

$$Z'(\mathbf{R}) = \frac{\mu \cos \mu R}{R} - \frac{\sin \mu R}{R^2}$$

$$W(\mathbf{R}) = \frac{\sinh \lambda R}{R}$$

$$W'(\mathbf{R}) = \frac{\lambda \cosh \lambda R}{R} - \frac{\sinh \lambda R}{R^2}$$



$$U(\mathbf{R}) = \frac{\sinh \kappa_3(\tilde{R}' - R)}{R}$$

$$U'(\mathbf{R}) = -\frac{\kappa_3 \cosh \kappa_3(\tilde{R}' - R)}{R} - \frac{\sinh \kappa_3(\tilde{R}' - R)}{R^2}$$

$$V(\mathbf{R}) = \frac{\sinh \kappa_4(\tilde{R}' - R)}{R}$$

$$V'(\mathbf{R}) = -\frac{\kappa_4 \cosh \kappa_4(\tilde{R}' - R)}{R} - \frac{\sinh \kappa_4(\tilde{R}' - R)}{R^2}$$

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TABLE 8.3. SUMMARY OF FUNCTIONS FOR PRINCIPAL REACTOR GEOMETRIES (*Continued*)

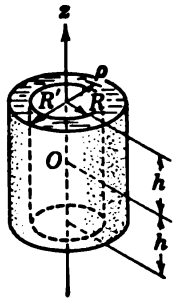
Cylinder (Side Reflector)

$$Z(\mathbf{R}) = J_0(l_1R)\cos l_2z$$
$$Z'(\mathbf{R}) = -l_1J_1(l_1R)\cos l_2z$$
$$W(\mathbf{R}) = I_0(l_3R)\cos l_2z$$
$$W'(\mathbf{R}) = l_3I_1(l_3R)\cos l_2z$$

$$l_1^2 \equiv \mu^2 - l_2^2 \qquad l_2 \equiv \frac{\pi}{2\tilde{h}} \qquad l_3^2 \equiv \lambda^2 + l_2^2$$

$$U(\mathbf{R}) = [I_0(l_4R)\,K_0(l_4\tilde{R}') - I_0(l_4\tilde{R}')\,K_0(l_4R)]\cos l_2z$$
$$U'(\mathbf{R}) = l_4[I_1(l_4R)\,K_0(l_4\tilde{R}') + I_0(l_4\tilde{R}')\,K_1(l_4R)]\cos l_2z$$
$$V(\mathbf{R}) = [I_0(l_5R)\,K_0(l_5\tilde{R}') - I_0(l_5\tilde{R}')\,K_0(l_5R)]\cos l_2z$$
$$V'(\mathbf{R}) = l_5[I_1(l_5R)\,K_0(l_5\tilde{R}') + I_0(l_5\tilde{R}')\,K_1(l_5R)]\cos l_2z$$

$$l_4^2 \equiv \kappa_3^2 + l_2^2 \qquad l_5^2 \equiv \kappa_4^2 + l_2^2$$

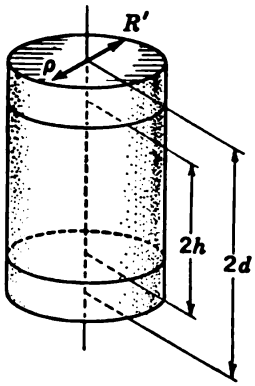


Cylinder (End Reflectors)

$$Z(\mathbf{R}) = J_0(m_1\rho)\cos m_2h$$
$$Z'(\mathbf{R}) = -m_2J_0(m_1\rho)\sin m_2h$$
$$W(\mathbf{R}) = J_0(m_1\rho)\cosh m_3h$$
$$W'(\mathbf{R}) = m_3J_0(m_1\rho)\sinh m_3h$$

$$m_1 \equiv \frac{2.405}{\tilde{R}'} \qquad m_2^2 \equiv \mu^2 - m_1^2 \qquad m_3^2 \equiv \lambda^2 + m_1^2$$

$$U(\mathbf{R}) = J_0(m_1\rho)\sinh m_4(\tilde{d} - h)$$
$$U'(\mathbf{R}) = -m_4J_0(m_1\rho)\cosh m_4(\tilde{d} - h) \qquad m_4^2 \equiv \kappa_3^2 + m_1^2$$
$$V(\mathbf{R}) = J_0(m_1\rho)\sinh m_5(\tilde{d} - h)$$
$$V'(\mathbf{R}) = -m_5J_0(m_1\rho)\cosh m_5(\tilde{d} - h) \qquad m_5^2 \equiv \kappa_4^2 + m_1^2$$

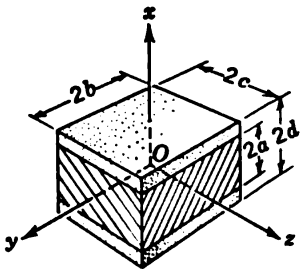


Block Geometry

$$Z(\mathbf{R}) = \cos n_1a\cos n_2y\cos n_3z$$
$$Z'(\mathbf{R}) = -n_1\sin n_1a\cos n_2y\cos n_3z$$
$$n_1^2 \equiv \mu^2 - n_2^2 - n_3^2$$
$$W(\mathbf{R}) = \cosh n_4a\cos n_2y\cos n_3z$$
$$W'(\mathbf{R}) = n_4\sinh n_4a\cos n_2y\cos n_3z$$

$$n_4^2 \equiv \lambda^2 + n_2^2 + n_3^2 \qquad n_3 \equiv \frac{\pi}{2\tilde{c}} \qquad n_2 \equiv \frac{\pi}{2\tilde{b}}$$

$$U(\mathbf{R}) = \sinh n_5(\tilde{d} - a)\cos n_2y\cos n_3z$$
$$U'(\mathbf{R}) = -n_5\cosh n_5(\tilde{d} - a)\cos n_2y\cos n_3z$$
$$V(\mathbf{R}) = \sinh n_6(\tilde{d} - a)\cos n_2y\cos n_3z$$
$$V'(\mathbf{R}) = -n_6\cosh n_6(\tilde{d} - a)\cos n_2y\cos n_3z$$
$$n_5^2 \equiv \kappa_3^2 + n_2^2 + n_3^2 \qquad n_6^2 \equiv \kappa_4^2 + n_2^2 + n_3^2$$



cal reflected reactor which has the following material properties:

Core: $D_1 = D_2 = 1$ cm
$\Sigma_R^{(1)} = 0.01\ \text{cm}^{-1}$
$p_C = 0.90$
$\Sigma_a^{(2)} = 0.05\ \text{cm}^{-1}$
$\Sigma_f = 0.0229\ \text{cm}^{-1}$

Reflector: $D_3 = D_4 = 1$ cm
$\Sigma_R^{(3)} = 0.01\ \text{cm}^{-1}$
$p_R = 0.90$
$\Sigma_a^{(4)} = 0.0049\ \text{cm}^{-1}$
$\tilde{T} = 22$ cm

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where $\tilde{T}$ denotes the extrapolated reflector thickness. Table 8.4 lists the critical radii of the core for this system based on the two models. For comparative purposes the results for the bare reactor, with the core properties listed above, have also been included. This table gives also the basic relations used in each calculation and the reference equation numbers.

TABLE 8.4 CRITICAL RADII OF SPHERICAL CORES IN VARIOUS APPROXIMATIONS
(Fast fissions neglected in all cases)

Method	Critical equation	Reference equations	Critical radius (cm)
1. Bare core, one-velocity (no fast leakage)	$\left(\frac{\nu\Sigma_f}{\Sigma_a}\right) p_{th}(1 + L^2B^2)^{-1} = 1$	(5.183)	76.8
	$B = \frac{\pi}{\tilde{R}}$		
2. Bare core, one-velocity (with fast leakage)	$\left(\frac{\nu\Sigma_f}{\Sigma_a}\right) p_{th}(1 + L^2B^2)^{-1}e^{-B^2\tau_{th}} = 1$	(6.83)	197
	$B = \frac{\pi}{\tilde{R}}$		
3. Bare core, two-group	$\left(\frac{\nu\Sigma_f^{(2)}}{\Sigma_a^{(2)}}\right) p_{th}(1 + L_2^2B^2)^{-1}(1 + L_1^2B^2)^{-1} = 1$	(8.155)	197
	$B = \frac{\pi}{\tilde{R}}$		
4. Reflected core,[a] one-velocity	$\left(\frac{\nu\Sigma_f}{\Sigma_a^{(C)}}\right) p_C(1 + L_C^2B^2)^{-1} = 1$	(8.73)	66.5
	$D_C(BR \cot BR - 1) = -D_R[\varkappa_R R \coth \varkappa_R(\tilde{R}' - R) + 1]$	(8.11)	
	$\varrho_{th} = 1$		
5. Reflected core, one-velocity (using migration area)	$\left(\frac{\nu\Sigma_f}{\Sigma_a}\right) p_C[1 + B^2(L_C^2 + \tau_{th}^C)]^{-1} = 1$	(8.85)	186
	$D_C(BR \cot BR - 1) = -D_R[\varkappa_R R \coth \varkappa_R(\tilde{R}' - R) + 1]$	(8.11)	
6. Reflected core, two-group	$\left(\frac{\nu\Sigma_f}{\Sigma_a^{(2)}}\right) p_C(1 + L_2^2\mu^2)^{-1}(1 + L_1^2\mu^2)^{-1} = 1$	(8.188) (8.171) (8.190)	180
	$\Delta = 0$		

[a] $\tilde{R}'$ = extrapolated reflector outer boundary.

The strictly one-velocity calculations, methods 1 and 4, are seen to give much smaller critical radii than the others; this indicates that the fast leakage is quite important in this system. Calculations 2 and 3 give identical results because of the large size of the bare core. It is interesting to note the comparative accuracy of the "migration-area" calculation, 5, which predicts, in this case, very nearly the same result as the two-group method.

The form of the flux predicted is also of interest; the one-velocity calculation, 4, yields the general functional shape shown in Fig. 8.20*a*. The two-group calculation, 6, yields a thermal-flux distribution, which, as

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shown by Eqs. (8.176) and (8.185) and Table 8.3, is a linear combination of two functions in both the core and the reflector:
For the thermal flux in a spherical core,

$$\phi_C(r) \equiv \phi_2(r) = \frac{\phi_C}{r}(\sin \mu r + c_0 \sinh \lambda r) \tag{8.192}$$

For the thermal flux in a spherical-shell reflector,

$$\phi_R(r) \equiv \phi_4(r) = \frac{\phi_R}{r}[\sinh \kappa_4(\tilde{R}' - r) + c_1 \sinh \kappa_3(\tilde{R}' - r)] \tag{8.193}$$

where ϕ_C, ϕ_R, c_0, and c_1 are constants to be determined by the calculation described above. The spatial form of these functions, as well as of the

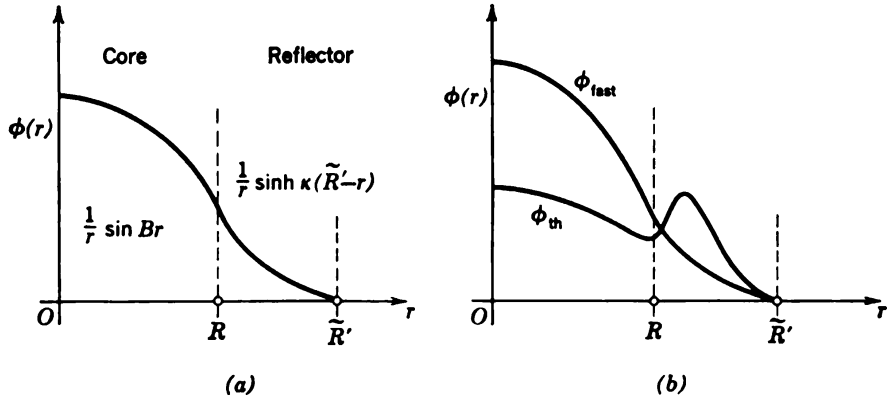

FIG. 8.20 Comparison of the flux distributions in core and reflector of a spherical reactor for the one- and two-group models.

fast flux, are shown in Fig. 8.20*b*. The dip in the thermal core flux and the peak in the reflector thermal flux result from the thermalization of neutrons in the reflector; as indicated by the slope of the curve at the core-reflector interface, there is a net flow of thermal neutrons from the reflector to the core. A numerical example of a reflected spherical reactor is considered in detail in Sec. 8.5.

g. Evaluation of Criticality Determinant. The evaluation of the criticality relation (8.188), although straightforward, is generally laborious and time-consuming. This is especially true in the case of cylindrical geometries wherein the functions Z, W, U, V and their derivatives which appear in the determinant each involves several terms. It is possible, however, to achieve some simplification in the calculation if one takes advantage of the fact that many of the terms in the determinant are relatively insensitive to variations in the fuel concentration (or the size of the core). This feature may be demonstrated by expanding the determinant and rearranging some of the resulting terms.

We begin by dividing the first column of the determinant by the func-

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tion Z, the second by W, the third by U, and the fourth by V. In this way we obtain the expression

$$\Delta = \begin{vmatrix} 1 & 1 & 1 & 0 \\ -D_1\zeta & -D_1\omega & -D_3\psi & 0 \\ a_1 & a_2 & -a_3 & -1 \\ -D_2a_1\zeta & -D_2a_2\omega & D_4a_3\psi & D_4\beta \end{vmatrix} = 0 \tag{8.194}$$

where $$\zeta \equiv \frac{Z'}{Z} \qquad \omega \equiv \frac{W'}{W} \qquad \psi \equiv \frac{U'}{U} \qquad \beta \equiv \frac{V'}{V} \tag{8.195}$$

Each of these functions is, of course, evaluated at the interface, e.g., in the case of the sphere, at $r = R$.

We next expand (8.194) in terms of the minors of the fourth column. It can be shown, after some algebra, that the resulting expression may be written in the form

$$\zeta = \frac{\omega\psi(a_3D_1D_4 + a_2D_2D_3) - D_1D_4\omega\beta(a_1 + a_3) + D_3D_4\psi\beta(a_1 - a_2)}{D_1D_2\omega(a_2 - a_1) - D_1D_4\beta(a_2 + a_3) + \psi(a_1D_2D_3 + a_3D_1D_4)} \tag{8.196}\star$$

In many cases of interest the right-hand side of this equation is relatively insensitive to the fuel concentration, the influence of which is felt through the value of the parameter μ. This dependence is easily seen from the quadratic equation (8.171) which defines μ. Note, in particular, that a change in the fuel concentration affects directly the value of $\Sigma_a^{(2)}$ and thereby the values of κ_2 and k_∞.

The computational procedure to be followed in using (8.196) consists in selecting various values of the fuel concentration, computing the corresponding values of μ (and λ) from (8.171), and evaluating each side of the relation (8.196). The right-hand side, call it $\Theta(\mu)$, and the function $\zeta(\mu)$ may then be plotted as a function of the fuel concentration, and the intersection of the two curves yields the critical concentration. Since the function $\Theta(\mu)$ will be, in general, a slowly varying function of μ, one or two estimates of Θ will suffice to determine the curve of Θ versus fuel concentration. The function $\zeta(\mu)$, on the other hand, is a rapidly varying function of μ and, moreover, has a relatively simple form. For example, in the case of a reflected spherical reactor,

$$\zeta = \mu \cot \mu R - \frac{1}{R} \tag{8.197}$$

Thus, the evaluation of Δ for a particular value of μ (i.e., fuel concentration) is reduced to one, or so, estimate of $\Theta(\mu)$ and a simple calculation of ζ.

The procedure suggested here is discussed in some detail in the sample calculation of Sec. 8.5, and appropriate curves of ζ and Θ are presented.

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8.5 Sample Computation Using Two-group Model

a. Description of a Nuclear Package Power Plant. As an illustration of the use of the two-group model in reactor calculations we consider a small nuclear power plant for generating steam and electricity. The essential nuclear and engineering features of this reactor are as follows: The core consists of a cylindrical array of MTR-type fuel elements mounted within a cylindrical pressure shell filled with water. The water, which serves both as coolant and moderator, enters the pressure shell through a set of pipes, as shown in Fig. 8.21*a*, and first flows downward

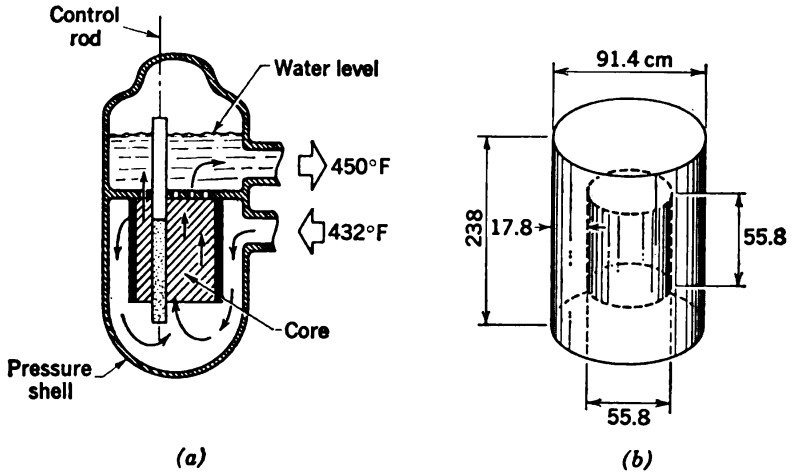

FIG. 8.21 Nuclear package power plant: (*a*) actual geometry; (*b*) idealized model.

in an annular region surrounding the fuel elements, then upward through the fuel elements, and finally out of the shell through another set of pipes. The temperature of the entrant water is 432°F, and that of the outlet, 450°F. The 450°F water, under pressure, then passes through a heat exchanger and produces lower-pressure steam for a turbogenerator system.

The inlet water which first passes through the annular region surrounding the core serves as a side reflector, and the water below and above the core, as an end reflector. The individual fuel elements, which are assembled vertically, consist of a box structure of parallel plates similar to the design shown in Fig. 5.28. The structural material of these elements is stainless steel, and the fuel-bearing plates are fabricated from an iron alloy of U^{235} clad on both sides with stainless steel. The fuel elements are 55.8 cm long, and the cylindrical cluster has a diameter of 55.8 cm. The annular region of water which forms the side reflector is 17.8 cm thick, and the end caps are 91.1 cm thick. The volume of the core cylinder is 136,400 cm^3, and the volume fractions of its constituents are

$$v_W = 0.797 \qquad v_{SS} = 0.110 \qquad v_{FA} = 0.093 \tag{8.198}$$

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where v_W refers to the water, v_{SS} to the stainless steel, and v_{FA} to the fuel-bearing alloy.

Our primary interest in this calculation is the determination of the critical mass of the hot clean reactor and the radial distribution of the fast and thermal flux throughout the core and reflector. An accurate analysis of this system must necessarily take into account the completely reflected cylindrical geometry shown in Fig. 8.21*b*. However, since this would entail a somewhat involved calculation, we will approximate the actual configuration by an equivalent reflected sphere of the same composition. This will reduce our computation appreciably and yet not obscure any of the essential steps in the application of the two-group model. A study of the effect of the corners in the completely reflected cylinder will be deferred until the next section.

For the purposes of the present calculation we assume that the equivalent spherical system[1] is one which has the same core volume as the actual geometry and a spherical-shell reflector 17.8 cm thick. On this basis the radius of the spherical core is $R = 31.93$ cm, and for convenience we take the extrapolated outer radius of the reflector to be $\tilde{R}' = 50$ cm. It is important to note that the simplification proposed here, and in particular the selection of the equivalent system, is not to be construed as necessarily representing a good approximation. Its only function is to clarify the presentation.

b. Group Constants. In the calculation which follows we will use the following set of data. For the densities of the various constituents of the core we take:

Material	*Density*
Water (450°F)	0.832 g/cm³
Stainless	7.75
Alloy	7.85
Fuel	10.9

where by the term "alloy" we mean the iron diluent which serves as the carrier for the fissionable material. According to the notation of (8.198), the volume fraction of the core occupied by alloy plus fuel is v_{FA}. For convenience in computation we define two additional volume fractions such that

$$v_{FA} \equiv v_F + v_A \tag{8.199}$$

where v_F is the volume fraction of the core occupied by the actual fissionable material and v_A the fraction occupied by the diluent. This notation will be useful when selecting various fuel concentrations (i.e., values of v_F) for evaluating the criticality determinant.

[1] Alternative procedures for determining equivalent systems is given by F. G. Prohammer, "A Comparison of One-dimensional Critical Mass Computations for Completely Reflected Reactors," ORNL-2007, Mar. 1, 1956.

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The nuclear concentrations of the components listed in the above table are easily computed. These are

$$\begin{aligned} N_W &= 0.02219 \times 10^{24} \text{ nuclei/cm}^3 & N_A &= 0.08443 \times 10^{24} v_A \\ N_{SS} &= 0.009169 \times 10^{24} & N_F &= 0.02459 \times 10^{24} v_F \end{aligned} \tag{8.200}$$

where we have used the relation (8.199) and the data of (8.198) and assumed that the average temperature of the water in the core is 450°F.

The group constants for this reactor have been computed from the methods described in Sec. 8.4b, using the data (8.200). These are listed in Table 8.5.

TABLE 8.5 TWO-GROUP CONSTANTS FOR PACKAGE POWER PLANT

Flux group	Core	Reflector
Fast	$\tau_C = 45.3$ cm^2 $D_1 = 1.44$ cm $p_C = 1.0$ $\Sigma_R^{(1)} = D_1/\tau_C = 0.03179$ cm^{-1}	$\tau_R = 40.0$ cm^2 $D_3 = 1.85$ cm $p_R = 1.0$ $\Sigma_R^{(3)} = 0.04625$ cm^{-1}
Thermal	$D_2 = 0.229$ cm	$D_4 = 0.204$ cm $L_4^2 = 16.64$ cm^2 $\Sigma_a^{(4)} = 0.01226$ cm^{-1}

The thermal-group absorption cross sections will be computed from the following data (for neutron temperature of 450°F):

$$\begin{aligned} \sigma_a^{(W)} &= 0.4518 \text{ barn} & \sigma_a^{(A)} &= 1.80 \text{ barns} \\ \sigma_a^{(SS)} &= 1.86 \text{ barns} & \sigma_a^{(F)} &= 444.1 \text{ barns} \end{aligned}$$

The corresponding macroscopic cross sections based on the nuclear concentrations (8.200) are

$$\begin{aligned} \Sigma_a^{(W)} &= 0.01002 \text{ cm}^{-1} & \Sigma_a^{(A)} &= 0.152 v_A \\ \Sigma_a^{(SS)} &= 0.01705 & \Sigma_a^{(F)} &= 10.92 v_F \end{aligned} \tag{8.201}$$

and the values of the various κ,

$$\begin{aligned} \kappa_1^2 &= \frac{\Sigma_R^{(1)}}{D_1} = \frac{1}{\tau_C} = 0.02208 \text{ cm}^{-2} & \kappa_1 &= 0.1486 \text{ cm}^{-1} \\ \kappa_3^2 &= \frac{\Sigma_R^{(3)}}{D_3} = \frac{1}{\tau_R} = 0.0250 \text{ cm}^{-2} & \kappa_3 &= 0.1581 \text{ cm}^{-1} \\ \kappa_4^2 &= \frac{\Sigma_a^{(4)}}{D_4} = 0.06010 \text{ cm}^{-2} & \kappa_4 &= 0.2452 \text{ cm}^{-1} \end{aligned} \tag{8.202}$$

c. Evaluation of Criticality Determinant. In the present section we trace through a sample calculation of the criticality determinant, using the procedure outlined in Sec. 8.4g. For this calculation we assume one value of the volume fraction of fuel v_F (i.e., the fuel concentration) and evaluate the functions ζ and Θ from the Eq. (8.196). In practice, several

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values of v_F will be required to establish the shape of the functions ζ and Θ. The value of v_F we select for the computation which follows is $v_F = 0.006$; since this does not correspond to the critical fuel concentration, ζ will not be equal to Θ in this case.

For $v_F = 0.006$, we obtain $v_A = 0.087$ from (8.199), and by (8.201)

$$\begin{aligned} \Sigma_a^{(A)} &= 0.01322 \text{ cm}^{-1} & \Sigma_a^{(2)} &= 0.1058 \text{ cm}^{-1} \\ \Sigma_a^{(F)} &= 0.06552 \text{ cm}^{-1} \quad \text{thus} & \kappa_2^2 &= \frac{\Sigma_a^{(2)}}{D_2} = 0.4620 \text{ cm}^{-2} \\ \Sigma_f &= 0.05496 \text{ cm}^{-1} & k_\infty &= \frac{\nu\Sigma_f p_C}{\Sigma_a^{(2)}} = 1.278 \end{aligned} \tag{8.203}$$

We compute next the roots of the quadratic equation (8.169), using (8.171). These are found to be

$$\mu = 0.07606 \qquad \lambda = 0.6999 \tag{8.204}$$

The various functions which appear in Eq. (8.196) may now be computed. We require

$$\begin{aligned} \omega &\equiv \frac{W'}{W} = \lambda \coth \lambda R - \frac{1}{R} = 0.6686 \\ \psi &\equiv \frac{U'}{U} = -\kappa_3 \coth \kappa_3 \tilde{T} - \frac{1}{R} = -0.1905 \\ \beta &\equiv \frac{V'}{V} = -\kappa_4 \coth \kappa_4 \tilde{T} - \frac{1}{R} = -0.2766 \end{aligned} \tag{8.205}$$

where we have used the definitions for W, U, V and their derivatives as listed in Table 8.3 for the case of the reflected-sphere reactor, and $\tilde{T} \equiv \tilde{R}' - R$. For completeness we list also the constants

$$\begin{aligned} a_1 &= \frac{D_1\mu^2 + \Sigma_R^{(1)}}{\nu\Sigma_f} = 0.2967 \\ a_2 &= \frac{-D_1\lambda^2 + \Sigma_R^{(1)}}{\nu\Sigma_f} = -4.983 \\ a_3 &= \frac{p_R\Sigma_R^{(3)}}{D_4(\kappa_3^2 - \kappa_4^2)} = -6.461 \end{aligned} \tag{8.206}$$

These data, (8.205) and (8.206), may now be used to compute the values of Θ and ζ from (8.196). For Θ we obtain

$$\begin{aligned} \Theta &\equiv \frac{\psi\omega(a_3D_1D_4 + a_2D_2D_3) - D_1D_4\omega\beta(a_1 + a_3) + D_3D_4\psi\beta(a_1 - a_2)}{D_1D_2\omega(a_2 - a_1) - D_1D_4\beta(a_2 + a_3) + \psi(a_1D_2D_3 + a_3D_1D_4)} \\ &= -0.1598 \end{aligned} \tag{8.207}$$

The function ζ we compute from

$$\zeta \equiv \frac{Z'}{Z} = \mu \cot \mu R - \frac{1}{R} = -0.1193 \tag{8.208}$$

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using the spherical functions given in Table 8.3. Since the results (8.207) and (8.208) are not identical, the determinant (8.188) is not zero, and therefore $v_F = 0.006$ does not correspond to the critical fuel concentration.

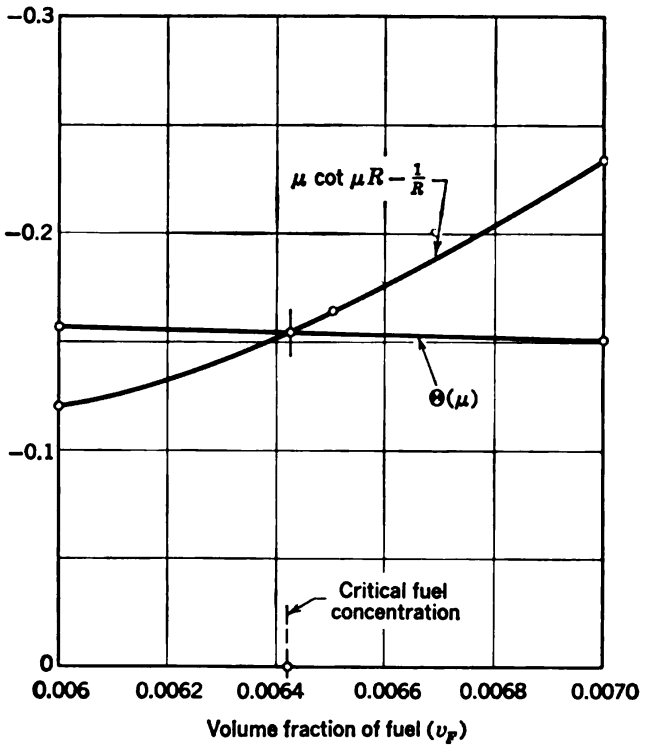

FIG. 8.22 Solution of criticality determinant.

Figure 8.22 gives a plot of the functions ζ and Θ over a range of fuel-volume fractions near the critical value. The solution to the present problem is seen to be

$$(v_F)_{\text{critical}} = 0.006425$$

It is of interest to note the shape of the functions ζ and Θ. Evidently, Θ is a very slowly varying function of v_F, whereas ζ is a very rapidly varying function. This behavior was previously mentioned in Sec. 8.4g.

The critical mass of U^{235} for the present system is easily computed from the volume fraction v_F. Thus

$$M_F = \rho_F v_F V = (10.9)(0.006425)(136{,}400) = 9.55 \text{ kg}$$

d. Flux Distributions. The calculation of the fast- and thermal-flux distributions requires the evaluation of the various constants which appear in the Eqs. (8.176) and (8.185). For this purpose we need also the criticality values of those nuclear constants of the system which are functions of the fuel concentration. These may be obtained by the

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procedure outlined in Sec. 8.5c. The appropriate values are found to be

$$\begin{gathered}\Sigma_a^{(A)} = 0.01316 \text{ cm}^{-1} \qquad \Sigma_a^{(F)} = 0.07016 \text{ cm}^{-1} \qquad \Sigma_a^{(2)} = 0.1104 \text{ cm}^{-1} \\ \kappa_2^2 = 0.4820 \text{ cm}^{-2} \qquad k_\infty = 1.316 \\ a_1 = 0.2842 \qquad a_2 = -4.859 \qquad a_3 = -6.461 \\ \mu = 0.08057 \qquad \lambda = 0.7146\end{gathered} \tag{8.209}$$

The flux distributions are given by the relations

$$\begin{gathered}\phi_1 = A_1 Z + C_1 W \qquad \phi_2 = a_1 A_1 Z + a_2 C_1 W \\ \phi_3 = S_1 U \qquad \phi_4 = T_1 V - a_3 S_1 U\end{gathered} \tag{8.210}\star$$

The constants A_1, C_1, T_1, and S_1 are determined from the set of equations (8.187). Since, as we have already recognized, these equations are homogeneous, it is not possible to obtain explicit relations for all four constants. The most information we can obtain from these equations is the ratios between the constants. Thus, the computational procedure is to solve for three of the unknowns in terms of the fourth. If this procedure is followed in the case of Eqs. (8.187*a*) and (8.187*b*), the following relationship is obtained between C_1 and A_1:

$$C_1 = \frac{Z}{W}\left(\frac{\zeta D_1 - \psi D_3}{\psi D_3 - \omega D_1}\right) A_1 \tag{8.211}\star$$

The functions which appear in this expression are given in the notation of Eqs. (8.195), and in the usual way, these are to be evaluated at the core-reflector interface. The result (8.211) may be used to determine S_1 from the relation

$$S_1 = \frac{1}{U}(ZA_1 + WC_1) \tag{8.212}\star$$

[which comes from Eq. (8.187*a*)] and T_1 by

$$T_1 = \frac{1}{V}[A_1 Z(a_1 + a_3) + C_1 W(a_2 + a_3)] \tag{8.213}\star$$

These three expressions, then, give C_1, S_1, and T_1 in terms of A_1.

We now apply these general results to the present problem. For this purpose we use the data (8.209) and let $A_1 = 12.41$. This value of A_1 is based on the normalization of the function ϕ_1; i.e., we obtain A_1 by requiring that $\phi_1(0) \equiv 1$. Then

$$\begin{gathered}A_1 = 12.41 \qquad C_1 = -1.548 \times 10^{-10} \\ S_1 = 0.6986 \qquad T_1 = -0.8135\end{gathered} \tag{8.214}$$

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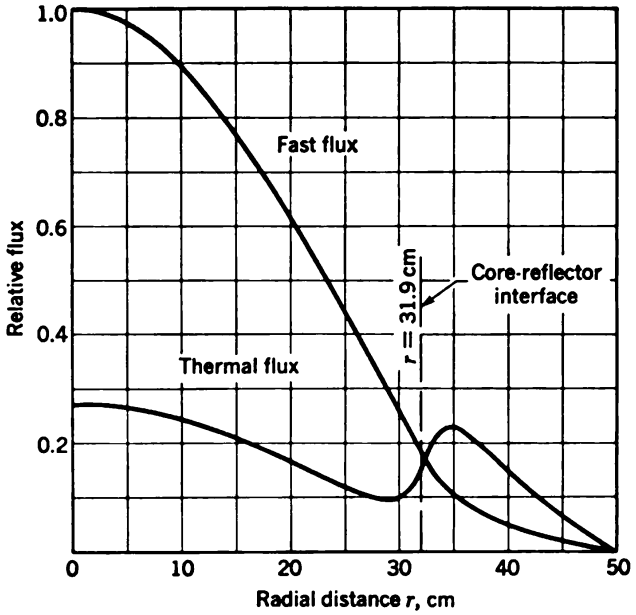

FIG. 8.23 Radial distribution of fast and thermal flux in reflected sphere reactor.

The substitution of these results into the flux relations (8.210) yields, for the *radial* distributions,

$$\begin{aligned}
\phi_1(r) &= 12.41\,\frac{\sin \mu r}{r} - 1.548 \times 10^{-10}\,\frac{\sinh \lambda r}{r} \\
\phi_2(r) &= 3.527\,\frac{\sin \mu r}{r} + 7.552 \times 10^{-10}\,\frac{\sinh \lambda r}{r} \\
\phi_3(r) &= 0.6986\,\frac{\sinh \kappa_3(\tilde{R}' - r)}{r} \\
\phi_4(r) &= -0.8135\,\frac{\sinh \kappa_4(\tilde{R}' - r)}{r} + 4.513\,\frac{\sinh \kappa_3(\tilde{R}' - r)}{r}
\end{aligned} \tag{8.215}$$

These functions have been plotted in Fig. 8.23. It is of interest to note the peak in the thermal flux near the core-reflector interface.

8.6 Completely Reflected Systems

a. One-group Model. A general analytical solution of the completely reflected reactor is presently not available. This creates no particular hardship when only a rough reactor calculation is required. In these cases the effect of the "corners" in reflected configurations, which is generally small, can be entirely neglected since only a very small portion of the total neutron population occupies these regions. Thus to ignore the corners introduces small errors into the critical-mass calculations even though the estimated local flux distributions may be in considerable error. When more accurate critical-mass estimates are required, these

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effects cannot be ignored, and in the absence of analytical solutions, it has been necessary to develop various approximate methods. These methods are generally of an iterative type and involve the use of the "equivalent-bare-reactor" concept mentioned previously. In the discussion which follows we describe two such methods which have been applied with some success to the one-velocity model and to the two-group model.

As a first example, consider the analysis of a completely reflected cylindrical reactor on the basis of the one-velocity model. A typical

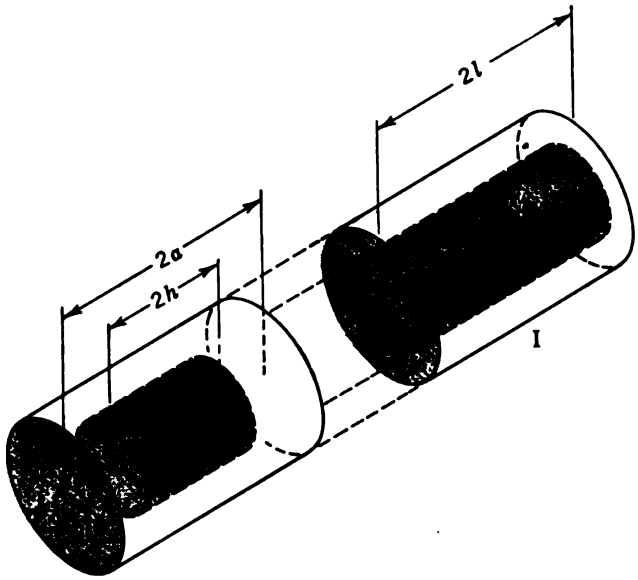

FIG. 8.24 Actual reactor and equivalent side-reflected cylinder.

system of this type is shown in Fig. 8.5 (cf. Fig. 8.21). Let the dimensions of this reactor be as follows: core, radius ρ_0 and height $2h$; reflector, radius ρ_1 and height $2a$. Given values for these dimensions and the composition of the reflector, the problem is to determine the critical fuel concentration in the core.

To solve this problem we study first a cylindrical reactor with *side reflector* only which has the same reflector material, the same fuel concentration, and the same core and reflector radii (ρ_0 and ρ_1, respectively) as the actual system. Such a system I is shown in Fig. 8.24. The only feature of this geometry as yet undetermined is its length ($2l$). Thus our next objective is to find l such that the two systems shown in Fig. 8.24 are critical with the same fuel *concentration.*

One plausible method of obtaining a first guess for l is that of finding the height of a *completely bare* reactor, III (radius ρ_1), which is critical with the same fuel concentration as that in the core of a critical cylinder reflected on the *ends* only (reflector material taken the same as that for

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the actual system) and having the dimensions shown in Fig. 8.25, II. Thus the value of l is determined by requiring that the two systems shown in Fig. 8.25 be critical with the same fuel concentration. In this approximation the procedure for calculating the fuel concentration for the completely reflected cylinder would be as follows:

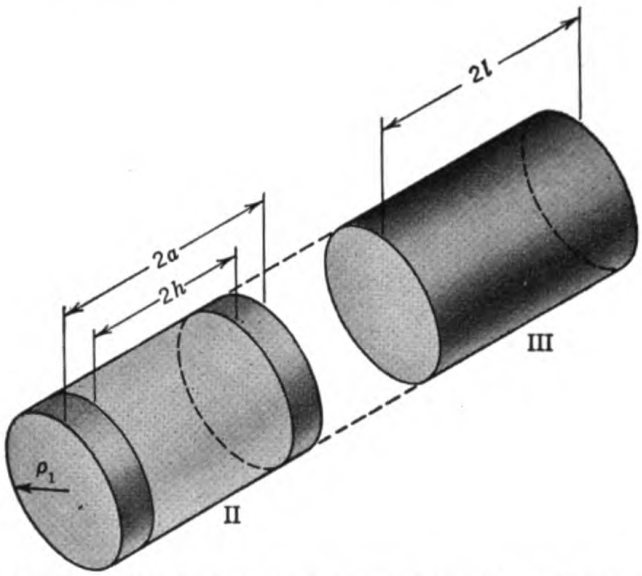

FIG. 8.25 Equivalent end-reflected and bare cylinders.

1. Calculate the buckling B_{II}^2 for the end-reflected cylinder; thus,

$$B_{\text{II}}^2 = \left(\frac{\nu_1}{\tilde{\rho}_1}\right)^2 + \mu_C^2 \tag{8.216}$$

with μ_C given by (8.54).

2. Find l such that system III will be critical with the same buckling; thus, compute

$$B_{\text{III}}^2 \equiv \left(\frac{\pi}{2l}\right)^2 + \left(\frac{\nu_1}{\tilde{\rho}_1}\right)^2 \equiv B_{\text{II}}^2 \tag{8.217}$$

where B_{II}^2 is obtained from (8.216). Equation (8.217) yields an expression for l:

$$l = \frac{\pi}{2\mu_C} \tag{8.218}$$

3. Computation of critical fuel concentration of system I. With the value of l obtained from (8.218), find

$$B_{\text{I}}^2 = \left(\frac{\pi}{2l}\right)^2 + \lambda_C^2 \tag{8.219}$$

where λ_C is given by (8.42). Since the fuel concentration of the actual

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system is to be the same as that of system I, we set

$$B^2_{\text{actual}} \equiv B^2_{\text{I}} \tag{8.220}$$

and compute N_F from the relation

$$\frac{1}{D_C}[\nu\Sigma_f - \Sigma_a^{(C)}] = B^2_{\text{actual}} \tag{8.221}$$

It should be noted in using this procedure that there does indeed exist an l for which Eqs. (8.219) and (8.220) hold accurately; however, the value obtained by steps 1 and 2 is only an approximation, and one whose validity could be ascertained only with great difficulty.

b. Two-group Model. The general approach used in the case of the one-velocity model is easily extended for use with the standard two-group method described in Sec. 8.4. As in that calculation, the primary objective here is the estimation of the reflector savings in the direction of each axis of symmetry of the reactor. One scheme which has been devised for this purpose employs an iterative procedure involving the analysis of a sequence of elementary reactors each of which incorporates only one feature of the actual reactor geometry. In each of these calculations it is assumed that only one face (or pair of faces) is reflected while the others are taken to be bare. For example, reactor 1 in the sequence could be reflected on the x_1 face and bare on the x_2 and x_3 faces. Reactor 2 could be reflected on the x_2 face and be bare on the x_1 and x_3 faces, and so on. The important point here is that each of these elementary systems has a sufficiently simple geometry to allow an analytical solution. This is carried out with the aid of the two-group model, and each calculation yields an estimate of the reflector savings for the face(s) which has been reflected. By progressing down the sequence of reactors, a complete cycle is eventually made around the actual reactor and a reflector savings obtained for every face.

It should be evident from these introductory remarks that the crux of the entire matter is the selection of the dimensions (or bucklings) of the bare faces for each of the elementary reactors in the sequence. It is at this point that we introduce the iterative procedure mentioned above. In carrying out the calculations for the first complete cycle around the reactor, it is necessary to make a first rough estimate of the reflector savings (i.e., bucklings) for all three faces. These data may then be used to obtain an improved estimate for the savings. For example, in analyzing reactor 1, the rough estimates for the bucklings in the x_2 and x_3 directions may be used to compute a better value for the savings in the x_1 direction. The same procedure may be followed in the case of the other two axes x_2 and x_3. This first cycle yields, then, an improved value for the bucklings in all three directions. By repeating this procedure for several cycles, one can easily obtain a convergent value for the

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bucklings (reflector savings) for each of the principal axes. It has been found in practice that this procedure does indeed converge very rapidly and the results appear to be independent of the initial guess which may be used in computing reactor 1. Unfortunately it has not yet been shown precisely what reactor is analyzed by this rationale since an exact solution of a completely reflected system is not presently available. However, criticality estimates obtained by this procedure have been found to agree surprisingly well with experimental measurements on the ORNL Swimming Pool Reactor.

The application of the two-group model to each of the elementary reactors in the iterative sequence described above requires the solution of the criticality equation (8.169). For this calculation it is convenient to write this relation in the form

$$1 = \frac{k_\infty^c}{(1 + L_2^2B^2)(1 + \tau_C B^2)} \tag{8.222}$$

where

$$L_2^2 \equiv \frac{1}{\kappa_2^2} = \frac{D_2}{\Sigma_a^{(2)}} \qquad L_1^2 \equiv \frac{1}{\kappa_1^2} = \frac{D_1}{\Sigma_R^{(1)}} \equiv \tau_C \tag{8.223}$$

and k_∞^c is the infinite-medium multiplication constant required to maintain the prescribed system (viz., geometry) at steady state. Note that this quantity is entirely equivalent to the number ν_c previously introduced for handling noncritical systems. Now, in the case of a noncritical system, it is also possible to write (8.222) in the form

$$k = \frac{k_\infty}{(1 + L_2^2B^2)(1 + \tau_C B^2)} \tag{8.224}$$

Combining (8.222) and (8.224) we obtain [cf. Eq. (8.189)]

$$k_\infty^c = \frac{k_\infty}{k}$$

For each of the elementary reactor calculations in the iteration we require an expression for the buckling B^2. This quantity is given by the small positive root μ^2 of (8.222) [cf. (8.173)]. The solution for B^2 is

$$B^2 = \frac{1}{2\tau_C L_2^2}\left\{-M^2 \pm \left[M^4 + 4\tau_C L_2^2\left(\frac{k_\infty}{k} - 1\right)\right]^{\frac{1}{2}}\right\} \tag{8.225}$$

where we have used (8.189) for k_∞ and the relation $M^2 \equiv \tau_C + L_2^2$. An approximate expression for the two roots (which will be useful for "hand" calculations) can be obtained by expanding the radical in a Taylor series. Thus, one obtains

$$\mu^2 \simeq \frac{1}{M^2}\left(\frac{k_\infty}{k} - 1\right)\left[1 - \frac{L_2^2\tau_C}{M^4}\left(\frac{k_\infty}{k} - 1\right)\right] \tag{8.226}\star$$

$$\lambda^2 \simeq \frac{M^2}{L_2^2\tau_C} + \frac{1}{M^2}\left(\frac{k_\infty}{k} - 1\right) \tag{8.227}\star$$

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The application of these results to a reactor calculation utilizing the iterative procedure is demonstrated below.

As an illustrative example consider a completely reflected rectangular-block reactor of the type shown in Fig. 8.8. A cross-sectional view of this reactor along any principal axis would have the form shown in Fig. 8.26*a*. The configuration which will be used in the iteration, however,

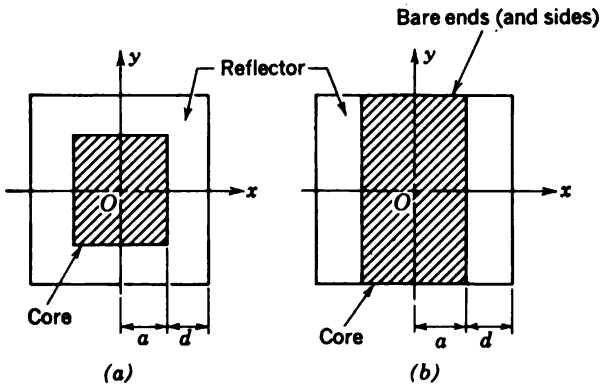

FIG. 8.26 Cross section through rectangular-block reactor: (*a*) actual cross section; (*b*) simplified reactor.

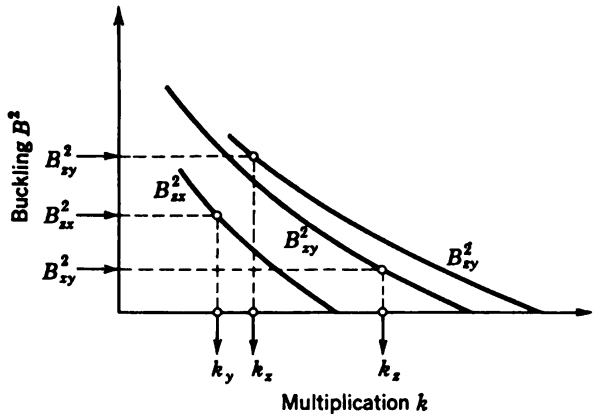

FIG. 8.27 Buckling as function of multiplication for rectangular-block reactor reflected on one pair of faces only.

is shown in Fig. 8.26*b*. In the case of the rectangular-block reactor we will require three such figures, one for each of the principal axes. In each figure the actual core width and reflector thickness will be correctly shown for one axis only. The other two pairs of faces are assumed to be bare. Figure 8.26*b* shows the proper sectional view for the x axis. With these three elementary geometries as a basis, the first step in the analysis is to compute the multiplication constant k_i for each axis x_i, assuming various values for the bucklings (i.e., bare widths) of the other two pairs of sides. The computational procedures outlined in Sec. 8.4 are directly applicable here. The next step in the calculation is to graph these results in a suitable form, as shown in Fig. 8.27. The notation which has been

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used in the figure is as follows:

Axis	Half-width of core	Buckling
x	a	$B_x = \pi/2a$
y	b	$B_y = \pi/2b$
z	c	$B_z = \pi/2c$

with

$$B_{xy}^2 \equiv B_x^2 + B_y^2 \qquad B_{yz}^2 \equiv B_y^2 + B_z^2 \qquad B_{xz}^2 = B_x^2 + B_z^2 \tag{8.228}$$

and the "total" buckling

$$B^2 \equiv B_x^2 + B_y^2 + B_z^2 \tag{8.229}$$

The calculations for the elementary reactors in the iterative sequence are carried out with direct application of buckling curves of the type shown in Fig. 8.27 and the use of the expression (8.226) for μ^2. The computational procedure might be as follows:

1. Select a set of equivalent bare core dimensions (i.e., bucklings). Denote these a_1, b_1, and c_1. (Note that this is equivalent to assuming a set of reflector savings.)

2. Compute the corresponding values of $B_{xy}^{(1)}$, $B_{yz}^{(1)}$, and $B_{xz}^{(1)}$, where the index (1) refers to "first estimate."

3. From a set of buckling curves of the type shown in Fig. 8.27, find the resulting values of $k_x^{(1)}$, $k_y^{(1)}$, and $k_z^{(1)}$.

4. Substitute these three values for k into Eq. (8.226) and compute the corresponding values of B^2 (here called μ^2). Note that the k_∞ which appears in (8.226) is a fixed number throughout this calculation and is determined by the selected fuel loading.

5. Use these three values of B^2 to obtain improved estimates for the B_{ij}. These may be computed from

$$\begin{aligned} [B_x^{(2)}]^2 &\equiv B^2[k_x^{(1)}] - [B_{yz}^{(1)}]^2 \\ [B_y^{(2)}]^2 &\equiv B^2[k_y^{(1)}] - [B_{xz}^{(1)}]^2 \\ [B_z^{(1)}]^2 &\equiv B^2[k_z^{(1)}] - [B_{xy}^{(1)}]^2 \end{aligned} \tag{8.230}$$

where it is understood that $B_i^{(2)}$ is the second estimate of the buckling in the direction x_i and is obtained from the value of the total buckling $B^2[k_i^{(1)}]$ based on the estimate $k_i^{(1)}$ determined in step 3. It should be recognized that each of these $B_i^{(2)}$ implies a second estimate of the reflector savings in the x_i directions, since, for example, from $B_x^{(2)}$ we can compute a_2, and $(a_2 - a)$ is by definition the reflector savings for the x direction.

6. With these new values of $B_i^{(2)}$ the entire procedure is repeated, starting with step 2 above.

The completion of the four basic steps in this procedure constitutes a single cycle in the analysis, and each cycle is expected to yield an

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improved estimate of the reflector savings over the preceding cycle. The number of complete cycles to be carried out is determined by the accuracy desired, which in turn may be measured by how well the three roots B^2 in step 4 agree. Even a rough guess for a_1, b_1, and c_1 based on the one-velocity relations (8.20) and (8.22) should yield convergence to three significant figures in B^2 within three cycles.

We demonstrate this general procedure by working through a numerical example. As an illustration we will compute the multiplication constant

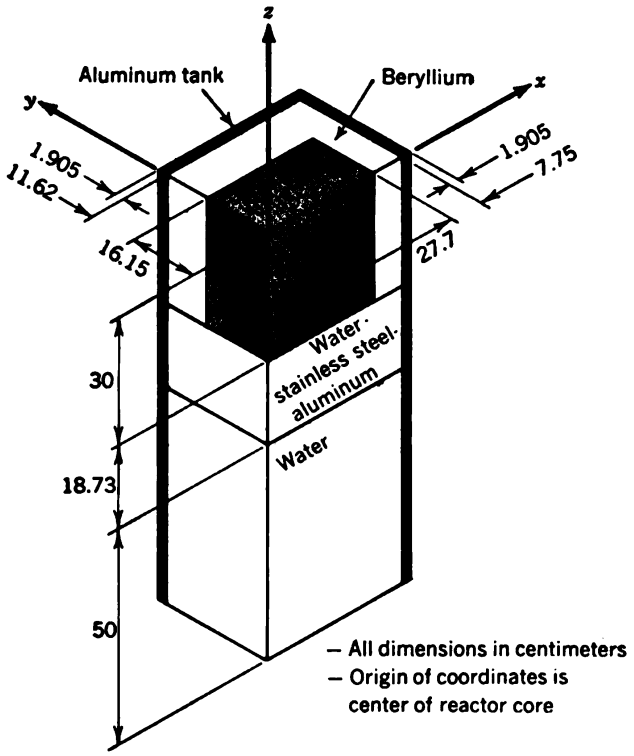

FIG. 8.28 One-eight section of core-reflector geometry of Oak Ridge Research Reactor.

for a specific fuel loading of the Oak Ridge Research Reactor. The basic nuclear and engineering features of this reactor which are essential to this study are as follows: (1) The standard core consists of a 4 by 7 vertical array of MTR-type fuel elements (cf. Fig. 5.27). The half-width in the x direction is $a = 27.7$ cm, in the y direction $b = 16.15$ cm, and in the z (vertical) direction $c = 30.0$ cm. (2) This central region is surrounded on four sides by a 7.75-cm-thick beryllium reflector. (3) The beryllium in turn is contained in a 1.905-cm-thick aluminum tank. (4) Both the top and bottom of the core are reflected by a double layer of metal and water. The first layer is a mixture of water, stainless steel, and aluminum and is 18.73 cm thick, and the second layer is 50 cm of pure water.

A sketch of the core-reflector configuration of the ORR is shown in a $\frac{1}{8}$ section in Fig. 8.28. Note that the origin of the coordinate system is placed at the center of the reactor core.

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The principal nuclear data for the two-group model are listed below:

$k_\infty = 1.677$	$\tau_C = 64$ cm^2	$\epsilon p = 1$
$\Sigma_a^{(2)} = 0.07992$ cm^{-1}	$D_2 = 0.269$ cm	$D_1 = 1.32$ cm
$V_C = 107{,}000$ cm^3	$L_2^2 = 3.366$ cm^2	$M = 67.37$ cm^2

The first step in the calculation is the computation of the multiplication curves for each of the principal axes of the reactor. These have

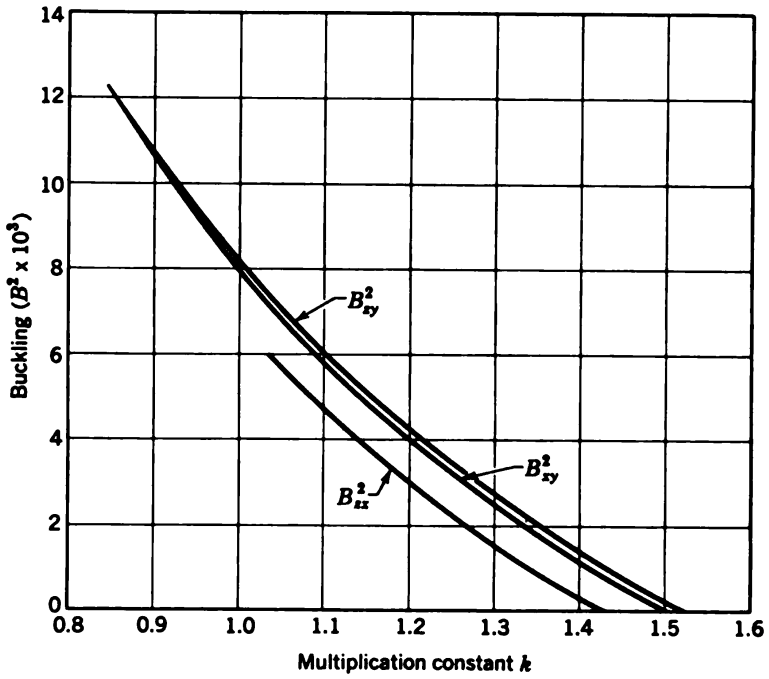

FIG. 8.29 Buckling-versus-multiplication curves for Oak Ridge Research Reactor (standard core loading).

been carried out by means of the scheme described above; however, in the present study it was not necessary to do these by hand since a three-group three-region (3G3R) code was available for the ORACLE. This was especially convenient in this case since the system is in effect a three-region geometry in both x and y directions, as well as in the z direction, if one takes the aluminum container into account. The existing code was applied (after a suitable reduction to a two-group model) to this geometry, and the multiplication constant k was computed for each axis for various values of the buckling for the remaining two directions. These results are shown in Fig. 8.29.

The next step in the calculation is the determination of the buckling B^2 from the multiplication curves. This involves the iterative procedure discussed above. The actual iteration was begun with the assumption that the reflector savings for the first three reactors were all zero. That is, the values used for the $B_{ij}^{(1)}$ in step 2 of the iteration were taken from

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the bare-core dimensions. Even with this very crude guess it was found that the value of B^2 converged to four-figure accuracy in four cycles. The results of these computations are given in Table 8.6. The figures in the left-hand column denote the cycle number. Note that in this particular formulation each row represents a complete reactor calculation so that in effect we have computed 12 reactors. The numbers in the extreme right are the bucklings computed from (8.226) for the value of k recorded at the immediate left. Finally, note that the B_x^2, B_y^2, and B_z^2 values listed under cycle 1 correspond to the bare-core dimensions $a = 27.7$, $b = 16.15$, and $c = 30.0$ cm.

TABLE 8.6 ITERATION FOR THE BUCKLING OF THE ORR REACTOR

1	$B_x^2 = 0.003216$	$B_{yz}^2 = 0.01220$	$k_x = 0.8570$	$B^2 = 0.01387$
	$B_y^2 = 0.009460$	$B_{xz}^2 = 0.005958$	$k_y = 1.037$	$B^2 = 0.008963$
	$B_z^2 = 0.002742$	$B_{xy}^2 = 0.01268$	$k_z = 0.8350$	$B^2 = 0.01425$
2	$B_x^2 = 0.001670$	$B_{yz}^2 = 0.004504$	$k_x = 1.187$	$B^2 = 0.006007$
	$B_y^2 = 0.002934$	$B_{xz}^2 = 0.003240$	$k_y = 1.187$	$B^2 = 0.006007$
	$B_z^2 = 0.001570$	$B_{xy}^2 = 0.004604$	$k_z = 1.163$	$B^2 = 0.006422$
3	$B_x^2 = 0.001503$	$B_{yz}^2 = 0.004585$	$k_x = 1.180$	$B^2 = 0.006127$
	$B_y^2 = 0\ 002767$	$B_{xz}^2 = 0.003321$	$k_y = 1.183$	$B^2 = 0.006075$
	$B_z^2 = 0.001818$	$B_{xy}^2 = 0.004270$	$k_z = 1.183$	$B^2 = 0.006075$
4	$B_x^2 = 0.001542$	$B_{yz}^2 = 0.004559$	$k_x = 1.182$	$B^2 = 0.006092$
	$B_y^2 = 0.002754$	$B_{xz}^2 = 0.003347$	$k_y = 1.182$	$B^2 = 0.006092$
	$B_z^2 = 0.001805$	$B_{xy}^2 = 0.004296$	$k_z = 1.182$	$B^2 = 0.006092$

It is seen from the results of Table 8.6 that to four-figure accuracy the multiplication constant for the reactor is $k = 1.182$; this corresponds to an infinite-medium multiplication $k_\infty = 1.677$. To find the k_∞ (i.e., the fuel loading) for criticality, several other guesses of k_∞ need to be made, the corresponding k computed, and the criticality value obtained from a graph or by interpolation.

As remarked previously, it is not known to what solution the proposed rationale converges. There is some reason to believe, however, that the end result corresponds to the actual reactor. Since an "exact" solution of the completely reflected reactor is not available, a check of the present method was attempted by comparing the computed critical mass for the Oak Ridge Swimming Pool reactor with the known experimentally determined value. This calculation was carried out on the ORACLE using a two-group two-region code derived from the (3G3R) code previously mentioned. However, one important modification was introduced. A subroutine was added to the standard code which carried out the iterative procedure described above. This modification reduced the entire study to the single task of computing the group constants. These data for the

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Swimming Pool reactor, along with the principal dimensions of the system, are given below:

TABLE 8.7 TWO-GROUP CONSTANTS FOR ORNL SWIMMING POOL REACTOR

Flux group	Core	Reflector
Fast	$\tau_C = 64.0$ cm^2 $D_1 = 1.320$ cm $p_C = 1$ $\Sigma_R^{(1)} = 0.02062$ cm^{-1}	$\tau_R = 32.7$ cm^2 $D_3 = 1.19$ cm $p_R = 1.$ $\Sigma_R^{(3)} = 0.0363$ cm^{-1}
Thermal	$D_2 = 0.269$ cm	$D_4 = 0.1602$ cm $\Sigma_a^{(4)} = 0.01916$ cm^{-1}

Reactor geometry:

Core: 4 by 7 array of MTR fuel elements (40.48 by 38.450 by 61.28 cm).

Reflector: Effectively infinite depth of water all around. For calculation assume 20 cm water reflector in all directions.

The multiplication constant for the system described above was computed by the iteration program for several values of the fuel loading (k_∞). By this procedure a critical mass $M_C = 3.050$ kg was obtained. The known experimental value is $M_{exp} = 3.190$ kg. The two-group two-region model using the iterative procedure gives, then, a critical-mass estimate 4.4 per cent low.

8.7 Feynman-Welton Method

a. Introduction. The usefulness of the two-group model in determining the space and energy distribution of neutrons in reflected systems was discussed in Sec. 8.4. Although this model cannot give a detailed picture of the complete neutron-energy spectrum, it is nevertheless a valuable tool for obtaining good estimates of the thermal-flux distribution and of the critical mass. These computations can be performed by hand; however, the labor required is not small, even when one applies the approximate method for evaluating the criticality determinant (Sec. 8.4g). The effort required is further increased if one wishes to account also for the effects of a completely reflected core (Sec. 8.6).

For many practical applications, such as preliminary studies, the primary information required is a reliable estimate of the critical mass. Since the two-group model is well suited to this purpose, it is clear that a formulation of the method which eliminates a large portion of the computational labor would be most useful. Such a method has been devised by R. P. Feynman andT. A. Welton. The Feynman-Welton method draws directly from the results of the one-velocity integral theory. The method effectively separates the space and energy dependence of the neutron-flux function and describes the spatial distribution of each

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energy group in terms of its fundamental mode. However, these simplifications are achieved at the expense of some accuracy; nevertheless, numerical results obtained by this method agree surprisingly well with more reliable models, especially in applications to water-moderated systems.

The particular merits of the method are: (1) it allows a two-group treatment of a reflected reactor with significantly less effort than required by the standard method; (2) it is well suited to small, fast reactor assemblies; and (3) it is especially convenient for making quick criticality calculations.

The method also has several shortcomings. Although it has been shown[1] that the model developed for the bare reactor and for the reactor with nonmoderating reflector has a sound theoretical basis, there is some question about the validity of the model developed for the reactor with a moderating reflector. The treatment in this case is based primarily on a physical approach and is somewhat more intuitive in nature than the treatment of the simpler systems. For this portion of the development, we draw directly from the work of Feynman and Welton.[2] It should be mentioned also that the results obtained for the reactor with moderating reflector apply only to spherically symmetric geometries. These calculations are further limited by the requirement that the reflector be infinite in extent. Although finite reflectors may also be treated by this method, the computational procedure rapidly becomes involved and a great deal of the usefulness of the method is lost.

b. Summary of Method for Application to Two-group Model. The method developed by Feynman and Welton is a general technique for treating the space and energy distribution of neutrons in a chain-reacting system. The derivation of the general theory is the main topic of this section. The most interesting attribute of the method, however, is the relative ease with which the two-group version may be applied to numerical computation. Since an alternative treatment of the two-group model for a thermal reactor has already been discussed in detail, it will be instructive to begin with a comparison of the results obtained by the two methods. This comparison will reveal the unique features of the Feynman-Welton method and serve also to relate some of the new ideas involved with concepts previously established. For this purpose we require a summary of the fundamental relations developed by Feynman

[1] B. Davison, "Transport Theory of Neutrons," National Research Council of Canada, LT-18, Chalk River, Ontario, January, 1947.

[2] See, for example, U.S. Atomic Energy Commission, "Reactor Handbook: Physics," p. 450, McGraw-Hill Book Company, Inc., New York, 1955; and R. P. Feynman and T. A. Welton, "The Calculation of Critical Masses Including the Effects of the Distribution of Neutron Energies," Los Alamos Scientific Laboratory, Report LA-524, Jan. 21, 1947, declassified.

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and Welton; these are presented in a heuristic fashion, accompanied by physical interpretations and arguments. The detailed derivation of all relations is presented in subsequent portions of this section.

Consider then a thermal reactor consisting of a spherical core of radius a and an infinite moderating reflector. For the present two-group model we label the fast neutrons in the core by the index $i = 1$ and those in the reflector by $i = 3$. Thermal neutrons in the core are labeled $i = 2$ and those in the reflector, $i = 4$. By definition, all fissions are caused by thermal neutrons ($i = 2$), and all neutrons born from fission appear in the fast group. We write the neutron-balance relation for this system in terms of the *removal density* (number of neutrons of group i removed per unit time per unit volume around $\mathbf{r}$): $A_i(\mathbf{r}) \equiv \Sigma_r^{(i)}\phi_i(\mathbf{r})$, where $\phi_i(\mathbf{r})$ is the neutron flux for group i and $\Sigma_r^{(i)}$ denotes the removal cross section for group i. For the thermal group ($i = 2, 4$), $\Sigma_r^{(i)}$ is the usual absorption cross section, and for the fast group ($i = 1, 3$), it may be approximated by (8.157). Then, in the core (where $\int_C$ denotes integration over the core)

$$A_1(\mathbf{r}) = \int_C S_1(\mathbf{r}')\, P_1(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \tag{8.231}$$

$$A_2(\mathbf{r}) = \int_C S_2(\mathbf{r}')\, P_2(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' + \int_C S_1(\mathbf{r}')\, P_{12}(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \tag{8.232}$$

where $P_i(\mathbf{r};\mathbf{r}')\, d\mathbf{r}$ is the probability that a neutron of energy group i born at $\mathbf{r}'$, after any number of *elastic*-scattering collisions in both core and reflector, is eventually *removed* as a group i neutron in $d\mathbf{r}$ about $\mathbf{r}$ in the core. The function $P_{12}(\mathbf{r};\mathbf{r}')d\mathbf{r}$ is the probability that a group 1 neutron born at $\mathbf{r}'$ in the core escapes removal in the core, is slowed down in the reflector, and finally returns to the core to be removed as a group 2 neutron in $d\mathbf{r}$ about $\mathbf{r}$. The source functions $S_i(\mathbf{r}')\, d\mathbf{r}'$ give the number of group i neutrons produced in volume $d\mathbf{r}'$ about $\mathbf{r}'$ per unit time. For a thermal reactor these are

$$S_1(\mathbf{r}) = \frac{\nu\Sigma_f^{(2)} A_2(\mathbf{r})}{\Sigma_r^{(2)}} \qquad S_2(\mathbf{r}) = A_1(\mathbf{r}) \tag{8.233}$$

where $\Sigma_f^{(2)}$ is the fission cross section for group 2 neutrons. If these are applied to the balance equations we note that the removal rate of fast neutrons is exactly balanced by the production rate from the fissions produced by the thermal neutrons. Clearly the function P_1 accounts for the leakages of fast neutrons in and out of the core. The removal rate of the thermals, however, has two components. The first is due to fast neutrons slowing down to thermal in the core, and the second, to those which have slowed down in the reflector and then diffused into the core.

In general the removal densities $A_i(\mathbf{r})$ are complicated functions, but in the present calculation we represent these by rough approximations based on the fundamental mode from the one-velocity model. In the

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so-called *lower approximation* used by Feynman and Welton, these are taken to be of the same functional form for all energies; thus,

$$A_i(\mathbf{r}) \equiv A_0^{(i)}\psi_0(\mathbf{r}) \tag{8.234}$$

where $A_0^{(i)}$ is some constant and $\psi_0(\mathbf{r})$ is an elementary function ($\sin B_i r/r$ for a spherically symmetric system). These functions are so selected that they provide a good representation of the spatial distribution of the neutrons in each group throughout the central region of the core (see Fig. 8.30). It is to be expected, then, that appreciable discrepancies will appear at the core-reflector interface. This means that the neutron current across the interface cannot be well estimated from the derivative of $\psi_0(\mathbf{r})$. To correct for this deficiency we introduce the second integral in Eq. (8.232). It is shown in a subsequent analysis that in first approximations the kernel P_{12} is independent of $\mathbf{r}$ and $\mathbf{r}'$ and may be given by $P_{12}(\mathbf{r};\mathbf{r}') \simeq w/V_C$, where V_C is the volume of the core and w is a constant. Then

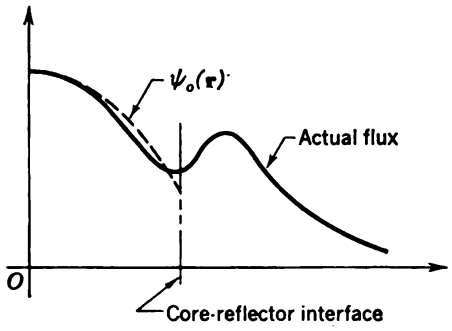

FIG. 8.30 Comparison of $\psi_0(\mathbf{r})$ and actual neutron flux.

$$\int_C S_1(\mathbf{r}')\, P_{12}(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \simeq \frac{w}{V_C}\int_C S_1(\mathbf{r}')\, d\mathbf{r}' = \frac{w\nu\Sigma_f^{(2)}}{V_C\Sigma_r^{(2)}}\int_C A_2(\mathbf{r}')\, d\mathbf{r}' \tag{8.235}$$

Since the integral of S_1 over the core gives the total production of fast neutrons, we can identify w simply as the fraction of all fast neutrons produced which are removed in the core as thermals, after slowing down in the reflector. Moreover, Eq. (8.235) states that the distribution of these removals in the core is spatially uniform.

Our immediate objective is to obtain the criticality condition appropriate to the neutron-balance relations (8.231) and (8.232). For this purpose we require an additional expression for $\psi_0(\mathbf{r})$ which can be used to elminate the integrals in the balance equations. Let us define, therefore,

$$\psi_0(\mathbf{r}) \equiv \mu_0^{(i)}\int_C \psi_0(\mathbf{r}')\, P_i(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \tag{8.236}$$

where $\mu_0^{(i)}$ is the number of neutrons produced in group i per *removal* of a neutron from group i; clearly, $\mu_0^{(i)}$ is analogous to the physical quantity η. A physical interpretation of the above relation is easily established if both sides of the equation are multiplied through by $A_0^{(i)}$. Then the right-hand side is immediately recognized as the number of neutrons produced per unit time in $d\mathbf{r}'$ at $\mathbf{r}'$, namely, $\mu_0^{(i)}A_0^{(i)}\psi_0(\mathbf{r}')\, d\mathbf{r}'$, times the probability that a neutron born at $\mathbf{r}'$ is removed at $\mathbf{r}$, and this product is

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integrated over the core volume. The result is the total number of removals per unit volume per unit time at r, namely, $A_0^{(i)}\psi_0(\mathbf{r})$, as specified by the left-hand side of (8.236). This equation is evidently a direct application of the kernel method discussed in Sec. 5.2d.

If we use the expression (8.234), the integral relation (8.236), and the source function (8.233) in (8.231), the result is an equation in $A_0^{(1)}$ and $A_0^{(2)}$. A second relation between these quantities is obtained by making similar substitutions in Eq. (8.232); in this case, however, it is necessary to integrate over r so as to remove the quantity $\int_C \psi_0(\mathbf{r})\, d\mathbf{r}$ implied in the term (8.235). The resulting pair of equations in the unknowns $A_0^{(1)}$ and $A_0^{(2)}$ is homogeneous, and a functional relationship must be established between the coefficients in order that there exist a nontrivial solution. This is easily obtained by eliminating the $A_0^{(i)}$; the result is the *criticality condition:*

$$\nu = \frac{\mu_0^{(1)}\mu_0^{(2)}\Sigma_r^{(2)}}{\Sigma_f^{(2)}[1 + w\mu_0^{(1)}\mu_0^{(2)}]} \tag{8.237}$$

To solve this relation for the critical concentration implied in $\Sigma_r^{(2)}$ and $\Sigma_f^{(2)}$ we require a relation between the $\mu_0^{(i)}$ and the dimensions of the system. This is obtained from the following differential equation:

$$-D_i\nabla^2\psi_0(\mathbf{r}) + \Sigma_r^{(i)}\psi_0(\mathbf{r}) = \mu_0^{(i)}\Sigma_r^{(i)}\psi_0(\mathbf{r}) \tag{8.238}$$

where the function $\psi_0(\mathbf{r})$ is approximately related to $\phi_i(\mathbf{r})$ by virtue of (8.234); thus,

$$A_0^{(i)}\psi_0(\mathbf{r}) = \Sigma_r^{(i)}\phi_i(\mathbf{r}) \tag{8.239}$$

Note that this equation is satisfied by the ψ_0 of (8.236) by the general proof given for (7.256); a specific demonstration of this property of the ψ_0 is given shortly. If Eq. (8.238) is integrated over the reactor core, the resulting relation may be written in the form

$$\frac{\mu_0^{(i)}\Sigma_r^{(i)}}{\Sigma_r^{(i)} + D_iB_i^2} = 1 \tag{8.240}$$

where, in the case of a spherical reactor with infinite reflector, the bucklings B_i are obtained from the familiar one-velocity relations:

$$\begin{aligned} B_1a \cot B_1a &= 1 - \frac{D_3}{D_1}(\kappa_3a + 1) \qquad \text{with } \kappa_3^2 \equiv \frac{\Sigma_r^{(3)}}{D_3} \simeq \frac{1}{\tau_{\text{th}}^{(3)}} \\ B_2a \cot B_2a &= 1 - \frac{D_4}{D_2}(\kappa_4a + 1) \qquad \kappa_4^2 \equiv \frac{\Sigma_r^{(4)}}{D_4} \end{aligned} \tag{8.241}$$

It is interesting to note that each of the B_i is computed from an equation which involves nuclear constants of group i type neutrons alone. Moreover, the $\psi_0(\mathbf{r})$ used to describe the spatial distribution of neutrons in each energy group, although of the same functional form, have different zeros

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or "wavelength" since in general $B_1 \neq B_2$. The coupling between the two energy groups is achieved through the criticality condition (8.237).

The constant w is computed from the expression

$$w \equiv \left[1 - \frac{1}{\mu_0^{(1)}}\right] \mathcal{P}_{42}$$

where, for the spherically symmetric system,

$$\mathcal{P}_{42} \equiv \left[1 - \frac{1}{\mu_0^{(2)}}\right] \frac{a^2 \Sigma_r^{(2)} \kappa_3^2 \kappa_4^2}{3\Sigma_r^{(4)} (\kappa_3 + \kappa_4)(1 + a\kappa_3)(1 + a\kappa_4)}$$

These relations are developed later in connection with derivation of the reflector thermal source term given in Eq. (8.235).

The computation of the critical concentration for a given core size begins with the solution for the B_i from (8.241). These are then used in (8.240) to obtain the $\mu_0^{(i)}$. The substitution of these results into (8.237) yields the fuel concentration. The details of this model are presented at the end of the present section, and suitable formulas are developed for a direct computation of the critical concentration.

c. One-velocity Model. Since the one-velocity integral theory plays an essential role in the development of the Feynman-Welton method, we will direct attention first to a study of some of the important properties of the integral equations which are involved. It will be convenient in developing some of the pertinent ideas to begin with an elementary system with which we have had some previous experience. Let us consider, therefore, a critical one-velocity reflected reactor in which both core and reflector are homogeneous media. The basic differential equation for the flux in the core of this system is

$$-D\nabla^2\phi(\mathbf{r}) + \Sigma_a\phi(\mathbf{r}) = \nu\Sigma_f\phi(\mathbf{r})$$

We showed previously in our study of the kernel method (see Sec. 5.2d) that the solution to this equation could be written in the integral form

$$\phi(\mathbf{r}) = \nu\Sigma_f \int_C \phi(\mathbf{r}') K(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \tag{8.242}$$

where the integration extends over the core and $K(\mathbf{r};\mathbf{r}')$ denotes the flux at $\mathbf{r}$ due to a unit point source at $\mathbf{r}'$. It follows from this definition that $K(\mathbf{r};\mathbf{r}')$ is the solution to the differential equation

$$-D\nabla^2 K(\mathbf{r};\mathbf{r}') + \Sigma_a K(\mathbf{r};\mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') \tag{8.243}\star$$

In general $K(\mathbf{r};\mathbf{r}')$ will be a complicated function, but one which can usually be obtained by straightforward methods. An example of such a kernel is the function ϕ_+ (and ϕ_-) which we constructed for use in the problem of the slab with the first-scattering sources [cf. Eqs. (5.266)]. Another, even more elementary example of the kernel K is given by the function (5.84) which applies to the special case of the homogeneous

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infinite medium. Throughout most of the work that follows we will not require a detailed definition of the kernels which are specified, but ask only that such functions exist. Thus the introduction of the kernels is in main part a convenient formalism which allows an especially simple presentation of the neutron-balance equations.

The basic form of the balance relations used in the development of the present method is a somewhat modified version of Eq. (8.242). In most instances the cross-section factor which appears in front of the integral will be combined into the definition of the kernel. Thus in our later analyses, we will deal with kernel functions which have a different physical interpretation from those given the function K above. For the present we will not be concerned with the detailed physical features of these systems but will focus our attention on certain mathematical properties of the basic integral equation. Since the equations which arise in subsequent analyses will be very similar mathematically to the form (8.242), we will be able to deduce the fundamental properties of these equations from a knowledge of the properties of Eq. (8.242) and of its kernel.

Equation (8.242) is a homogeneous integral equation[1,2] and possesses solutions only for particular values of the parameter ν. Let us denote these values ν_n, where $n = 0, 1, \ldots$, and let the corresponding solutions to (8.242) be the functions $\psi_n(\mathbf{r})$. Then

$$\psi_n(\mathbf{r}) = \nu_n \Sigma_f \int_C \psi_n(\mathbf{r}') \, K(\mathbf{r};\mathbf{r}') \, d\mathbf{r}' \qquad (8.244)\star$$

The quantities ν_n are the eigenvalues of this equation and the ψ_n the corresponding eigenfunctions. We now demonstrate that the set ψ_n is orthogonal on the assumption that the kernel K is symmetric[3] in the variables $\mathbf{r}$ and $\mathbf{r}'$. That is, if $K(\mathbf{r};\mathbf{r}')$ is the flux at $\mathbf{r}$ due to a unit source at $\mathbf{r}'$, then this must be identical to $K(\mathbf{r}';\mathbf{r})$, the flux at $\mathbf{r}'$ due to a unit source at $\mathbf{r}$. Thus, we require that

$$K(\mathbf{r};\mathbf{r}') \equiv K(\mathbf{r}';\mathbf{r}) \qquad (8.245)\star$$

The first step of the proof is to multiply through equation (8.244) by $\psi_m(\mathbf{r})$ and integrate over the region C. We obtain

$$\int_C \psi_m(\mathbf{r}) \, \psi_n(\mathbf{r}) \, d\mathbf{r} = \nu_n \Sigma_f \int_C \psi_m(\mathbf{r}) \, d\mathbf{r} \int_C \psi_n(\mathbf{r}') \, K(\mathbf{r};\mathbf{r}') \, d\mathbf{r}' \qquad (8.246)$$

[1] A discussion of integral equations is given by R. Courant and D. Hilbert, "Methods of Mathematical Physics," vol. I, pp. 351–371, 122–136. Interscience Publishers, Inc., New York, 1953.

[2] Also, H. Margenau and G. M. Murphy, "The Mathematics of Physics and Chemistry," pp. 509–512, D. Van Nostrand Company, Inc., Princeton, N.J., 1943.

[3] The symmetry property of the one-velocity kernel K is easily demonstrated in the case of both the bare and reflected slab and sphere geometries. For example, note that $\phi_+(x;s) = \phi_-(s;x)$ in Eq. (5.266).

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By interchanging the indices n and m we have also

$$\int_C \psi_n(\mathbf{r})\,\psi_m(\mathbf{r})\,d\mathbf{r} = \nu_m\Sigma_f \int_C \psi_n(\mathbf{r})\,d\mathbf{r} \int_C \psi_m(\mathbf{r}')\,K(\mathbf{r};\mathbf{r}')\,d\mathbf{r}'$$
$$= \nu_m\Sigma_f \int_C \psi_m(\mathbf{r}')\,d\mathbf{r}' \int_C \psi_n(\mathbf{r})\,K(\mathbf{r};\mathbf{r}')\,d\mathbf{r} \quad (8.247)$$

where the last step follows from a change in the order of integration. Since we have required symmetry in K, (8.247) may also be written

$$\int_C \psi_n(\mathbf{r})\,\psi_m(\mathbf{r})\,d\mathbf{r} = \nu_m\Sigma_f \int_C \psi_m(\mathbf{r}')\,d\mathbf{r}' \int_C \psi_n(\mathbf{r})\,K(\mathbf{r}';\mathbf{r})\,d\mathbf{r} \quad (8.248)$$

But, the right-hand side of this equation, except for the factor ν_m, is identical to (8.246); therefore,

$$\int_C \psi_m(\mathbf{r})\,\psi_n(\mathbf{r})\,d\mathbf{r} = 0 \qquad m \neq n$$

where in general $\nu_n \neq \nu_m$. If $n \equiv m$, this integral will be different from zero, and by a suitable construction of the set ψ_n, we can fix its value to be unity. In this way, we find that

$$\int_C \psi_n(\mathbf{r})\,\psi_m(\mathbf{r})\,d\mathbf{r} = \delta_{mn} \quad (8.249)\star$$

where δ_{mn} denotes the Kronecker delta.

The orthonormal set ψ_n may be used to obtain a suitable series expansion for both the flux $\phi(\mathbf{r})$ and the kernel $K(\mathbf{r};\mathbf{r}')$. In order that such expansions exist, it is necessary that the set ψ_n be complete.[1] We will assume that this is the case. Then, we may write

$$\phi(\mathbf{r}) \equiv \sum_n \phi_n\psi_n(\mathbf{r}) \quad (8.250)$$

The ϕ_n are the coefficients of the expansion and can be computed from the relation

$$\phi_n = \int_C \psi_n(\mathbf{r})\,\phi(\mathbf{r})\,d\mathbf{r} \quad (8.251)$$

which follows from the application of the orthogonality properties of the ψ_n. In the same way, we can describe an expansion for the kernel K; thus,

$$K(\mathbf{r};\mathbf{r}') \equiv \sum_n K_n(\mathbf{r}')\,\psi_n(\mathbf{r}) \quad (8.252)$$

The coefficients $K_n(\mathbf{r}')$ are obtained in the manner of the ϕ_n, that is,

$$K_n(\mathbf{r}') = \int_C \psi_n(\mathbf{r})\,K(\mathbf{r};\mathbf{r}')\,d\mathbf{r} \quad (8.253)$$

[1] Courant and Hilbert, *Op. cit.*, pp. 359–360.

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The right-hand side of this equation may be evaluated by the application of (8.244), since by the symmetry property it follows that

$$\int_C \psi_n(\mathbf{r})\, K(\mathbf{r};\mathbf{r}')\, d\mathbf{r} = \int_C \psi_n(\mathbf{r})\, K(\mathbf{r}';\mathbf{r})\, d\mathbf{r} = \frac{\psi_n(\mathbf{r}')}{\nu_n \Sigma_f} \tag{8.254}$$

The substitution of this result into (8.253) yields

$$K_n(\mathbf{r}') = \frac{\psi_n(\mathbf{r}')}{\nu_n \Sigma_f} \tag{8.255}$$

And, the expansion for the kernel takes the form

$$K(\mathbf{r};\mathbf{r}') = \sum_n \frac{\psi_n(\mathbf{r}')\, \psi_n(\mathbf{r})}{\nu_n \Sigma_f} \tag{8.256}\star$$

There is an alternate expansion for this kernel which can be obtained from the original differential equation (8.243). For this purpose let us assume that the coefficients $K_n(\mathbf{r}')$ in (8.252) may be written in the form [cf. (8.255)]

$$K_n(\mathbf{r}') \equiv \frac{\psi_n(\mathbf{r}')}{\lambda_n} \tag{8.257}$$

and compute the quantities λ_n from the requirement that the expansion

$$K(\mathbf{r};\mathbf{r}') \equiv \sum_n \frac{\psi_n(\mathbf{r}')\, \psi_n(\mathbf{r})}{\lambda_n} \tag{8.258}$$

satisfy Eq. (8.243). Direct substitution of this expression into (8.243) yields the relation

$$(D\nabla^2 - \Sigma_a) \sum_n \frac{\psi_n(\mathbf{r}')\, \psi_n(\mathbf{r})}{\lambda_n} = -\delta(\mathbf{r} - \mathbf{r}')$$

If we multiply through by $\psi_m(\mathbf{r}')$ and integrate over the region C, applying the orthogonality property (8.249), we obtain

$$\frac{D}{\lambda_m} \nabla^2 \psi_m(\mathbf{r}) - \frac{\Sigma_a}{\lambda_m} \psi_m(\mathbf{r}) = -\psi_m(\mathbf{r}) \tag{8.259}$$

But we know that the function $\phi(\mathbf{r})$ satisfies both Eq. (8.242) and

$$\nabla^2 \phi(\mathbf{r}) + B^2 \phi(\mathbf{r}) = 0$$

In the same way, the ψ_m satisfy (8.244), and

$$\nabla^2 \psi_m(\mathbf{r}) + B_m^2 \psi_m(\mathbf{r}) = 0 \tag{8.260}\star$$

These properties of the eigenfunctions ψ_m were displayed previously in the study of the integral-equation methods (cf. Sec. 7.4d). The expression (8.260) may be used to eliminate the Laplacian in (8.259); thus, we

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obtain the relation

$$\lambda_m = \Sigma_a + DB_m^2 \tag{8.261}$$

and substitution into (8.258) yields

$$K(\mathbf{r};\mathbf{r}') = \sum_n \frac{\psi_n(\mathbf{r}')\,\psi_n(\mathbf{r})}{\Sigma_a + DB_n^2} \tag{8.262}$$

Upon comparing this result with (8.256), we find that

$$\frac{\nu_n \Sigma_f}{\Sigma_a + DB_n^2} = 1 \tag{8.263}\star$$

Thus the eigenvalues ν_n and B_n are related through an equation identical in form to the criticality condition for the one-velocity reactor. This leads us to attempt a physical interpretation of the ν_n. Several such observations, along with other properties of the eigenvalues and eigenfunctions, are listed below:

1. The eigenvalues for a real, symmetric kernel are all real.[1]

2. The eigenfunctions ψ_n have an oscillatory character, and all (except one[2]) take on negative values in certain regions of the space C. We denote by ψ_0 the function which has no nodes in C, by ψ_1 the function with the next fewest number, and so on. In the case of the bare-slab reactor in diffusion theory we found, for example, that

$$\psi_0(x) = \cos\frac{\pi x}{2\tilde{a}} \qquad \psi_1(x) = \cos\frac{3\pi x}{2\tilde{a}} \qquad \psi_2(x) = \cos\frac{5\pi x}{2\tilde{a}} \cdots$$

Note that ψ_0 has no nodes, ψ_1 has two, and ψ_2 has four.

3. The eigenfunctions $\psi_n(\mathbf{r})$ represent the spatial distributions of the flux according to these modes, and the coefficients ϕ_n of Eq. (8.250) give the *weighting function* for each mode, i.e., the fraction of the neutron population distributed in each space form ψ_n.

4. The eigenvalues ν_n may be ordered; thus

$$\nu_0 < \nu_1 < \nu_2 < \cdots \tag{8.264}$$

We are not interested here in establishing a formal proof of this statement. For this the reader is referred to the literature on integral equations and eigenfunctions.[3] It is possible, nevertheless, to offer some physical arguments based on properties 2 and 3 and Eq. (8.244) which can give some insight into the basis of the inequality (8.264). For this purpose it will be helpful to think of Eq. (8.244) in terms of a physical system.

[1] Margenau and Murphy, *op. cit.*, p. 511.

[2] Courant and Hilbert, *op. cit.*, p. 458.

[3] See, for example, P. M. Morse and H. Feshbach, "Methods of Theoretical Physics," vol. I, pp. 719–725, McGraw-Hill Book Company, Inc., New York, 1953; also, Courant and Hilbert, *op. cit.*, vol. I, pp. 451–455.

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Thus if ψ_n describes the spatial distribution of neutrons in a particular system, then ν_n may be interpreted as the average number of neutrons per fission required to maintain this distribution for all time. We observed previously, in statement 2 above, that ψ_0 has no nodes, and the neutron yield per fission for this system we denote by ν_0. The distribution ψ_1, on the other hand, may exhibit several nodes, and with each there will be a region in which $\psi_1 < 0$. We may think of these regions as occupied by certain particles, the presence of which can cancel an equal number of actual neutrons. Thus to maintain this system in steady state ν_1 must be greater than ν_0. The same argument may be applied to ν_2. However, ν_2 is also greater than ν_1 since ψ_2 is a more rapidly oscillating function (i.e., has more nodal surfaces) than ψ_1, and the "leakage" of mutually annihilating particles into one another will be more pronounced. From this point of view it is clear that increasing the number of nodal surfaces will increase the value of ν required to maintain a particular space mode ψ. Thence follows (8.264).

5. The probability of leakage from the core of neutrons distributed in the mode ψ_n is given by

$$\text{Leakage} = 1 - \frac{1}{\eta_n} \qquad \text{where } \eta_n \equiv \frac{\nu_n \Sigma_f}{\Sigma_a} \tag{8.265}$$

and Σ_a denotes the absorption cross section of the mixture. The above relation follows immediately from the interpretation given for ν_n in statement 4 above. Since

$$\begin{aligned} \eta_n &\equiv \text{average number of neutrons produced per absorption} \\ \frac{1}{\eta_n} &= \text{average number of absorptions per neutron produced} \end{aligned} \tag{8.266}$$

Then

$$\frac{\text{Leakage from core}}{\text{Total neutrons produced}} = \frac{\Sigma_f \int_C \nu_n \psi_n \, d\mathbf{r} - \int_C \Sigma_a \psi_n \, d\mathbf{r}}{\Sigma_f \int_C \nu_n \psi_n \, d\mathbf{r}} = 1 - \frac{1}{\eta_n}$$

d. Multivelocity Reactor with Energy-independent Cross Sections. The analysis of multivelocity reactors is carried out by considering a sequence of reactor systems of increasing generality. The systems studied in this section, both bare and reflected, will be limited by the assumption that all cross sections in the core (and reflector) are energy independent. Studies of more general reactors which include energy-dependent cross sections and moderating reflectors are considered in subsequent sections.

In the present study of multivelocity reactors it will be convenient to separate all scattering collisions into two categories. To the first category we assign all nuclear scattering phenomena which cause substantial

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changes in the kinetic energy of the neutron. These processes we represent by the cross-section symbol $\Sigma_\star$. In the second category we place all other scattering events which yield relatively small (and presumably negligible) changes in neutron energy, and to these we assign the cross-section symbol Σ_e. Precisely which of the various nuclear scattering processes are to be included in each category is determined, of course, by the expected neutron-energy spectrum of the reactor in question. For example, if the system contains an appreciable amount of hydrogenous materials, then in this model it would be appropriate to assign the usual scattering cross section (Σ_s) of the hydrogenous materials to the group represented by $\Sigma_\star$. All other materials in this system would cause relatively small changes in neutron energy in scattering, and one could properly include the scattering cross sections of these materials in the group represented by Σ_e. On the other hand, if the system of interest is a fast reactor, it would be in order to associate the $\Sigma_\star$ with the inelastic cross sections of the various materials present and the Σ_e with the usual "elastic" cross sections Σ_s.

A formal definition of these two scattering cross sections is as follows:

$$\Sigma_\star(E) \equiv \begin{array}{l}\text{average number of "scattering" collisions per unit}\\ \text{path of travel by neutron of energy } E \text{ which could}\\ \text{cause a "substantial" change in its energy}\end{array} \tag{8.267}$$

$$\Sigma_e(E) \equiv \begin{array}{l}\text{average number of "scattering" collisions per unit}\\ \text{path of travel of neutron of energy } E \text{ which cause}\\ \text{a negligible change in its energy}\end{array} \tag{8.268}$$

The term "scattering" has been included in these definitions to differentiate these nuclear events from other processes, such as fission reactions, which can yield neutrons at markedly different energies.

The particular merit of this formulation of the scattering phenomena is that it allows one to treat a large variety of reactor problems by means of the energy (velocity) group methods developed for the present model. Thus by a suitable choice of the scattering–cross section assignments it is possible to study thermal, epithermal, or fast reactors.

The cross sections introduced above may be combined with the cross sections for the absorption phenomena to obtain a new definition for the total cross section; thus,

$$\Sigma_t \equiv \Sigma_\gamma + \Sigma_f + \Sigma_\star + \Sigma_e \tag{8.269}$$

where Σ_γ and Σ_f have the usual meanings (2.40). It is convenient to introduce also a *removal* cross section Σ_r where

$$\Sigma_r \equiv \Sigma_a + \Sigma_\star \tag{8.270}$$

so that

$$\Sigma_t = \Sigma_r + \Sigma_e \tag{8.271}$$

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Note that the cross section Σ_r will be useful for computing the number of neutrons lost from an energy group, since both the Σ_a and $\Sigma_\star$ are "removal" processes.

1. *Bare Homogeneous Reactor.* We begin our treatment of multivelocity reactors with a study of the bare reactor in which all cross sections are energy independent. Neutrons emerging from fission and $\Sigma_\star$ reactions will, in general, be distributed according to some specified energy spectrum, however.

For this analysis we introduce the following functions:

$$\begin{aligned} \phi(\mathbf{r},E)\,d\mathbf{r}\,dE &\equiv \text{total track length per unit time in } d\mathbf{r} \text{ about } \mathbf{r} \\ &\quad \text{due to neutrons with energies in } dE \text{ about } E \\ S(\mathbf{r},E)\,d\mathbf{r}\,dE &\equiv \text{total number of neutrons produced per unit} \\ &\quad \text{time in } d\mathbf{r} \text{ about } \mathbf{r} \text{ with energies in } dE \text{ about } E \\ P_B(\mathbf{r};\mathbf{r}')\,d\mathbf{r} &\equiv \text{probability}^1 \text{ that a neutron produced at } \mathbf{r}' \text{ is} \\ &\quad \textit{removed} \text{ in } d\mathbf{r} \text{ about } \mathbf{r} \end{aligned} \tag{8.272}$$

The function $P_B(\mathbf{r};\mathbf{r}')$ defined here will be the kernel in the integral-equation formulation of the neutron-balance equation; note that it is independent of energy in this particular model in which the cross sections are energy independent. It should be noted that this definition differs from that given the function $K(\mathbf{r};\mathbf{r}')$ used in the preceding section. Whereas the kernel K represented the flux at $\mathbf{r}$ due to a unit source at $\mathbf{r}'$, the kernel P_B implies a function of the type K multiplied by Σ_r. This relationship may be seen more clearly if we consider, for example, the quantity $\Sigma_r K(\mathbf{r};\mathbf{r}')\,d\mathbf{r}$, a physical interpretation of which is

$$\begin{aligned} \Sigma_r K(\mathbf{r};\mathbf{r}')\,d\mathbf{r} &\equiv \text{number of neutrons removed per unit time in } d\mathbf{r} \\ &\quad \text{about } \mathbf{r} \text{ due to one neutron released per unit} \\ &\quad \text{time from } \mathbf{r}' \\ &\equiv \text{probability that neutron released at } \mathbf{r}' \text{ is} \\ &\quad \text{removed in } d\mathbf{r} \text{ about } \mathbf{r} \end{aligned} \tag{8.273}$$

The last statement follows from the fact that the flux K is due to a unit source.

In setting up the neutron-balance relation it will be convenient to use, in place of the flux, the *removal* (*collision*) *density* $A(\mathbf{r},E)$, where

$$A(\mathbf{r},E) \equiv \Sigma_r \phi(\mathbf{r},E) \tag{8.274}_\star$$

The neutron balance is then given by

$$A(\mathbf{r},E) = \int_{\text{reactor}} S(\mathbf{r}',E)\, P_B(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' \tag{8.275}$$

[1] The subscript B is added here to distinguish this function for the bare reactor from subsequent definitions of P given for multiregion systems.

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In terms of the diffusion-theory notation, the source function S takes the form

$$S(\mathbf{r},E) = \nu z(E)\Sigma_f \int_0^\infty \phi(\mathbf{r},E')\,dE' + \Sigma_\star \int_0^\infty \phi(\mathbf{r},E')\,h(E;E')\,dE' \equiv \int_0^\infty A(\mathbf{r},E')\,\mathcal{S}(E,E')\,dE' \tag{8.276}$$

where we have defined

$$\mathcal{S}(E,E') \equiv \frac{\nu z(E)\Sigma_f + \Sigma_\star h(E;E')}{\Sigma_r} \tag{8.277}$$

$$h(E;E')\,dE \equiv \begin{matrix}\text{probability that a neutron of energy } E' \text{ when}\\ \text{scattered by the process (s) represented by } \Sigma_\star\\ \text{emerges with energy in } dE \text{ about } E\end{matrix} \tag{8.278}$$

and $z(E)$ denotes the fission spectrum [cf. Eq. (4.153)]. The substitution of the relation (8.276) into (8.275) yields

$$A(\mathbf{r},E) = \int_0^\infty \mathcal{S}(E,E')\,dE' \int_{\text{reactor}} A(\mathbf{r}',E')\,P_B(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' \tag{8.279}$$

We look for solutions of this equation which have the form

$$A(\mathbf{r},E) \equiv A(E)\psi(\mathbf{r}) \tag{8.280}$$

Substitution of this relation into (8.279) gives immediately complete separation of the space and energy dependence:

$$A(E) = m\int_0^\infty \mathcal{S}(E,E')\,A(E')\,dE' \qquad (a)$$
$$\psi(\mathbf{r}) = \lambda \int_{\text{reactor}} \psi(\mathbf{r}')\,P_B(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' \qquad (b) \tag{8.281}$$

and the condition

$$m\lambda \equiv 1 \tag{8.282}$$

These results are entirely consistent with those of our previous analysis of the bare reactor using the Fermi age model (refer to Sec. 6.3). In this formulation, Eq. (8.281a) describes the neutron-energy spectrum and is easily recognized as the integral equation for the collision density in an infinite homogeneous system. If we select, for example [cf. Eq. (4.36)],

$$h(E;E') = \frac{1}{(1-\alpha)E'}$$

and let $\Sigma_\star \equiv \Sigma_s$ so that $\Sigma_r = \Sigma_s + \Sigma_a \equiv \Sigma_t$, then Eq. (8.281$a$) takes on the form of Eqs. (4.105) when the fission neutrons are represented by a monoenergetic source. The solution of these equations in the asymptotic range is discussed in Sec. 4.4b. In the more general situation involving a distributed fission spectrum $z(E)$, the methods and general results of Sec. 4.6 are directly applicable for obtaining solutions in the asymptotic

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range and lead to the critical equation in the usual way [see, for example, Eq. (4.162)].

Equation (8.281*b*) gives the spatial distribution of the neutrons throughout the reactor at all energies. The solutions to this equation are given by a set of eigenfunctions $\psi_n(\mathbf{r})$ with eigenvalues λ_n [cf. Eq. (8.244)]:

$$\psi_n(\mathbf{r}) = \lambda_n \int_{\text{reactor}} \psi_n(\mathbf{r}') \, P_B(\mathbf{r};\mathbf{r}') \, d\mathbf{r}' \tag{8.283}$$

For an actual system having the nuclear and geometric properties assumed above, only one of the solutions (8.283) is physically acceptable. This solution is the fundamental mode $\psi_0(\mathbf{r})$. It is clear that, for $n > 0$, the steady-state spatial distribution would require that $A(\mathbf{r},E) < 0$ in certain regions of the reactor. The determination of the neutron-flux spatial distribution in a finite system by means of the integral-equation formulation was demonstrated in Sec. 5.7c for the case of an infinite slab. Those results may be applied directly to the multiplying medium problem. In the case of the infinite-slab reactor, of width $2a$, Eq. (8.283) takes the form

$$\psi_0(x) = \lambda_0 \int_{-a}^{a} \psi_0(x') \, P_B(x;x') \, dx' \tag{8.284}$$

for the fundamental mode. If we adopt the formalism of Eq. (5.266), then the one-velocity kernel $P_B(x;x')$ may be written

$$P_B(x;x') = \Sigma_t \phi(x;x') \tag{8.285}$$

where $\phi(x;x')$ is the flux at x due to a unit source at x'. Substitution into Eq. (8.284) yields

$$\psi_0(x) = \lambda_0 \Sigma_t \int_{-a}^{a} \psi_0(x') \, \phi(x;x') \, dx' \equiv \nu\Sigma_f \int_{-a}^{a} \psi_0(x') \, \phi(x;x') \, dx' \tag{8.286}$$

where the last equality follows from the general form[1] (8.242). If we apply the solutions (5.266) for the kernel $\phi(x;x')$, then it is easily shown that the $\psi_0(x)$ given by (8.286) also satisfies the equation $\psi_0''(x) + B_0^2\psi_0(x) = 0$, where $B_0^2 = (\nu\Sigma_f - \Sigma_a)/D$. Thus in this simple case $\psi_0(x)$ has the usual trigonometric form, and the condition on the eigenvalue is the usual criticality relation for the one-velocity model. The critical size o this reactor is obtained by computing the lowest eigenvalue m_0 of Eq. (8.281*a*) and adjusting the size of the reactor until the lowest eigenvalue λ_0 of (8.286) satisfies the relation

$$\lambda_0 = \frac{1}{m} \tag{8.287}$$

where in this case $\lambda_0 = \nu\Sigma_f/\Sigma_t$ and the cross sections are related to the reactor size through the B_0^2 equation.

[1] Note that the functions K and $\phi(x;x')$ have identical definitions.

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2. *Reflected Reactor with No Moderation in Reflector.* As in the preceding case, all cross sections in both core and reflector are independent of energy although the spectrums $z(E)$ and $h(E;E')$ need not be. All cross sections in the core Σ_f, $\Sigma_\gamma^{(C)}$, $\Sigma_\star^{(C)}$, and $\Sigma_e^{(C)}$ are in general different from zero. In the reflector $\Sigma_\gamma^{(R)}$ and $\Sigma_e^{(R)}$ are different from zero, but $\Sigma_f = \Sigma_\star^{(R)} = 0$.

For this system we define the kernel $P(\mathbf{r};\mathbf{r}')$ such that

$$P(\mathbf{r};\mathbf{r}')\,d\mathbf{r} \equiv \begin{matrix}\text{probability that neutron born at } \mathbf{r}' \\ \text{is removed in } d\mathbf{r} \text{ about } \mathbf{r}\end{matrix} \tag{8.288}$$

In this definition $\mathbf{r}'$ must be in the core; however, this kernel allows neutrons created in the core to wander out into the reflector, be "elastically scattered" about and finally *removed* in either the core or the reflector.

The sources in the reactor are given by

$$\begin{aligned} \mathcal{S}_C(E,E') &\equiv \frac{\nu z(E)\Sigma_f + \Sigma_\star^{(C)} h(E;E')}{\Sigma_r^{(C)}} \\ \mathcal{S}_R(E,E') &\equiv 0 \end{aligned} \tag{8.289}$$

The neutron-balance equation is of the form (8.275), with the P_B now replaced by P given in (8.288). The substitution of (8.289), along with the general relation (8.276), into (8.275) yields

$$A(\mathbf{r},E) = \int_0^\infty \mathcal{S}_C(E,E')\,dE' \int_C A(\mathbf{r}',E')\,P(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' \tag{8.290}$$

where $\int_C$ denotes integration over the core. The solution to this equation is accomplished by the separation of variables, as in the bare-reactor calculation. The results are

$$\begin{aligned} A(E) &\equiv m \int_0^\infty \mathcal{S}_C(E,E')\,A(E')\,dE' \qquad (a) \\ \psi(\mathbf{r}) &= \lambda \int_C \psi(\mathbf{r}')\,P(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' \qquad (b) \end{aligned} \tag{8.291}$$

Equation (8.291b) has a set of eigenfunctions $\psi_n(\mathbf{r})$ and eigenvalues λ_n. The ψ_n are defined as orthogonal over the *core only*.

The solution for the critical size proceeds as before.

e. Multivelocity Reactor with Energy-dependent Cross Sections. The basic ideas developed for the reactor with constant cross sections are applied here to more general systems. In the present study we allow for the energy dependence of all cross sections and focus our attention on reactors which are either (1) bare, or (2) have reflectors which do not moderate neutrons. For all reactors in class 1, the cross sections of the core Σ_f, Σ_γ, $\Sigma_\star$, and Σ_e will be, in general, some specified functions of energy. For systems described by class 2, we allow in addition that

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the reflector cross sections $\Sigma_\gamma^{(R)}$ and $\Sigma_e^{(R)}$ be energy dependent, but that $\Sigma_f^{(R)} = \Sigma_\star^{(R)} \equiv 0$.

In the present treatment we study in detail reactors of class 2. The results derived for these systems are easily modified to describe those of class 1. As in the study of the constant cross-section systems of Sec. 8.7d, we begin the analysis with a statement of the neutron-balance condition. For this purpose we require the functions $\phi(\mathbf{r},E)$ and $A(\mathbf{r},E)$ defined by (8.272). In addition we introduce the kernel

$$P(\mathbf{r};\mathbf{r}',E)\,d\mathbf{r} \equiv \begin{array}{l}\text{probability that a neutron born at } \mathbf{r}' \text{ with energy} \\ E \text{ is removed in } d\mathbf{r} \text{ about } \mathbf{r} \text{ without intervening} \\ \text{energy loss}\end{array} \tag{8.292}$$

Thus this new kernel includes the possibility that the neutron of energy E created at $\mathbf{r}'$ may experience any number of *elastic* scattering collisions (described by Σ_e) in both core and reflector before it is finally removed in $d\mathbf{r}$ about $\mathbf{r}$.

The neutron-balance equation may be expressed in terms of this function. If as before we define a source $S(\mathbf{r},E)$ according to (8.272), then the removal rate is given by

$$A(\mathbf{r},E) = \int_C S(\mathbf{r}',E)\,P(\mathbf{r};\mathbf{r}',E)\,d\mathbf{r}' \tag{8.293}\star$$

In the diffusion-theory notation, it is clear that the source may be defined by [cf. (8.276)]

$$\begin{aligned} S(\mathbf{r}',E) &\equiv \nu z(E) \int_0^\infty \Sigma_f(E')\,\phi(\mathbf{r}',E')\,dE' \\ &\qquad + \int_0^\infty \Sigma_\star(E')\,\phi(\mathbf{r}',E')\,h(E;E')\,dE' \\ &\equiv \int_0^\infty A(\mathbf{r}',E')\,\mathcal{S}(E,E')\,dE' \end{aligned} \tag{8.294}$$

where [cf. (8.277)]

$$\mathcal{S}(E,E') = \frac{\nu z(E)\,\Sigma_f(E') + \Sigma_\star(E')\,h(E;E')}{\Sigma_r^{(C)}(E')} \tag{8.295}\star$$

The substitution of these expressions into (8.293) yields

$$A(\mathbf{r},E) = \int_0^\infty \mathcal{S}(E,E')\,dE' \int_C A(\mathbf{r}',E')\,P(\mathbf{r};\mathbf{r}',E)\,d\mathbf{r}' \tag{8.296}\star$$

It is clear from the above expressions that, since there are no "sources" of neutrons in the reflector, the spatial integration extends over the core volume only.

The immediate objective of the present section is the determination of the solutions to Eq. (8.296) or (8.293). It will be convenient in undertaking this analysis to introduce a somewhat modified version of the basic relation (8.293), which may be derived from a different physical

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point of view. For this purpose we construct a fictitious one-velocity system in which all neutrons have energy E. The cross sections (Σ') of this system are related to the cross sections of the actual system through the following definitions:

$$\begin{array}{lll} \textit{In the reflector:} & \Sigma'_\gamma \equiv \Sigma_\gamma & \Sigma'_e \equiv \Sigma_e \\ \textit{In the core:} & (1)\ \Sigma'_f \equiv \Sigma_f & \Sigma'_e \equiv \Sigma_e \\ & (2)\ \Sigma'_\star \equiv 0 & \\ & (3)\ \Sigma'_\gamma \equiv \Sigma_\gamma + \Sigma_\star & \end{array} \tag{8.297}$$

We note immediately from these expressions that the removal cross section of the fictitious system (Σ'_r) is identical to that of the actual system, since

$$\Sigma'_r \equiv \Sigma'_f + \Sigma'_\gamma + \Sigma'_\star = \Sigma'_f + \Sigma'_\gamma \equiv \Sigma_f + \Sigma_\gamma + \Sigma_\star \equiv \Sigma_r \tag{8.298}$$

The only significant difference between the two systems, then, is that the "slowing-down" cross section of the fictitious system ($\Sigma'_\star$) is zero.

Now, let us introduce also a quantity to define the neutron yield for the fictitious system; thus,

$$\mu(E) \equiv \text{average number of neutrons } \textit{produced per removal} \tag{8.299}\star$$

It is clear from these definitions that the source $S(\mathbf{r},E)$ in the fictitious system is due only to the removal reactions. We can write, therefore,

$$S(\mathbf{r},E) \equiv \mu(E)\ \Sigma'_r(E)\ \phi(\mathbf{r},E) \equiv \mu(E)\ A(\mathbf{r},E) \tag{8.300}$$

The substitution of this relation into (8.293) yields

$$A(\mathbf{r},E) = \mu(E) \int_C A(\mathbf{r}',E)\ P(\mathbf{r};\mathbf{r}',E)\ d\mathbf{r}' \tag{8.301}\star$$

It is important to recognize that, in this formulation, the variable E which appears in the integral equation serves only as a parameter. The principal variable(s) is $\mathbf{r}$. Thus the analytical machinery developed previously for the one-velocity model (Sec. 8.7c) is directly applicable to the present system. It is possible, therefore, to construct a set of eigenfunctions $\psi_n(\mathbf{r},E)$ for Eq. (8.301) which are solutions to the equation

$$\psi_n(\mathbf{r},E) = \mu_n(E) \int_C \psi_n(\mathbf{r}',E)\ P(\mathbf{r};\mathbf{r}',E)\ d\mathbf{r}' \tag{8.302}$$

where the $\mu_n(E)$ are the corresponding eigenvalues [cf. Eq. (8.244)]. With the present point of view it is easy to show that the properties of the eigenfunctions $\psi_n(\mathbf{r},E)$ and of the kernel $P(\mathbf{r};\mathbf{r}',E)$ are identical to those of the $\psi_n(\mathbf{r})$ and $K(\mathbf{r};\mathbf{r}')$ developed in Sec. 8.7c. Consider first the orthogonality properties. From physical requirements it is clear that the kernel $P(\mathbf{r};\mathbf{r}',E)$ must be symmetric in $\mathbf{r}$ and $\mathbf{r}'$. If we bear in mind that E is to be treated as a parameter, then all the derivation leading to the orthogonality relation (8.249) is directly applicable to the functions

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$\psi_n(\mathbf{r},E)$ and $P(\mathbf{r};\mathbf{r}',E)$, and we obtain immediately the property

$$\int_C \psi_n(\mathbf{r}',E)\,\psi_m(\mathbf{r}',E)\,d\mathbf{r}' = \delta_{mn} \tag{8.303}\star$$

The series expansion for the kernel $P(\mathbf{r};\mathbf{r}',E)$ in terms of the eigenfunctions $\psi_n(\mathbf{r},E)$ is also easily demonstrated. As in the one-velocity derivation we postulate that the $\psi_n(\mathbf{r},E)$ form a complete set. It follows, therefore, that we can construct an expression for $A(\mathbf{r},E)$ in the form

$$A(\mathbf{r},E) \equiv \sum_n A_n(E)\,\psi_n(\mathbf{r},E) \tag{8.304}$$

where the coefficients may be computed from the relation

$$A_n(E) = \int_C \psi_n(\mathbf{r},E)\,A(\mathbf{r},E)\,d\mathbf{r} \tag{8.305}$$

In the same way we define, on the basis of our experience with the kernel K, the series expansion [cf. Eq. (8.258)]

$$P(\mathbf{r};\mathbf{r}',E) \equiv \sum_n P_n(E)\,\psi_n(\mathbf{r}',E)\,\psi_n(\mathbf{r},E) \tag{8.306}$$

If we use this expression in (8.302), it is easily shown that the coefficients $P_n(E)$ are given by

$$P_n(E) = \frac{1}{\mu_n(E)} \tag{8.307}$$

Thus,

$$P(\mathbf{r};\mathbf{r}',E) = \sum_n \frac{\psi_n(\mathbf{r}',E)\,\psi_n(\mathbf{r},E)}{\mu_n(E)} \tag{8.308}\star$$

The two basic results obtained above, (8.303) and (8.308), are generally valid and may be applied directly to the physical system described by (8.296). Our only purpose in introducing the fictitious system was to offer some aid in grasping the essential ideas and analytical tools which may be used to solve the neutron-balance equation. Let us now apply these results to Eq. (8.296). The substitution of (8.304) into (8.296) yields

$$\sum_n A_n(E)\,\psi_n(\mathbf{r},E) = \int_0^\infty \mathbb{S}(E,E')\,dE' \int_C P(\mathbf{r};\mathbf{r}',E) \sum_n A_n(E')\,\psi_n(\mathbf{r}',E')\,d\mathbf{r}'$$

We multiply this equation by $\psi_m(\mathbf{r},E)$ and integrate over the core. With the aid of (8.303), we obtain

$$\begin{aligned} A_m(E) &= \int_0^\infty \mathbb{S}(E,E')\,dE' \sum_n A_n(E') \int_C \psi_m(\mathbf{r},E)\,d\mathbf{r} \int_C P(\mathbf{r};\mathbf{r}',E)\,\psi_n(\mathbf{r}',E')\,d\mathbf{r}' \\ &= \int_0^\infty \mathbb{S}(E,E')\,dE' \sum_n A_n(E') \int_C \psi_n(\mathbf{r}',E')\,d\mathbf{r}' \int_C P(\mathbf{r};\mathbf{r}',E)\,\psi_m(\mathbf{r},E)\,d\mathbf{r} \end{aligned} \tag{8.309}$$

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Since the kernel P is symmetric, we may apply (8.302) to the $\mathbf{r}$ integration; this yields $\psi_m(\mathbf{r}',E)/\mu_m(E)$. Equation (8.309) then reduces to the form

$$A_m(E) = \frac{1}{\mu_m(E)} \int_0^\infty \mathcal{S}(E,E')\, dE' \sum_n A_n(E') \int_C \psi_n(\mathbf{r}',E')\, \psi_m(\mathbf{r}',E)\, d\mathbf{r}' \tag{8.310}$$

If we call

$$A_{nm}(E,E') \equiv \int_C \psi_n(\mathbf{r}',E')\, \psi_m(\mathbf{r}',E)\, d\mathbf{r}' \tag{8.311}$$

then (8.310) may be written

$$A_m(E) = \frac{1}{\mu_m(E)} \sum_n \int_0^\infty A_n(E')\, \mathcal{S}(E,E')\, A_{nm}(E,E')\, dE' \tag{8.312}\star$$

It should be noted that, for $E = E'$, (8.311) reduces to the orthogonality relation (8.303), i.e.,

$$A_{nm}(E,E) = \delta_{mn} \tag{8.313}$$

The eigenfunctions $\psi_n(\mathbf{r},E)$ are computed from (8.302) and represent the solutions to a set of *one-velocity problems*, each defined by a specific value (or increment) of E. Once these one-velocity problems have been solved, the $A_{nm}(E,E')$ may be computed from (8.311), and these in turn used to solve the integral equation (8.312) in the single variable E. Thus the more difficult integral equation (8.296) involving two independent sets of variables $\mathbf{r}$ and E is replaced by a system [Eqs. (8.302)] of integral equations in $\mathbf{r}$ (for each E) and another [Eqs. (8.312)] in E. The solution of the neutron-balance equation (8.296) is greatly simplified in this way, especially since in many cases of interest the infinite series in (8.312) is well represented by the first few (frequently the first) terms.

This formulation of the general space-energy problem is well adapted for use in the energy-group method, as will be demonstrated in subsequent analyses.

f. Upper and Lower Approximations. Various approximations have been suggested for solving the neutron-balance equation (8.296) which we have reduced to the set (8.302) and (8.312). These involve the use of approximate expressions for either of the two functions $A(\mathbf{r},E)$ or $P(\mathbf{r};\mathbf{r}',E)$. In the present treatment we apply the approximation to the removal density $A(\mathbf{r},E)$. Specifically, we assume that

$$A(\mathbf{r},E) \equiv \sum_n A_n(E)\, \psi_n(\mathbf{r},E) \simeq A_0(E)\, \psi_0(\mathbf{r},E) \tag{8.314}$$

Thus we have set all $A_n \equiv 0$ for $n > 0$. The corresponding expression

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for the flux takes the form

$$\phi(\mathbf{r},E) \equiv \frac{A(\mathbf{r},E)}{\Sigma_r(E)} \simeq \frac{A_0(E)\,\psi_0(\mathbf{r},E)}{\Sigma_r(E)} \tag{8.315}$$

If we use the relation (8.314) in (8.312), we obtain the single integral equation

$$A_0(E) = \frac{1}{\mu_0(E)} \int_0^\infty A_0(E')\,\mathcal{S}(E,E')\,A_{00}(E,E')\,dE' \tag{8.316}\star$$

Numerical calculations have shown that the use of this relation in reactor calculations yields an overestimate of the critical mass. This may be seen from the fact that in estimating the removal rate (effectively the neutron "absorption" rate) we have ignored the contributions of the higher modes in (8.314). Thus the neutron production rate in the core has been underestimated, and the resulting computation of the critical mass will be larger than that required to maintain the system at steady state. For this reason we call (8.316) the *upper approximation*.

A second estimate of $A_0(E)$ may be obtained by assuming, further, that

$$\psi_0(\mathbf{r},E) \simeq \psi_0(\mathbf{r},E') \equiv \psi_0(\mathbf{r}) \tag{8.317}$$

That is, we assert that the fundamental modes of neutrons of all energies are in first approximation nearly the same. If we use this assumption in (8.311), we obtain

$$A_{00}(E,E') \simeq 1 \tag{8.318}$$

and it follows that (8.316) reduces to the form

$$A_0(E) = \frac{1}{\mu_0(E)} \int_0^\infty A_0(E')\,\mathcal{S}(E,E')\,dE' \tag{8.319}\star$$

This relation is called the *lower approximation*, since it is believed to yield an underestimate of the critical mass. The comparison of mass estimates based on these two approximations with known experimentally determined values would indicate that this is indeed true. Furthermore, there is reason to believe that both approximations are not far removed from the correct value. The validity of these assertions is best determined by comparison with experiment in each case.

Since both estimates appear to be fairly reliable, we will use the lower approximation in all subsequent analyses in view of the fact that the relation (8.319) is more easily applied to reactor calculations. We con-

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tinue with the calculation on this basis and introduce the source function (8.295) into (8.319).

$$\mu_0(E)\ A_0(E) = \nu z(E) \int_0^\infty \frac{\Sigma_f(E')\ A_0(E')\ dE'}{\Sigma_r^{(C)}(E')} + \int_0^\infty \frac{\Sigma_\star(E')\ h(E;E')\ A_0(E')\ dE'}{\Sigma_r^{(C)}(E')} \tag{8.320}$$

This equation may also be written in terms of the flux. Let

$$\phi(\mathbf{r},E) \equiv \sum_n \phi_n(E)\ \psi_n(\mathbf{r},E) \tag{8.321}$$

Since $A(\mathbf{r},E)$ is directly related to the flux and has been defined by the series (8.304), it is easy to show that

$$A_n(E) = \Sigma_r(E)\ \phi_n(E) \tag{8.322}$$

We apply this relation to (8.320) and obtain

$$\Sigma_r(E)\ \phi_0(E)\ \mu_0(E) = z(E) + \int_0^\infty \Sigma_\star(E')\ h(E;E')\ \phi_0(E')\ dE' \tag{8.323}\star$$

where we have normalized the neutron production, i.e., we have set

$$\nu \int_0^\infty \Sigma_f(E')\ \phi_0(E')\ dE' = 1 \tag{8.324}$$

g. Two-group Model for Bare Homogeneous Reactor. Throughout the remainder of the present treatment of the Feynman-Welton method we adopt the energy-group description of the neutron-energy spectrum in a multivelocity reactor in place of the continuous spectrum previously considered. In practical situations wherein the Feynman-Welton model is most useful, reactors reflected by moderating materials, the two-group formulation is the more easily and effectively applied.

The basic relations for the two-group model are developed by treating a sequence of progressively more general reactor problems. We begin with the bare homogeneous reactor. Since the energy-group relations are derived from the continuous-energy-spectrum results, we summarize these equations below:

$$A(\mathbf{r},E) = \int_C S(\mathbf{r}',E)\ P_B(\mathbf{r};\mathbf{r}',E)\ d\mathbf{r}' \tag{8.325}$$

$$S(\mathbf{r}',E) \equiv \int_0^\infty \frac{A(\mathbf{r}',E')}{\Sigma_r(E')} [\nu z(E)\ \Sigma_f(E') + \Sigma_\star(E')\ h(E;E')]\ dE' \tag{8.326}$$

where $P_B(\mathbf{r};\mathbf{r}',E)$ is the kernel for the bare reactor and comparable to $P(\mathbf{r};\mathbf{r}',E)$ defined by (8.292).

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In the two-group formulation the various functions which appear in the above equations take the special forms given below. The index $i = 1$ denotes fast neutrons, $i = 2$, slow.

$$A_i(\mathbf{r}) \equiv \text{number of removals per unit volume per unit time around space point } \mathbf{r} \text{ of neutrons in } i\text{th energy group} \tag{8.327}$$

$$S_i(\mathbf{r}) \equiv \text{number of neutrons of energy group } i \text{ born per unit volume per unit time around point } \mathbf{r} \tag{8.328}$$

$$P_B^{(i)}(\mathbf{r};\mathbf{r}')\, d\mathbf{r} \equiv \text{probability that neutron of energy group } i \text{ born at } \mathbf{r}' \text{ is eventually removed as an } i\text{-group neutron in } d\mathbf{r} \text{ about } \mathbf{r} \tag{8.329}$$

$$\Sigma_{12} \equiv \text{average number of scattering collisions per unit path of travel by neutron of energy group 1 which reduce it to an energy group 2 neutron} \tag{8.330}$$

Note that the group cross section Σ_{12} is in fact the "slowing-down" cross section (analogous to Σ_s) from the fast group to the slow. With these definitions Eqs. (8.325) and (8.326) may be written

$$A_i(\mathbf{r}) = \int_C P_B^{(i)}(\mathbf{r};\mathbf{r}')\, S_i(\mathbf{r}')\, d\mathbf{r}' \tag{8.331}$$

with

$$S_1(\mathbf{r}) = \nu z_1[\gamma_1 A_1(\mathbf{r}) + \gamma_2 A_2(\mathbf{r})] \tag{8.332}$$

$$S_2(\mathbf{r}) = \nu z_2[\gamma_1 A_1(\mathbf{r}) + \gamma_2 A_2(\mathbf{r})] + \frac{\Sigma_{12} A_1(\mathbf{r})}{\Sigma_r^{(1)}} \tag{8.333}$$

where z_i denotes the number of neutrons of energy group i produced per fission reaction and

$$\gamma_i \equiv \frac{\Sigma_f^{(i)}}{\Sigma_r^{(i)}} \tag{8.334}$$

As indicated previously we apply the lower approximation to this calculation. First, we have from (8.314),

$$A_i(\mathbf{r}) = A_0^{(i)} \psi_0^{(i)}(\mathbf{r}) \tag{8.335}$$

Then, from (8.317),

$$\psi_0^{(1)}(\mathbf{r}) = \psi_0^{(2)}(\mathbf{r}) \equiv \psi_0(\mathbf{r}) \tag{8.336}$$

The application of these relations to the integral equation (8.302) yields, for the fundamental modes [where $\mu_0^{(i)}$ is defined as in Eq. (8.236)],

$$\psi_0^{(i)}(\mathbf{r}) = \mu_0^{(i)} \int_C \psi_0^{(i)}(\mathbf{r}')\, P_B^{(i)}(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \tag{8.337}$$

If the kernel $P_B^{(i)}(\mathbf{r};\mathbf{r}')$ were known in detail, the solution of this equation would yield the spatial distributions for the neutrons in group i. These results in turn could be used in Eqs. (8.331) through (8.333), and a simultaneous solution would give the criticality condition for the reactor.

As a rule, the calculation of the $P_B^{(i)}$ will be somewhat involved, like-

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wise, that of the $\psi_0^{(i)}(\mathbf{r})$. These computations became even more complicated when dealing with reflected systems. It is possible, however, to obtain reasonable approximations to these functions, in particular to the spatial modes $\psi_0^{(i)}$ and the eigenvalues $\mu_0^{(i)}$, by the application of the diffusion-theory or infinite-medium transport equation results [cf. Eq. (7.254)]. In the present treatment we consider in detail only the computational procedure based on the diffusion-theory solutions. The use of the transport-theory results in the calculation of the eigenvalues is easily shown and involves some minor changes in the procedure developed for the diffusion theory.

The application of the diffusion theory to the calculation of the eigenfunctions and eigenvalues for the Feynman-Welton method requires a restatement of the neutron-balance equation in terms of the removal cross section $\Sigma_r^{(i)}$ and the neutron yield per removal $\mu_0^{(i)}$. It is obvious that the appropriate form is [cf. equation preceding (8.242) and Eq. (8.238)]

$$-D_i\nabla^2\phi_0^{(i)}(\mathbf{r}) + \Sigma_r^{(i)}\phi_0^{(i)}(\mathbf{r}) = \mu_0^{(i)}\Sigma_r^{(i)}\phi_0^{(i)}(\mathbf{r}) \tag{8.338}$$

where $\phi_0^{(i)}(\mathbf{r})$ is the fundamental mode for the flux in the ith group. In the kernel formulation [cf. Eq. (8.242)], the neutron balance would be given by [with $P_B^{(i)} \equiv \Sigma_r^{(i)}K_B^{(i)}$]

$$\phi_0^{(i)}(\mathbf{r}) = \mu_0^{(i)}\Sigma_r^{(i)} \int_C \phi_0^{(i)}(\mathbf{r}')\, K_B^{(i)}(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \tag{8.339}$$

It follows directly from the results of Sec. 8.7c [cf. Eq. (8.263)] that the eigenvalues $\mu_0^{(i)}$ must satisfy the relation

$$\frac{\mu_0^{(i)}\Sigma_r^{(i)}}{\Sigma_r^{(i)} + D_iB_i^2} = 1 \tag{8.240}$$

where from (8.338)

$$\nabla^2\phi_0^{(i)}(\mathbf{r}) + B_i^2\phi_0^{(i)}(\mathbf{r}) = 0 \tag{8.340}$$

Note that the mode index $n = 0$ has been omitted from the buckling B_i to simplify the notation. Thus, as a first approximation to the Eq. (8.337), we use the solution (space form) given by (8.340) and compute the eigenvalues $\mu_0^{(i)}$ from (8.240), keeping in mind that the B_i are given by the usual relations for the bare reactor (e.g., in the case of the sphere, $B_i \equiv B = \pi/\tilde{R}$).

1. *Application to Thermal Reactor.* A simple, direct illustration of the use of the results obtained above is demonstrated in the analysis of the reactor in which all fissions are due to thermal neutrons alone. For this example we make the following choice of group cross sections:

$$\begin{array}{llll} \Sigma_f^{(1)} = 0 & z_1 = 1 & \Sigma_\gamma^{(1)} = 0 & \Sigma_r^{(1)} = \Sigma_{12} \\ \Sigma_f^{(2)} = \Sigma_f & z_2 = 0 & \Sigma_{12} \neq 0 & \Sigma_r^{(2)} = \Sigma_\gamma^{(2)} + \Sigma_f \end{array} \tag{8.341}$$

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The corresponding expressions for the source functions given by (8.332) and (8.333) are

$$S_1(\mathbf{r}) = \nu\gamma_2 A_2(\mathbf{r}) \qquad S_2(\mathbf{r}) = A_1(\mathbf{r}) \tag{8.342}$$

and by (8.331)

$$\begin{aligned} A_1(\mathbf{r}) &= \nu\gamma_2 \int_C A_2(\mathbf{r}')\, P_B^{(1)}(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \\ A_2(\mathbf{r}) &= \int_C A_1(\mathbf{r}')\, P_B^{(2)}(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \end{aligned} \tag{8.343}$$

But, by (8.335) and (8.336),

$$A_i(\mathbf{r}) = A_0^{(i)}\psi_0(\mathbf{r}) \tag{8.344}$$

so that Eqs. (8.343) reduce to the form

$$\begin{aligned} A_0^{(1)}\psi_0(\mathbf{r}) &= \nu\gamma_2 A_0^{(2)} \int_C \psi_0(\mathbf{r}')\, P_B^{(1)}(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \\ A_0^{(2)}\psi_0(\mathbf{r}) &= A_0^{(1)} \int_C \psi_0(\mathbf{r}')\, P_B^{(2)}(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \end{aligned} \tag{8.345}$$

These integrals may be evaluated by means of (8.337). The resulting relations are

$$A_0^{(1)} = \frac{\nu\Sigma_f A_0^{(2)}}{\mu_0^{(1)}\Sigma_r^{(2)}} \qquad A_0^{(2)} = \frac{A_0^{(1)}}{\mu_0^{(2)}} \tag{8.346}$$

The ratio of these two equations gives the *criticality condition* for the reactor; thus,

$$\frac{\nu\Sigma_f}{\Sigma_r^{(2)}} = \mu_0^{(1)}\mu_0^{(2)} \tag{8.347}\star$$

The eigenvalues are computed from (8.240)

$$\mu_0^{(1)} = \frac{\Sigma_{12} + D_1B^2}{\Sigma_{12}} \qquad \mu_0^{(2)} = \frac{\Sigma_r^{(2)} + D_2B^2}{\Sigma_r^{(2)}} \tag{8.348}$$

where we have used the fact that the $B_i \equiv B$, a constant independent of the neutron energy, if the energy dependence of the extrapolation distance is ignored. If the above relations are used in the criticality condition, we obtain

$$\frac{\nu\Sigma_f\Sigma_{12}}{[\Sigma_{12} + D_1B^2][\Sigma_r^{(2)} + D_2B^2]} = 1 \tag{8.349}$$

It is interesting to note that this result is identical in form to the criticality condition for the bare reactor derived previously by the standard two-group method of Sec. 8.4b [cf. Eqs. (8.155)], since (8.349) may also be written

$$\frac{\nu\Sigma_f}{\Sigma_a^{(2)}(1 + L_1^2B^2)(1 + L_2^2B^2)} = 1 \tag{8.350}$$

We have used here the identity $\Sigma_r^{(2)} = \Sigma_\gamma^{(2)} + \Sigma_f \equiv \Sigma_a^{(2)}$, in the usual notation, and $L_1^2 \equiv D_1/\Sigma_{12}$ and $L_2^2 \equiv D_2/\Sigma_a^{(2)}$.

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2. *Application to Fast Reactor.* We develop next the general criticality relation for the bare fast reactor. In this analysis we retain, as generally different from zero, all the group constants for both the slow and fast neutrons. The general source functions (8.332) and (8.333) are used, therefore, in (8.331). The neutron-balance conditions are then

$$A_1(\mathbf{r}) = \nu z_1 \left[\gamma_1 \int_C A_1(\mathbf{r}') \, P_B^{(1)}(\mathbf{r};\mathbf{r}') \, d\mathbf{r}' + \gamma_2 \int_C A_2(\mathbf{r}') \, P_B^{(1)}(\mathbf{r};\mathbf{r}') \, d\mathbf{r}' \right] \tag{8.351}$$

$$A_2(\mathbf{r}) = \nu z_2 \left[\gamma_1 \int_C A_1(\mathbf{r}') \, P_B^{(2)}(\mathbf{r};\mathbf{r}') \, d\mathbf{r}' + \gamma_2 \int_C A_2(\mathbf{r}') \, P_B^{(2)}(\mathbf{r};\mathbf{r}') \, d\mathbf{r}' \right] + \frac{\Sigma_{12}}{\Sigma_r^{(1)}} \int_C A_1(\mathbf{r}') \, P_B^{(2)}(\mathbf{r};\mathbf{r}') \, d\mathbf{r}'$$

The application of (8.344) and (8.337) yields the set

$$\begin{aligned} A_0^{(1)} &= \frac{\nu z_1}{\mu_0^{(1)}} [\gamma_1 A_0^{(1)} + \gamma_2 A_0^{(2)}] \\ A_0^{(2)} &= \frac{\nu z_2}{\mu_0^{(2)}} [\gamma_1 A_0^{(1)} + \gamma_2 A_0^{(2)}] + \frac{\Sigma_{12} A_0^{(1)}}{\Sigma_r^{(1)} \mu_0^{(2)}} \end{aligned} \tag{8.352}$$

The first of these equations gives the simpler relationship between $A_0^{(1)}$ and $A_0^{(2)}$:

$$A_0^{(1)} = \frac{\nu z_1 \gamma_2 A_0^{(2)}}{\mu_0^{(1)} - \nu z_1 \gamma_1} \tag{8.353}$$

And, the simultaneous solution of the pair (8.352) yields the criticality relation

$$\nu = \frac{\mu_0^{(1)} \mu_0^{(2)}}{z_1[\gamma_1 \mu_0^{(2)} + \gamma_2 \Sigma_{12}/\Sigma_r^{(1)}] + z_2 \gamma_2 \mu_0^{(1)}} \tag{8.354}\star$$

As in the case of the thermal reactor previously analyzed, an approximate estimate of the eigenvalues may be obtained from the simple diffusion theory results. The procedure is straightforward and involves the computation of the B^2 from the usual relations given in Table 5.3.

h. Two-group Model of Reactor with Moderating Reflector. The most general reactor problem so far considered has been the reflected multivelocity reactor with energy-dependent cross sections but with no slowing down in the reflector. A general method for treating this system, and other more elementary ones, has been discussed in some detail. Further, it has been shown that, by introducing various approximations into the expression for the removal density, the general results could be reduced to a more tractable set of equations. A final simplification was added in the case of the bare reactor with the formulation of the two-group model.

It has already been mentioned that the extension of these general methods to the analysis of reactors with moderating reflectors is a diffi-

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cult task. It is possible, however, to develop directly an approximate treatment of this problem based on the two-group model.[1] For this purpose we draw from the relations derived for the bare reactor (8.351) and introduce appropriate modifications in the definition of the kernel to account for the presence of the reflector. In addition we provide an extra term in the slow-neutron balance equation which represents the contributions to this group from the slowing-down reactions in the reflector. The calculation of this term is based on a direct physical interpretation of the neutron transport and slowing-down processes.

The analysis of the reactor with moderating reflector is carried out for the case of the infinite reflector. As in the bare-reactor calculation we denote the fast group in the core by the index $i = 1$ and the slow group by $i = 2$. In the reflector, the fast group is denoted by $i = 3$ and the slow by $i = 4$, so as to be consistent with the two-group notation of Sec. 8.4.

We begin with the neutron-balance equations, which we write in the form

$$A_1(\mathbf{r}) = \int_C S_1(\mathbf{r}')\, P_1(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \tag{8.231}$$

$$A_2(\mathbf{r}) = \int_C S_2(\mathbf{r}')\, P_2(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' + \int_C S_1(\mathbf{r}')\, P_{12}(\mathbf{r};\mathbf{r}')\, d\mathbf{r}' \tag{8.232}$$

where the $S_i(\mathbf{r})$ are given by (8.332) and (8.333) and the kernels $P_i(\mathbf{r};\mathbf{r}')$ are defined by

$$P_i(\mathbf{r};\mathbf{r}')\, d\mathbf{r} \equiv \begin{array}{l}\text{probability that a neutron of energy group } i \text{ born} \\ \text{at } \mathbf{r}', \text{ after any number of elastic collisions in both} \\ \text{core and reflector, is eventually removed as a} \\ \text{group } i \text{ neutron in } d\mathbf{r} \text{ about } \mathbf{r}\end{array} \tag{8.355}$$

It should be noted that these functions correspond to the $P(\mathbf{r};\mathbf{r}',E)$ of (8.292). The kernel $P_{12}(\mathbf{r};\mathbf{r}')$ is defined

$$P_{12}(\mathbf{r};\mathbf{r}')\, d\mathbf{r} \equiv \begin{array}{l}\text{probability that a fast neutron born at } \mathbf{r}' \text{ in the} \\ \text{core escapes removal in the core, is slowed down} \\ \text{in the reflector, and finally diffuses back to the} \\ \text{core to be removed as a slow neutron in } d\mathbf{r} \text{ about } \mathbf{r}\end{array} \tag{8.356}$$

This is the only new quantity which appears in the set (8.231) and (8.232). In determining this function it is convenient to introduce the slow-neutron removal density in the reflector, $A_4(\mathbf{r}')$ and a new kernel for slow neutrons, $Q_2(\mathbf{r}';\mathbf{r})$, which we define:

$$Q_2(\mathbf{r}';\mathbf{r}) \equiv \begin{array}{l}\text{total track length of slow neutrons per unit time per} \\ \text{unit volume around } \mathbf{r}' \text{ in the reflector produced by a} \\ \text{unit source of slow neutrons at } \mathbf{r} \text{ in the core}\end{array} \tag{8.357}$$

[1] R. P. Feynman and T. A. Welton, "The Calculation of Critical Masses Including the Effects of the Distribution of Neutron Energies," Los Alamos Report LA-524, Ch. III, Jan. 21, 1947, declassified.

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Note that this function is symmetric in $\mathbf{r}$ and $\mathbf{r}'$; i.e., $Q_2(\mathbf{r};\mathbf{r}')$ is the track length at $\mathbf{r}$ in the core produced by a unit source at $\mathbf{r}'$ in the reflector. The removal density $A_4(\mathbf{r}')$ may be written in terms of Q_2. Thus if $S_2(\mathbf{r})$ is the distribution of slow-neutron sources in the core, we can write

$$A_4(\mathbf{r}') \equiv \Sigma_r^{(4)}\phi_4(\mathbf{r}') = \Sigma_r^{(4)} \int_C S_2(\mathbf{r})\, Q_2(\mathbf{r}';\mathbf{r})\, d\mathbf{r} \tag{8.358}$$

The source distribution may be represented by an infinite series of the slow-neutron eigenfunctions $\psi_n^{(2)}(\mathbf{r})$; thus,

$$S_2(\mathbf{r}) \equiv \sum_n S_n^{(2)}\psi_n^{(2)}(\mathbf{r}) \tag{8.359}$$

The substitution of this expression into (8.358) yields

$$A_4(\mathbf{r}') = \Sigma_r^{(4)} \sum_n S_n^{(2)} \int_C \psi_n^{(2)}(\mathbf{r})\, Q_2(\mathbf{r}';\mathbf{r})\, d\mathbf{r}$$

It is convenient to identify the individual terms of this expansion as some set $A_n^{(4)}(\mathbf{r}')$; thus,

$$A_n^{(4)}(\mathbf{r}') \equiv S_n^{(2)}\Sigma_r^{(4)} \int_C \psi_n^{(2)}(\mathbf{r})\, Q_2(\mathbf{r}';\mathbf{r})\, d\mathbf{r} \tag{8.360}$$

Note that $A_4(\mathbf{r}')$ is then given by the sum

$$A_4(\mathbf{r}') \equiv \sum_n A_n^{(4)}(\mathbf{r}') \equiv \sum_n A_n^{(4)}\psi_n^{(4)}(\mathbf{r}') \tag{8.361}$$

where the last definition introduces a set of dimensionless functions comparable to the set $\psi_n^{(2)}(\mathbf{r})$ defined for the core [cf. Eq. (8.359)]. With this definition (8.360) may be written

$$A_n^{(4)}\psi_n^{(4)}(\mathbf{r}') = S_n^{(2)}\Sigma_r^{(4)} \int_C \psi_n^{(2)}(\mathbf{r})\, Q_2(\mathbf{r}';\mathbf{r})\, d\mathbf{r} \tag{8.362}$$

These results may be used to obtain suitable expansions for the kernel Q_2. If we define

$$Q_2(\mathbf{r}';\mathbf{r}) \equiv \sum_m Q_m^{(2)}(\mathbf{r}')\, \psi_m^{(2)}(\mathbf{r}) \tag{8.363}$$

then the substitution of this expansion into (8.362), followed by the application of the orthogonality properties of the $\psi_n^{(2)}(\mathbf{r})$, yields the following expression for the coefficients $Q_n^{(2)}(\mathbf{r}')$:

$$Q_n^{(2)}(\mathbf{r}') = \frac{A_n^{(4)}\psi_n^{(4)}(\mathbf{r}')}{S_n^{(2)}\Sigma_r^{(4)}} \tag{8.364}$$

Now, from neutron-conservation requirements, the total number of removals per unit time of slow neutrons in mode n in the reflector must be precisely the number of these neutrons escaping per unit time from

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This expression may also be derived on the basis of simple physical arguments. Since $P_2(\mathbf{r}'';\mathbf{r})$ is the probability that a slow neutron born at $\mathbf{r}$ in the core is removed at $\mathbf{r}''$ in the core, the integral of P_2 over all $\mathbf{r}''$ gives the probability that a slow neutron born at $\mathbf{r}$ is removed somewhere in the core. It follows, therefore, that one minus this quantity, the brackets in Eq. (8.371), is the probability that the neutron is removed somewhere in the reflector. Moreover, the probability that a slow neutron is removed at some point $\mathbf{r}'$ in the reflector is just $\psi^{(4)}(\mathbf{r}')$ divided by its integral over the reflector volume. The product of this ratio and the preceding probability are by definition $\Sigma_r^{(4)}Q_2(\mathbf{r}';\mathbf{r})$, thence, (8.371).

The kernel P_{12} may be given in terms of Q_2. If $S_4(\mathbf{r}'';\mathbf{r}')$ is the distribution of slow-neutron sources in the reflector produced by the slowing down of fast neutrons arriving from the core, then by the definition of P_{12} given in (8.356),

$$P_{12}(\mathbf{r};\mathbf{r}') = \Sigma_r^{(2)} \int_R S_4(\mathbf{r}'';\mathbf{r}')\, Q_2(\mathbf{r};\mathbf{r}'')\, d\mathbf{r}''$$

where we have employed the symmetry property of Q_2 noted in the discussion following Eq. (8.357). In order to compute S_4 we introduce a fast-neutron kernel analogous to Q_2:

$$Q_1(\mathbf{r}'';\mathbf{r}') \equiv \begin{array}{l}\text{total track length of fast neutrons per unit time per}\\ \text{unit volume around } \mathbf{r}'' \text{ in the reflector produced by a}\\ \text{unit source of fast neutrons at } \mathbf{r}' \text{ in the core}\end{array} \tag{8.372}$$

If Σ_{34} is the slowing-down cross section in the reflector, then it follows that $\Sigma_{34}Q_1(\mathbf{r}'';\mathbf{r}')$ is the number of fast neutrons arriving in the slow group per unit time per unit volume about $\mathbf{r}''$ in the reflector because of the presence of the unit source of fast neutrons at $\mathbf{r}'$ in the core. We have, then,

$$P_{12}(\mathbf{r};\mathbf{r}') = \Sigma_{34}\Sigma_r^{(2)} \int_R Q_1(\mathbf{r}'';\mathbf{r}')\, Q_2(\mathbf{r};\mathbf{r}'')\, d\mathbf{r}'' \tag{8.373}$$

The kernels Q_1 and Q_2 can be approximated by the form (8.371). By following an analogous line of reasoning to that leading to the result (8.371), we obtain

$$Q_1(\mathbf{r}'';\mathbf{r}') = \frac{\psi^{(3)}(\mathbf{r}'')}{\Sigma_r^{(3)} \int_R \psi^{(3)}(\mathbf{r}'')\, d\mathbf{r}''} \left[1 - \int_C P_1(\mathbf{r};\mathbf{r}')\, d\mathbf{r}\right] \tag{8.374}$$

$$Q_2(\mathbf{r};\mathbf{r}'') = \frac{\psi^{(4)}(\mathbf{r}'')}{\Sigma_r^{(4)} \int_R \psi^{(4)}(\mathbf{r}'')\, d\mathbf{r}''} \left[1 - \int_C P_2(\mathbf{r}';\mathbf{r})\, d\mathbf{r}'\right] \tag{8.375}$$

where the function $\psi^{(3)}(\mathbf{r})$ represents the distributions of the fast-removal reactions (and therefore also the flux) in the reflector. This function is the fast-neutron counterpart to the $\psi^{(4)}(\mathbf{r})$ developed previously for

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the slow neutrons and may be derived by similar arguments. The substitution of the above approximations for Q_1 and Q_2 into Eq. (8.373) yields

$$P_{12}(\mathbf{r};\mathbf{r}') = \frac{\Sigma_{34}\mathcal{P}_{42}\left[1 - \int_C P_1(\mathbf{r};\mathbf{r}')\,d\mathbf{r}\right]\left[1 - \int_C P_2(\mathbf{r}';\mathbf{r})\,d\mathbf{r}'\right]}{\Sigma_r^{(3)}V_C\left[1 - \frac{1}{\mu_0^{(2)}}\right]} \tag{8.376}$$

where we have called

$$\mathcal{P}_{42} \equiv \frac{\Sigma_r^{(2)}V_C\left[1 - \frac{1}{\mu_0^{(2)}}\right]\int_R \psi^{(3)}(\mathbf{r}'')\,\psi^{(4)}(\mathbf{r}'')\,d\mathbf{r}''}{\Sigma_r^{(4)}\int_R \psi^{(3)}(\mathbf{r})\,d\mathbf{r}\int_R \psi^{(4)}(\mathbf{r})\,d\mathbf{r}} \tag{8.377}$$

Finally, we introduce the Feynman-Welton lower approximation for the kernels P_i, which consists of setting all $\mu_n^{(i)} \equiv \mu_0^{(i)}$. From (8.370) and (8.369) we obtain

$$P_i(\mathbf{r};\mathbf{r}') \simeq \frac{1}{\mu_0^{(i)}}\,\delta(\mathbf{r} - \mathbf{r}') \tag{8.378}$$

and substitution into (8.376) yields

$$P_{12}(\mathbf{r};\mathbf{r}') = \frac{\Sigma_{34}\mathcal{P}_{42}[1 - (1/\mu_0^{(1)})]}{\Sigma_r^{(3)}V_C} \equiv \frac{w}{V_C} \tag{8.379}$$

with

$$w \equiv \left[1 - \frac{1}{\mu_0^{(1)}}\right]\frac{\Sigma_{34}}{\Sigma_r^{(3)}}\,\mathcal{P}_{42} \tag{8.380}$$

It is interesting to note that the result (8.379) is independent of both $\mathbf{r}$ and $\mathbf{r}'$. Thus we find that in this approximation the probability of a fast neutron from the core diffusing out into the reflector, slowing down there, and finally returning to the core to be absorbed there as a slow neutron is the same for all combinations of origin and terminus.

In order to obtain the criticality condition for this system we observe first that in analogy to (8.301) we can write

$$A_i(\mathbf{r}) = \mu^{(i)}\int_C A_i(\mathbf{r}')\,P_i(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' \tag{8.381}$$

By using the lower approximation (8.344), this equation reduces to

$$\psi_0(\mathbf{r}) = \mu_0^{(i)}\int_C \psi_0(\mathbf{r}')\,P_i(\mathbf{r};\mathbf{r}')\,d\mathbf{r}' \tag{8.236}$$

The substitution of these forms into Eq. (8.231), along with the source S_1 from (8.332), yields

$$A_0^{(1)} = \frac{\nu z_1}{\mu_0^{(1)}}\left[\gamma_1 A_0^{(1)} + \gamma_2 A_0^{(2)}\right] \tag{8.382}$$

The corresponding expression for (8.232) is obtained by using (8.333)

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for S_2 and (8.379) for P_{12} and then integrating over all r. The result may be written:

$$A_0^{(2)} = \left[\frac{\nu z_2}{\mu_0^{(2)}} + \nu z_1 w\right][\gamma_1 A_0^{(1)} + \gamma_2 A_0^{(2)}] + \frac{\Sigma_{12} A_0^{(1)}}{\Sigma_r^{(1)} \mu_0^{(2)}} \tag{8.383}$$

Equation (8.382) gives a relation between $A_0^{(1)}$ and $A_0^{(2)}$ which is identical to (8.353). The criticality condition is obtained by eliminating $A_0^{(1)}$ and $A_0^{(2)}$ by means of (8.382) and (8.383). The result is found to be

$$\nu = \frac{\mu_0^{(1)} \mu_0^{(2)} \Sigma_r^{(1)} \Sigma_r^{(2)}}{z_1[w \Sigma_f^{(2)} \Sigma_r^{(1)} \mu_0^{(1)} \mu_0^{(2)} + \Sigma_f^{(1)} \Sigma_r^{(2)} \mu_0^{(2)} + \Sigma_{12} \Sigma_f^{(2)}] + z_2 \Sigma_f^{(2)} \Sigma_r^{(1)} \mu_0^{(1)}} \tag{8.384}\star$$

The only quantities in this expression which are as yet undetermined are the eigenvalues $\mu_0^{(1)}$ and $\mu_0^{(2)}$ and the functions $\psi^{(3)}(\mathbf{r})$ and $\psi^{(4)}(\mathbf{r})$ which appear through the quantity w. As in the case of the bare-reactor calculation, the determination of these quantities is a relatively easy matter if one is willing to accept the one-velocity diffusion theory or the infinite-medium transport equation results for the spatial distributions and the characteristic equations. It is expected that for those situations in which this method will be most useful, the one-velocity diffusion-theory solutions should be entirely adequate. We proceed on this basis.

As an illustrative example consider a spherical reactor of core radius a and an infinite reflector. In the case of the one-velocity diffusion-theory model we found that the appropriate characteristic equation was the transcendental relation (8.11). For the reactor with an infinite reflector, this relation reduces to

$$Ba \cot Ba = 1 - \frac{D_R}{D_C}(\kappa_R a + 1) \tag{8.385}$$

This equation is to be solved simultaneously with the B^2 relations (8.5). In the notation of the present method, the B^2 equations take the form (8.240). Thus in computing the eigenvalues $\mu_0^{(3)}$, we determine first B_1 and B_2 from the relations

$$\begin{aligned} B_1 a \cot B_1 a &= 1 - \frac{D_3}{D_1}(\kappa_3 a + 1) \\ B_2 a \cot B_2 a &= 1 - \frac{D_4}{D_2}(\kappa_4 a + 1) \end{aligned} \tag{8.241}$$

where D_1 and D_3 denote the fast-group diffusion coefficients in the core and reflector, respectively, and D_2 and D_4 the slow-group diffusion coefficients in core and reflector. Also,

$$\kappa_3^2 \equiv \frac{\Sigma_r^{(3)}}{D_3} \qquad \kappa_4^2 \equiv \frac{\Sigma_r^{(4)}}{D_4} \tag{8.386}$$

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With the B_1 and B_2 values computed from (8.241) Eq. (8.240) may then be used to obtain $\mu_0^{(1)}$ and $\mu_0^{(2)}$. All this assumes, of course, that the various D_i and κ_i are independent of the $\mu_0^{(i)}$. If they are not, then a graphical solution may be applied.

The flux distributions $\psi^{(3)}$ and $\psi^{(4)}$ in the reflector are easily obtained from the diffusion theory. For nonmultiplying media these are found to be

$$\psi^{(3)}(r) = \frac{e^{-\kappa_3 r}}{r} \qquad \psi^{(4)}(r) = \frac{e^{-\kappa_4 r}}{r} \tag{8.387}$$

Note that a proportionality constant is not required in either case since these would be eliminated in any event in the calculation of $\mathcal{P}_{42}$. It can be shown that the substitution of (8.387) into the integrals of (8.377) yields

$$\frac{\int_R \psi^{(3)}\psi^{(4)}\,d\mathbf{r}}{\int_R \psi^{(4)}\,d\mathbf{r} \int_R \psi^{(3)}\,d\mathbf{r}} \equiv \frac{4\pi \int_a^\infty e^{-r(\kappa_3+\kappa_4)}\,dr}{(4\pi)^2 \int_a^\infty re^{-\kappa_3 r}\,dr \int_a^\infty re^{-\kappa_4 r}\,dr}$$

$$\equiv \frac{\kappa_3^2\kappa_4^2}{4\pi(\kappa_3 + \kappa_4)(1 + a\kappa_3)(1 + a\kappa_4)} \tag{8.388}$$

If we use also $V_C = \frac{4}{3}\pi a^3$, then for the spherical reactor with infinite reflector,

$$\mathcal{P}_{42} = \frac{a^3}{3}\frac{\Sigma_r^{(2)}}{\Sigma_r^{(4)}}\left[1 - \frac{1}{\mu_0^{(2)}}\right]\frac{\kappa_3^2\kappa_4^2}{(\kappa_3 + \kappa_4)(1 + a\kappa_3)(1 + a\kappa_4)} \tag{8.389}\star$$

i. Application to Thermal Reactor with Moderating Reflector. An interesting and useful example of the application of the two-group Feynman-Welton method is the criticality calculation for the thermal reactor with moderating reflector. For our analytical model we select a spherically symmetric system with infinite reflector and the group cross sections listed in Table 8.8.

TABLE 8.8 TWO-GROUP CROSS SECTIONS FOR FEYNMAN-WELTON MODEL OF THERMAL REACTOR

Energy group	Core	Reflector
Fast	$\Sigma_f^{(1)} = 0$ $\quad z_1 = 1$ $\Sigma_\gamma^{(1)} = 0$ $\quad \Sigma_r^{(1)} \equiv \Sigma_{12}$	$\Sigma_\gamma^{(3)} = 0$ $\Sigma_r^{(3)} \equiv \Sigma_{34}$
Thermal	$\Sigma_f^{(2)} \equiv \Sigma_f$ $z_2 = 0$ $\Sigma_r^{(2)} \equiv \Sigma_a^{(2)} = \Sigma_\gamma^{(2)} + \Sigma_f$	$\Sigma_r^{(4)} \equiv \Sigma_a^{(4)} = \Sigma_\gamma^{(4)}$

With these data, the criticality condition (8.384) takes the form

$$\nu = \frac{\mu_0^{(1)}\mu_0^{(2)}\Sigma_a^{(2)}}{w\Sigma_f\mu_0^{(1)}\mu_0^{(2)} + \Sigma_f} \tag{8.390}\star$$

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An alternate expression may be obtained by eliminating the eigenvalues with the aid of the relations

$$\mu_0^{(1)} = 1 + L_1^2 B_1^2 \qquad \mu_0^{(2)} = 1 + L_2^2 B_2^2 \tag{8.391}$$

which follow from (8.240). Here, $L_1^2 \equiv D_1/\Sigma_{12}$ and $L_2^2 \equiv D_2/\Sigma_a^{(2)}$. Then

$$\nu = \frac{\Sigma_a^{(2)}(1 + L_1^2 B_1^2)(1 + L_2^2 B_2^2)}{\Sigma_f[1 + w(1 + L_1^2 B_1^2)(1 + L_2^2 B_2^2)]} \tag{8.392}$$

It is of interest to note that if Σ_{12}, D_1, and D_2 are independent of the fuel concentration (at least in first approximation) then it is possible to obtain an explicit expression for the critical concentration directly from the criticality relation. Under these circumstances, the B_i given by (8.241) are also independent of the fuel concentration and the values computed from these equations may be substituted immediately into the criticality condition. Now, for the present system [from Eq. (8.380)],

$$w = \left[1 - \frac{1}{\mu_0^{(1)}}\right] \mathcal{P}_{42} = \left(\frac{L_1^2 B_1^2}{1 + L_1^2 B_1^2}\right) \mathcal{P}_{42} \tag{8.393}$$

where we have used (8.391). It is convenient to write also

$$\mathcal{P}_{42} \equiv \Sigma_a^{(2)} \left(\frac{L_2^2 B_2^2}{1 + L_2^2 B_2^2}\right) C \qquad C \equiv \frac{a^3 \kappa_3^2 \kappa_4^2}{3\Sigma_a^{(4)}(\kappa_3 + \kappa_4)(1 + a\kappa_3)(1 + a\kappa_4)} \tag{8.394}$$

Here C denotes all the various factors in (8.389) which are independent of the fuel concentration. If we use these two expressions in (8.392) and solve for the critical fuel concentration N_F, we obtain

$$N_F = \frac{D_2 B_2^2 + \Sigma_a^{(20)}}{\nu\sigma_f[(1 + CD_2 L_1^2 B_1^2 B_2^2)/(1 + L_1^2 B_1^2)] - \sigma_a^{(2F)}} \tag{8.395}\star$$

The symbol $\Sigma_a^{(20)}$ denotes the thermal absorption cross sections of the nonfuel components of the core and $\Sigma_a^{(2F)}$, the thermal absorption cross section of the fuel. Thus the present method permits a direct solution for N_F and avoids entirely the trial-and-error computations required for the standard two-group method of Sec. 8.4. It should be evident from the procedure outlined above that the computational effort involved in the Feynman-Welton two-group model is only a small fraction of that required in the standard method. This fact may be more clearly demonstrated by considering a numerical example.

As an example, let us compute the critical mass for the nuclear package power plant analyzed in Sec. 8.5. Since the group constants selected for the present method are defined in the exact manner as those of the standard method, the data listed in Table 8.5 are directly applicable to this calculation. The only information we lack is the values for the B_i and the constant C. The direct substitution of the appropriate

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numerical data into the relation (8.394) and the set (8.241) yields

$$B_1^2 = 0.007444 \text{ cm}^{-2} \qquad B_2^2 = 0.007878 \text{ cm}^{-2} \qquad C = 61.9 \text{ cm} \quad (8.396)$$

If we apply these results to (8.395) it is not difficult to show that the volume fraction of fuel corresponding to the critical concentration is $v_F = 0.006398$. This result is to be compared to the value $v_F = 0.006425$ obtained in Sec. 8.5. The present computation yields a critical mass of 9.512 kg, which differs from the previous computation by less than 1 per cent. Thus the Feynman-Welton calculation allows not only an extremely quick computation, but one that apparently agrees very well with the results from the standard two-group model in the case of a water-moderated system.

8.8 Multigroup Treatment of Reflected Reactors

a. Integration over the Lethargy Intervals. In preceding sections the treatment of the neutron population of reflected reactors based on one group of diffusing neutrons (thermal) or on two groups of diffusing neutrons (one thermal and one fast) has been examined in some detail. It is natural to examine the possibility of a more general treatment involving many groups of diffusing neutrons such that the one- and two-group models would be special cases of the more general theory. However, it is to be noted that the one- and two-group models are really based on fairly crude assumptions and are expected to be applicable to only a limited class of reactors; even so, the use of either model is justified only by previous successful application in reactor calculations. Much more confidence in a many-group model would result if it could be derived from a more basic treatment of neutron problems. At least one such approach has been sufficiently successful to justify detailed consideration; this approach, known as the *multigroup method*, is based on the age-diffusion equations. Of course, using the age-diffusion equations as a starting point for obtaining multigroup equations limits the application of the results to those problems to which the original equations are applicable; however, the limitations of the age-diffusion model can be clearly brought out by the underlying approximations needed to derive them from the Boltzmann equation (as discussed in Sec. 7.3) so that some understanding of the range of applicability of the multigroup method is obtained. Further, since in all reactors the neutrons must go through a slowing-down process, it would be expected that any method properly based on the age-diffusion equations would yield better results than the one- or two-group methods. Attempts have been made to base multigroup calculations on the original Boltzmann equation, but such methods are not as yet sufficiently well developed to justify detailed presentation here. On the other hand, the multigroup method based on the age-

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diffusion equation has been widely used in reactor calculations and has given satisfactory results.

In obtaining the multigroup equations[1] we consider a multiregion reactor to which the following set of slowing-down diffusion equations is to be applied; each region is taken to be homogeneous and uniform, and the equations apply to each region separately with appropriate constants; proper boundary conditions (to be discussed later) are to be applied throughout the system:

$$\begin{aligned} -D(u)\nabla^2\phi(\mathbf{r},u) + \Sigma_a(u)\,\phi(\mathbf{r},u) &= S(\mathbf{r},u) - \frac{\partial}{\partial u}\,q(\mathbf{r},u) \\ & 0 \le u \le u_{\text{th}} \quad (a) \\ q(\mathbf{r},u) = \xi\Sigma_t(u)\,\phi(\mathbf{r},u) & \quad 0 \le u < u_{\text{th}} \quad (b) \\ -D_{\text{th}}\nabla^2\phi_{\text{th}}(\mathbf{r}) + \Sigma_a^{\text{th}}\phi_{\text{th}}(\mathbf{r}) = q(\mathbf{r},u_{\text{th}}) & \quad u = u_{\text{th}} \quad (c) \end{aligned} \tag{6.54}$$

with

$$S(\mathbf{r},u) = \nu\left[\int_0^{u_{\text{th}}} \Sigma_f(u')\,\phi(\mathbf{r},u')\,du' + \Sigma_f^{\text{th}}\phi_{\text{th}}(\mathbf{r})\right]\chi(u)\,du \tag{6.90}$$

This set of equations has already been applied (in Sec. 6.3) to a single-region reactor, and it was possible to obtain a solution there by the method of separation of space and lethargy variables; as has been discussed previously (Sec. 8.1b) this is not possible, in general, in the multiregion case. Thus we are led to consider techniques of numerical integration of the equations above. The method under consideration here is intended to be a natural one for calculations using (6.54) and (6.90) in the multiregion case performed on large-capacity fast computing machines.

Consider the zero of lethargy such that there are (for all practical purposes) no neutrons in the reactor whose energies are greater than that energy corresponding to lethargy zero (e.g., 10 Mev). Consider the lethargy range from $u = 0$ to $u = u_{\text{th}}$ as divided up into N intervals of (generally) unequal sizes:

$$\begin{aligned} \Delta u_i &= u_i - u_{i-1} \qquad u_N = u_{\text{th}} \\ u_0 &= 0 \qquad\qquad i = 1, 2, \ldots, N \end{aligned} \tag{8.397}$$

These lethargy intervals may be chosen for convenience, according to position along the lethargy scale and according to size.

Now in a given region integrate Eq. (6.54a) over the ith lethargy interval; thus

$$\begin{aligned} -\int_{u_{i-1}}^{u_i} D(u)\,\nabla^2\phi(\mathbf{r},u)\,du + \int_{u_{i-1}}^{u_i} \Sigma_a(u)\,\phi(\mathbf{r},u)\,du \\ = -\int_{u_{i-1}}^{u_i} \frac{\partial}{\partial u}\,q(\mathbf{r},u)\,du + \int_{u_{i-1}}^{u_i} S(\mathbf{r},u)\,du \end{aligned} \tag{8.398}$$

[1] The original form of this method (not exactly the one given here) was developed by the Physics Division of KAPL [see R. Ehrlich and H. Hurwitz, Jr., *Nucleonics*, **12** (2), 23 (1954)].

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In order to progress from Eq. (8.398) when confronted with the unknown variation of the neutron cross sections and the flux with lethargy, the following definitions are made:

$$\overline{\phi_i}(\mathbf{r}) \equiv \frac{1}{\Delta u_i}\int_{u_{i-1}}^{u_i} \phi(\mathbf{r},u)\,du \qquad (a)$$

$$\overline{D_i} \equiv \frac{\int_{u_{i-1}}^{u_i} D(u)\,\phi(\mathbf{r},u)\,du}{\int_{u_{i-1}}^{u_i} \phi(\mathbf{r},u)\,du} \qquad (b)$$

$$\overline{\Sigma_{a_i}} \equiv \frac{\int_{u_{i-1}}^{u_i} \Sigma_a(u)\,\phi(\mathbf{r},u)\,du}{\int_{u_{i-1}}^{u_i} \phi(\mathbf{r},u)\,du} \qquad (c) \qquad (8.399)$$

$$\overline{\Sigma_{f_i}} \equiv \frac{\int_{u_{i-1}}^{u_i} \Sigma_f(u)\,\phi(\mathbf{r},u)\,du}{\int_{u_{i-1}}^{u_i} \phi(\mathbf{r},u)\,du} \qquad (d)$$

The definition of $\overline{\Sigma_{f_i}}$ is intended for later use in connection with the treatment of the term $S(\mathbf{r},u)$. If we use Eq. (8.399) in (8.398) and perform the integration in the final term on the right-hand side of (8.398), we obtain

$$-\overline{D_i}\Delta u_i\nabla^2\overline{\phi_i}(\mathbf{r}) + \overline{\Sigma_{a_i}}\Delta u_i\overline{\phi_i}(\mathbf{r}) = q(\mathbf{r},u_{i-1}) - q(\mathbf{r},u_i) + \int_{u_{i-1}}^{u_i} S(\mathbf{r},u)\,du \qquad (8.400)$$

Consider the integration over the source term; if we apply (6.90),

$$\int_{u_{i-1}}^{u_i} S(\mathbf{r},u)\,du = \nu\int_{u_{i-1}}^{u_i} \chi(u)\,du\left[\sum_{k=1}^{N}\int_{u_{k-1}}^{u_k}\Sigma_f(u)\,\phi(\mathbf{r},u)\,du + \Sigma_f^{\text{th}}\phi_{\text{th}}(\mathbf{r})\right] \qquad (8.401)$$

Now define $\left[\text{note that from (6.88)}, \sum_{k=1}^{N}\chi_k\Delta u_k = 1\right]$,

$$\chi_i\Delta u_i \equiv \int_{u_{i-1}}^{u_i}\chi(u)\,du \qquad (8.402)$$

Then, applying (8.399d), we have

$$\int_{u_{i-1}}^{u_i} S(\mathbf{r},u)\,du = \nu\chi_i\Delta u_i\sum_{k=1}^{N+1}\overline{\Sigma_{f_k}\phi_k}(\mathbf{r})\Delta u_k \qquad (8.403)$$

The thermal fissions have been formally included by defining

$$\overline{\Sigma_{f_{N+1}}} \equiv \Sigma_f^{\text{th}} \qquad \overline{\phi_{N+1}}(\mathbf{r}) \equiv \phi_{\text{th}}(\mathbf{r}) \qquad \Delta u_{N+1} = 1 \qquad (8.404)$$

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The application of (8.403) to (8.400) yields

$$-\overline{D_i}\Delta u_i\nabla^2\overline{\phi_i}(\mathbf{r}) + \overline{\Sigma_{a_i}}\Delta u_i\overline{\phi_i}(\mathbf{r}) = q(\mathbf{r},u_{i-1}) - q(\mathbf{r},u_i) + \nu\chi_i\Delta u_i\sum_{k=1}^{N+1}\overline{\Sigma_{f_k}\phi_k}(\mathbf{r})\Delta u_k \quad (8.405)\star$$

Equation (8.405) may be regarded as a restatement of Eqs. (6.54*a*), (6.54*c*), and (6.90), since the thermal equation can be formally included using the definitions of Eq. (8.404) and the following limits:

$$\chi_{N+1} \equiv 0 \qquad q(\mathbf{r},u_{N+1}) \equiv 0 \quad (8.406)$$

Thus the index i in Eq. (8.405) takes on the values 1, 2, 3, . . . , N, $N + 1$. Note that no approximations have been made in obtaining Eq. (8.405) from (6.54) and (6.90). At this point, however, some approximations must be introduced to make Eq. (8.405) tractable.

b. The Simultaneous Set of Multigroup Equations. As a digression from the main purpose of this section, which is to demonstrate the technique for applying the age-diffusion equations to certain classes of reactors using fast computing machines, consider the use of Eq. (8.405) to obtain a generalization of the one- and two-group methods. The difficulty in Eq. (8.405) from this viewpoint is the presence of the terms in the slowing-down density at the lethargy limits of the ith interval; it would be desirable to have these expressed in terms of the average fluxes in neighboring intervals. As an example of such a relationship similar to that used in the two-group method, assume that a removal cross section for the ith interval (or group) can be defined such that

$$q(\mathbf{r},u_i) = \Sigma_{R_i}\overline{\phi_i}(\mathbf{r})\Delta u_i \quad (8.407)$$

Then Eq. (8.405) becomes

$$-\overline{D_i}\nabla^2\phi_i(\mathbf{r}) + (\overline{\Sigma_{a_i}} + \Sigma_{R_i})\phi_i(\mathbf{r}) = \Sigma_{R_{i-1}}\phi_{i-1}(\mathbf{r}) + \nu\chi_i\Delta u_i\sum_{k=1}^{N+1}\overline{\Sigma_{f_k}\phi_k}(\mathbf{r}) \quad (8.408)$$

with

$$\phi_i(\mathbf{r}) \equiv \overline{\phi_i}(\mathbf{r})\Delta u_i \quad (8.409)$$

This is, then, a set of $N + 1$ equations for $N + 1$ unknowns, the $\phi_i(\mathbf{r})$; since the equations are homogeneous it is to be expected that there will be a condition for solubility in the form of a determinantal equation [cf. Eq. (8.188) in the two-group case] which would serve as a criticality condition. The average cross sections needed for Eq. (8.408) present no more problem than in any other scheme based on Eq. (8.405) and accompanying definitions; since these calculations will be discussed later, let us assume here that they are known. However, the removal cross sections

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are introduced quite arbitrarily in Eq. (8.407) and, as in the two-group method, can be chosen to give good results in various cases.

As an example of the use of Eq. (8.408), consider the case of a reactor medium for which the materials and lethargy groups are so chosen that fissions are caused only by thermal neutrons and neutrons from fission appear only in the highest lethargy group, thus

$$\begin{aligned} \chi_1 \Delta u_1 &= 1 \qquad \chi_k = 0 \qquad \text{for } k > 1 \\ \overline{\Sigma_{f_{N+1}}} &= \Sigma_f^{\text{th}} \qquad \overline{\Sigma_{f_k}} = 0 \qquad \text{for } k < N+1 \end{aligned} \tag{8.410}$$

For this case, Eqs. (8.408) become [assuming $\phi_0(\mathbf{r}) \equiv 0$]

$$\begin{aligned} -\overline{D_1}\nabla^2\phi_1(\mathbf{r}) + (\overline{\Sigma_{a_1}} + \Sigma_{R_1})\phi_1(\mathbf{r}) &= \nu\Sigma_f^{\text{th}}\phi_{N+1}(\mathbf{r}) \\ -\overline{D_2}\nabla^2\phi_2(\mathbf{r}) + (\overline{\Sigma_{a_2}} + \Sigma_{R_2})\phi_2(\mathbf{r}) &= \Sigma_{R_1}\phi_1(\mathbf{r}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -\overline{D_k}\nabla^2\phi_k(\mathbf{r}) + (\overline{\Sigma_{a_k}} + \Sigma_{R_k})\phi_k(\mathbf{r}) &= \Sigma_{R_{k-1}}\phi_{k-1}(\mathbf{r}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -\overline{D_{N+1}}\nabla^2\phi_{N+1}(\mathbf{r}) + \overline{\Sigma_{a_{N+1}}}\phi_{N+1}(\mathbf{r}) &= \Sigma_{R_N}\phi_N(\mathbf{r}) \end{aligned} \tag{8.411}$$

The last equation (for the thermal flux) follows from the fact that by Eqs. (8.406) and (8.407) $\Sigma_{R_{N+1}} \equiv 0$. If Eqs. (8.411) are applied to a bare reactor, a solution may be obtained as usual by assuming that the spatial dependence of the flux is the same for all groups; thus let

$$\phi_k(\mathbf{r}) = \phi_k F(\mathbf{r}) \qquad \text{with } \nabla^2 F(\mathbf{r}) + B^2 F(\mathbf{r}) = 0 \tag{8.412}$$

thus the $F(\mathbf{r})$ are the usual eigenfunctions for bare reactors of various geometries and the B^2 are the corresponding eigenvalues. With Eqs. (8.412), Eqs. (8.411) become $N+1$ homogeneous equations for the $N+1$ coefficients ϕ_k; the condition for solubility is seen to be (with $\Sigma_a^{\text{th}} \equiv \overline{\Sigma_{a_{N+1}}}$, $D_{\text{th}} \equiv \overline{D_{N+1}}$, and $L_{\text{th}}^2 \equiv D_{\text{th}}/\Sigma_a^{\text{th}}$)

$$\frac{\nu\Sigma_f^{\text{th}}}{\Sigma_a^{\text{th}}}(1 + L_{\text{th}}^2 B^2)^{-1}\left[\prod_{i=1}^{N}\left(1 + \frac{\overline{\Sigma_{a_i}}}{\Sigma_{R_i}} + \frac{\overline{D_i}B^2}{\Sigma_{R_i}}\right)\right]^{-1} = 1 \tag{8.413}$$

If we compare this result with the accurate equation (6.78) for a bare reactor in the age-diffusion approximation (with $\epsilon = 1$), we find that the removal cross sections should be chosen so that

$$\left[\prod_{i=1}^{N}\left(1 + \frac{\overline{\Sigma_{a_i}}}{\Sigma_{R_i}} + \frac{\overline{D_i}B^2}{\Sigma_{R_i}}\right)\right]^{-1} \equiv p_{\text{th}} e^{-B^2\tau_{\text{th}}} \tag{8.414}$$

The comparison may be carried a step further if the ratios $\overline{\Sigma_{a_i}}/\Sigma_{R_i}$ and

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$\overline{D_i}B^2/\Sigma_{R_i}$ are all small compared with unity; then

$$\left[\prod_{i=1}^{N}\left(1+\frac{\overline{\Sigma_{a_i}}}{\Sigma_{R_i}}+\frac{\overline{D_i}B^2}{\Sigma_{R_i}}\right)\right]^{-1} \simeq \left[\prod_{i=1}^{N}\left(1+\frac{\overline{\Sigma_{a_i}}}{\Sigma_{R_i}}\right)\prod_{i=1}^{N}\left(1+\frac{\overline{D_i}B^2}{\Sigma_{R_i}}\right)\right]^{-1}$$

$$\simeq \exp\left(-\sum_{i=1}^{N}\frac{\overline{\Sigma_{a_i}}}{\Sigma_{R_i}}\right)\exp\left(-B^2\sum_{i=1}^{N}\frac{\overline{D_i}}{\Sigma_{R_i}}\right) \quad (8.415)$$

Thus the removal cross sections might well be chosen to be consistent with

$$\begin{aligned}\sum_{i=1}^{N}\frac{\overline{\Sigma_{a_i}}}{\Sigma_{R_i}} &= \int_0^{u_{\text{th}}}\frac{\Sigma_a(u)\,du}{\xi\Sigma_t(u)} = \sum_{i=1}^{N}\overline{\left(\frac{\Sigma_a}{\xi\Sigma_t}\right)_i}\Delta u_i \\ \sum_{i=1}^{N}\frac{\overline{D_i}}{\Sigma_{R_i}} &= \int_0^{u_{\text{th}}}\frac{D(u)\,du}{\xi\Sigma_t(u)} = \sum_{i=1}^{N}\overline{\left(\frac{D}{\xi\Sigma_t}\right)_i}\Delta u_i\end{aligned} \quad (8.416)$$

where we have used the definitions

$$\overline{\left(\frac{\Sigma_a}{\xi\Sigma_t}\right)_i}\Delta u_i \equiv \int_{u_{i-1}}^{u_i}\frac{\Sigma_a(u)\,du}{\xi\Sigma_t(u)} \quad \text{and} \quad \overline{\left(\frac{D}{\xi\Sigma_t}\right)_i}\Delta u_i \equiv \int_{u_{i-1}}^{u_i}\frac{D(u)\,du}{\xi\Sigma_t(u)} \quad (8.417)$$

Further, if the lethargy intervals are sufficiently small and the cross sections sufficiently slowly varying over the intervals that an average over a product or a quotient of cross sections is approximately equal to the product or quotient of the separate averages, both statements of Eq. (8.416) are consistent with the choice for each interval:

$$\Sigma_{R_i} = \frac{\overline{(\xi\Sigma_t)_i}}{\Delta u_i} \quad (8.418)$$

where $\overline{(\xi\Sigma_t)_i}$ is an appropriate average of $\xi\Sigma_t(u)$ over the ith lethargy interval. It is not intended here that Eq. (8.418) should be regarded as the best method for choosing the removal cross section for any actual calculation. In the actual case it may be an advantage to have the removal cross sections as adjustable parameters (within limits) so that as experience is gained various classes of reactors might be subject to fairly accurate calculation. Note that Eq. (8.418), along with the basic assumption of Eq. (8.407), leads to the result

$$q(\mathbf{r},u_i) = \overline{(\xi\Sigma_t)_i}\overline{\phi_i}(\mathbf{r}) \quad (8.419)$$

which seems to be consistent with Eq. (6.54*b*), at least for very small intervals Δu_i; however, Eq. (8.419) is by no means a necessary consequence of (6.54*b*). Indeed, the assumption of Eq. (8.407) is only an

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example of the many possible relationships which may be used. As another more complicated example, which will not be considered further, the slowing-down density at any lethargy might be related to the flux in all lethargy intervals at lower lethargies by

$$q(\mathbf{r},u_i) = \sum_{k=1}^{i} A_{k_i}\,\overline{\phi_k}(\mathbf{r})\Delta u_k \tag{8.420}$$

where the A_{k_i} may be chosen for consistency with an appropriate slowing-down kernel. It is even possible that a set of A_{k_i} might be so chosen that the resulting multigroup equations gave better representation of the slowing-down process than the original age-diffusion theory which basically assumes a continuous slowing-down process.

Let us now consider the application of the set of Eqs. (8.408) to a reflected reactor, assuming that all cross sections and constants have been chosen. For a two-region reactor, for each lethargy interval there are two sets of differential equations, one for the core and one for the reflector. An appropriate set of boundary conditions must be chosen, for example, analogously to that for the one-velocity reactor model; thus if $\phi_i^{(C)}(\mathbf{r})$ and $\phi_i^{(R)}(\mathbf{r})$ are the core and reflector fluxes, respectively, we might take the set

$$\begin{aligned}
&(1)\ \phi_i^{(C)}(\mathbf{r}) \text{ is nonsingular and nonnegative}\\
&(2)\ \phi_i^{(C)}(\mathbf{R}) = \phi_i^{(R)}(\mathbf{R})\\
&(3)\ -\bar{D}_i^{(C)}\nabla\phi_i^{(C)}(\mathbf{R}) = -\bar{D}_i^{(R)}\nabla\phi_i^{(R)}(\mathbf{R})\\
&(4)\ \phi_i^{(R)}(\tilde{\mathbf{R}}') = 0
\end{aligned} \tag{8.421}$$

where $\mathbf{R}$ defines the core-reflector interface, $\tilde{\mathbf{R}}'$ the extrapolated outer boundary of the reflector, and $\bar{D}_i^{(C)}$ and $\bar{D}_i^{(R)}$ are the average values of the diffusion coefficients in core and reflector, respectively, as defined by (8.399*b*).

One of the difficulties in using the set (8.408) is that they are simultaneous equations in the $\phi_i(\mathbf{r})$. For any number of groups larger than three, the equations become so cumbersome to handle that recourse is usually had to numerical integration in combination with iteration techniques. However, this usually implies large computing machines. Thus attention is focused throughout the remainder of this section on a direct approach to such a numerical integration-iteration solution of Eqs. (8.405) which is not dependent on the restrictive assumption of Eq. (8.407).

The analytic solutions to Eqs. (8.408) are so involved for any large number of groups and of such restricted use that a detailed discussion of these methods is not justified; however, brief mention will be made of two such methods.

1. *The Harmonics Method.* The functions $\phi_i(\mathbf{r})$ may be expanded in a set of normalized functions, which are orthogonal over the range including the core and the reflector; then substitution into Eqs. (8.408) and

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application of the orthogonality property give linear simultaneous equations in the coefficients of the expansion which may be solved algebraically. Along with the matching conditions at the core-reflector interface, these results will then yield a criticality condition.

2. *The Method of Characteristic Functions.* This is the generalization of the method used above for the two-group calculation. For purposes of simplification of the presentation we will confine our attention to the system described by Eqs. (8.411), a special case of Eqs. (8.408). The flux in each group, in each region, is expressed as a linear combination of functions appropriate for the particular geometry:

$$\begin{aligned} \phi_i^{(C)}(\mathbf{r}) &\equiv \sum_{k=1}^{N+1} A_{i_k} Z(\mu_k, \mathbf{r}) \qquad (a) \\ \phi_i^{(R)}(\mathbf{r}) &\equiv \sum_{k=1}^{N+1} a_{i_k} U(\varkappa_k, \mathbf{r}) \qquad (b) \end{aligned} \tag{8.422}$$

the functions Z and U are given in Table 8.3 for a few geometries. The parameters μ_k are obtained by assuming that the core fluxes are proportional to a space function, as in (8.412). Then the solutions for B^2 are found from

$$\prod_{i=1}^{N} \left(1 + L_i^2 B^2 + \frac{\overline{\Sigma_{a_i}}}{\Sigma_{R_i}}\right) = \frac{\nu \Sigma_f^{\text{th}}}{\Sigma_a^{\text{th}}(1 + L_{\text{th}}^2 B^2)} \tag{8.423}$$

where

$$L_i^2 \equiv \frac{\overline{D_i}}{\Sigma_{R_i}} \qquad \text{and} \qquad L_{\text{th}}^2 \equiv \frac{D_{\text{th}}}{\Sigma_a^{\text{th}}} \tag{8.424}$$

Note that the solutions to (8.423) for B^2 will, in general, be complex. The $\varkappa_k$ of Eq. (8.422*b*) are given by

$$\varkappa_k^2 \equiv \frac{(\overline{\Sigma_{a_i}} + \Sigma_{R_i})_R}{\bar{D}_i^{(R)}} \tag{8.425}$$

The a_{i_k} of Eq. (8.422*b*) are in part determined by requiring that the fluxes satisfy a set of equations similar to those of Eq. (8.411) but with source terms appropriate to the reflector; e.g., if the reflector contains no fissionable material, all the equations (8.411) may be taken over and applied to the reflector (with the proper constants) except that there would be a zero source term on the right-hand side of the equation for group one. In this special case the solutions for the reflector would be of the form:

$$\begin{aligned} \phi_1(\mathbf{r}) &= a_0 U(\varkappa_1, \mathbf{r}) \\ \phi_2(\mathbf{r}) &= a_1 U(\varkappa_2, \mathbf{r}) + \left(\frac{\Sigma_{R_1}/\overline{D_2}}{\varkappa_2^2 - \varkappa_1^2}\right) a_0 U(\varkappa_1, \mathbf{r}), \text{ etc.} \end{aligned} \tag{8.426}$$

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Equation (8.422*a*) contains $(N + 1)^2$ unknown coefficients; substitution of (8.422*a*) into (8.411) yields relationships among these $(N + 1)^2$ coefficients such that there remain only $(N + 1)$ independent coefficients. Along with the $(N + 1)$ coefficients from the reflector fluxes, this gives $2(N + 1)$ unknown coefficients. The $2(N + 1)$ homogeneous equations obtained by matching flux and current in each group at the core-reflector interface allow for a complete solution except for a scale factor in the flux, as usual. The condition for solubility of this set of $2(N + 1)$ equations is of the form

$$\Delta = 0 \tag{8.427}$$

where Δ, a $2(N + 1)$-rowed determinant, is the criticality condition for the reactor system.

This brief outline is sufficient to indicate the difficulties inherent in analytic solutions to the multigroup equations. Two group calculations, as discussed above, are laborious but can be systematized so that parametric studies of a series of reactors is feasible by hand calculations. Three-group calculations can be and are done by hand, but beyond this the analytic methods become so cumbersome that it is advantageous to go directly to numerical computations.

c. Multigroup Equations Adapted for Numerical Computation. We return to the main interest of this section by considering Eq. (8.405), which is now written in the form ($i = 1, 2, \ldots N, N + 1$):

$$-\overline{D_i}\Delta u_i \nabla^2 \overline{\phi_i}(\mathbf{r}) + \overline{\Sigma_{a_i}}\Delta u_i \overline{\phi_i}(\mathbf{r}) = q(\mathbf{r},u_{i-1}) - q(\mathbf{r},u_i) + \nu\overline{\delta_i}\Delta u_i F(\mathbf{r}) \tag{8.428}$$

Here the total number of fissions per unit volume per unit time caused by neutrons of all energies at space point $\mathbf{r}$ has been designated by $F(\mathbf{r})$, replacing the summation over lethargy groups of Eq. (8.405). This is done for convenience in treating the multigroup equations by the iterative technique. From the beginning, in the present method, the multigroup equations are not regarded as a set of simultaneous equations in the neutron fluxes, but as a set of equations which describe the flow of neutrons downward in energy, starting from a given initial distribution in space and energy. Thus, for a given source distribution $F(\mathbf{r})$, the Eqs. (8.428) give the complete history of one generation of neutrons. Then, for the reactor case, it is necessary to apply the condition that the spatial distribution of fissions calculated from the results of applying Eq. (8.428) must be the same as the source distribution with which the generation was started. In a reactor calculation, then, a certain source distribution which we shall call $F_0(\mathbf{r})$ is chosen (somewhat arbitrarily) to start the calculation; supposing that all the constants in all regions and groups are known, then the spatial distribution of fissions may be calculated. This distribution may then be called $F_1(\mathbf{r})$ and used as a source distribution to start a new generation. This process may be repeated until the calcu-

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lated distribution of fissions agrees with the starting source distribution for that generation within the required accuracy. This iterative technique makes good use of a general property of fast computing machines, namely, that they are most efficiently used by a calculation which requires a repetitive operation.

With the source distribution chosen, actual calculation from Eqs. (8.428) depends on a choice for the relationship between the slowing-down densities at the lethargy limits of the group and the average flux or average slowing-down density in the group. Either the average flux or the average slowing-down density may be used, since by virtue of Eq. (6.54*b*), Eqs. (8.428) may be written in terms of either. In the multigroup formulation the appropriate form of the coupling equation is developed from $\overline{\phi}_i(\mathbf{r})$ and $\overline{q}_i(\mathbf{r})$, where $\overline{\phi}_i(\mathbf{r})$ is given by (8.399*a*), and

$$\overline{q}_i(\mathbf{r}) \equiv \frac{1}{\Delta u_i}\int_{u_{i-1}}^{u_i} q(\mathbf{r},u)\,du = \frac{1}{\Delta u_i}\int_{u_{i-1}}^{u_i} \xi\Sigma_t(u)\,\phi(\mathbf{r},u)\,du \tag{8.429}$$

The last equality follows by the application of Eq. (6.54*b*). If we call

$$\overline{(\xi\Sigma_t)_i} \equiv \frac{\displaystyle\int_{u_{i-1}}^{u_i} \xi\Sigma_t(u)\,\phi(\mathbf{r},u)\,du}{\displaystyle\int_{u_{i-1}}^{u_i} \phi(\mathbf{r},u)\,du} \tag{8.430}$$

and apply (8.399*a*) for the denominator, then (8.429) may be written

$$\overline{q}_i(\mathbf{r}) = \overline{(\xi\Sigma_t)_i}\;\overline{\phi}_i(\mathbf{r}) \tag{8.431}$$

Some of the choices for this relationship which have received considerable attention are given below:

$$\begin{aligned}
&\overline{q}_i(\mathbf{r}) = f_* q(\mathbf{r},u_i) && (a)\\
&q(\mathbf{r},u_i) = \xi\Sigma_t(u_i)\overline{\phi}_i(\mathbf{r}) && (b)\\
&\overline{q}_i(\mathbf{r}) = \tfrac{1}{2}[q(\mathbf{r},u_i) + q(\mathbf{r},u_{i-1})] && (c)\\
&\overline{\phi}_i(\mathbf{r}) = \tfrac{1}{2}[\phi(\mathbf{r},u_i) + \phi(\mathbf{r},u_{i-1})] && \\
&\qquad \text{with } q(\mathbf{r},u_i) = \xi\Sigma_t(u_i)\,\phi(\mathbf{r},u_i) && (d)
\end{aligned} \tag{8.432}$$

In Eq. (8.432*a*) the quantity f_* is an adjustable parameter to be determined in part by iteration of the group calculation under certain criteria for selection of a value of f_* and in part by experience in calculations to give good results. In this sense it somewhat resembles the choice of Eq. (8.407) discussed above. Method (*a*) has been carefully tried at KAPL, and some discussion is given in the reference report.[1] Methods with adjustable parameters such as f_* in (8.432*a*) are particularly useful in calculations using a fairly small number of intervals for the range of energy above thermal; however, they all involve a certain amount of

[1] *Ibid.*

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trial and error, and for fast computing machines, it is more convenient to have a direct connecting equation such as (*b*), (*c*), or (*d*) of (8.432). It has been demonstrated that at least (*c*) and (*d*) of (8.432) can give good results for a finely divided lethargy range below thermal. For the present discussion we will assume (8.432*d*), a method which has been well tested in multigroup calculations at ORNL (especially).

Equation (8.432*d*) may be used in Eq. (8.428) to eliminate the slowing-down density *out of* the *i*th group, $q(\mathbf{r},u_i)$; thus

$$-\overline{D}_i\Delta u_i\nabla^2\overline{\phi}_i(\mathbf{r}) + [\overline{\Sigma_a}_i\Delta u_i + 2(\xi\Sigma_t)_{u_i}]\overline{\phi}_i(\mathbf{r}) = \left[1 + \frac{(\xi\Sigma_t)_{u_i}}{(\xi\Sigma_t)_{u_{i-1}}}\right] q(\mathbf{r},u_{i-1}) + \nu\chi_i\Delta u_i F(\mathbf{r}) \quad (8.433)$$

Here $(\xi\Sigma_t)_{u_i}$ designates the value of the quantity $\xi\Sigma_t$ at lethargy u_i; as discussed in an earlier chapter (see Secs. 4.5b and c), since it is really the quantity $\xi\Sigma_t$ which is to be calculated at each lethargy for a homogeneous mixture of materials, it is not proper to choose one particular value of ξ for the entire lethargy range.

We may now describe in general terms the procedure for the calculation based on Eq. (8.433). The lethargy range is divided up into intervals as described above; the constants required in Eq. (8.433) are calculated for each interval and for each region. A starting source distribution is chosen; let this be called $F_0(\mathbf{r})$. Then suppose that lethargy u_0 is so chosen that

$$q(\mathbf{r},u_0) = 0 \qquad \text{for all regions of the reactor} \quad (8.434)$$

The calculation begins with the first lethargy group. A differential equation of the form

$$-\bar{D}_1^{(n)}\Delta u_1\nabla^2\bar{\phi}_1^{(n)}(\mathbf{r}) + [\bar{\Sigma}_{a_1}^{(n)}\Delta u_1 + 2(\xi\Sigma_t)_{u_1}^{(n)}]\bar{\phi}_1^{(n)}(\mathbf{r}) = \nu\chi_1^{(n)}\Delta u_i F_0^{(n)}(\mathbf{r}) \quad (8.435)$$

is written for each region of the reactor, where the superscript (n) designates the region considered. Proper boundary conditions are chosen to connect the $\bar{\phi}_1^{(n)}(\mathbf{r})$ and their derivatives at interfaces between regions, and the set of space-wise differential equations is solved numerically by methods to be described later. With $\bar{\phi}_1^{(n)}(\mathbf{r})$ known, the slowing-down density out of the first group may be calculated; thus, from (8.432*d*)

$$q^{(n)}(\mathbf{r},u_1) = (\xi\Sigma_t)_{u_1}^{(n)}\phi^{(n)}(\mathbf{r},u_1) = (\xi\Sigma_t)_{u_1}^{(n)}[2\bar{\phi}_1^{(n)}(\mathbf{r}) - 0] \quad (8.436)$$

since the flux at u_0 is taken to be zero from (8.432*d*). Then this slowing-down density may be used as a source term for the equation for the second group:

$$-\bar{D}_2^{(n)}\Delta u_2\nabla^2\bar{\phi}_2^{(n)}(\mathbf{r}) + [\bar{\Sigma}_{a_2}^{(n)}\Delta u_2 + 2(\xi\Sigma_t)_{u_2}^{(n)}]\bar{\phi}_2^{(n)}(\mathbf{r}) = \left[1 + \frac{(\xi\Sigma_t)_{u_2}^{(n)}}{(\xi\Sigma_t)_{u_1}^{(n)}}\right] q^{(n)}(\mathbf{r},u_1) + \nu\chi_2^{(n)}\Delta u_2 F_0^{(n)}(\mathbf{r}) \quad (8.437)$$

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The $\bar{\phi}_2^{(n)}(\mathbf{r})$ are then found for each region, again by numerical integration and application of proper boundary conditions. The slowing-down density from the second group may be calculated from

$$q^{(n)}(\mathbf{r},u_2) = (\xi\Sigma_t)_{u_2}^{(n)}\phi^{(n)}(\mathbf{r},u_2) = (\xi\Sigma_t)_{u_2}^{(n)}\left[2\bar{\phi}_2^{(n)}(\mathbf{r}) - \frac{q^{(n)}(\mathbf{r},u_1)}{(\xi\Sigma_t)_{u_1}^{(n)}}\right] \quad (8.438)$$

The process thus proceeds from group to group until the $(N+1)$st group, i.e., the thermal group, is reached. For the thermal group Eq. (8.433) does not really apply and we must return to Eq. (8.428), along with Eqs. (8.404) and (8.406), to write

$$-\bar{D}_{N+1}^{(n)}\nabla^2\bar{\phi}_{N+1}^{(n)}(\mathbf{r}) + \Sigma_{a_{N+1}}^{(n)}\bar{\phi}_{N+1}^{(n)}(\mathbf{r}) = q^{(n)}(\mathbf{r},u_N) \quad (8.439)$$

with $q^{(n)}(\mathbf{r},u_N)$ to be calculated from

$$q^{(n)}(\mathbf{r},u_N) = (\xi\Sigma_t)_{u_N}^{(n)}\phi^{(n)}(\mathbf{r},u_N) = (\xi\Sigma_t)_{u_N}^{(n)}\left[2\bar{\phi}_N^{(n)}(\mathbf{r}) - \frac{q^{(n)}(\mathbf{r},u_{N-1})}{(\xi\Sigma_t)_{u_{N-1}}^{(n)}}\right] \quad (8.440)$$

As a result of this calculation the average fluxes in each group in each region, $\bar{\phi}_{i_0}^{(n)}(\mathbf{r})$, are known at every space point [the additional subscript zero indicates that these fluxes are derived from the assumed source distribution $F_0^{(n)}(\mathbf{r})$]. The comparison of the derived source distribution with the original one may now be made; therefore compare

$$\sum_{i=1}^{N+1}\Sigma_{f_i}^{(n)}\bar{\phi}_{i_0}^{(n)}(\mathbf{r})\Delta u_i \sim F_0^{(n)}(\mathbf{r})$$

and, if certain standards of comparison have been chosen, it may be decided whether the original source distribution was a sufficiently good choice. However, there exists a more valuable criterion. This is connected with the calculation of the multiplication constant for the system, which should be one of the results available from the computation. Suppose the following normalization had been chosen for $F_0^{(n)}(\mathbf{r})$:

$$\nu\sum_{i=1}^{N+1}\chi_i\Delta u_i\int_{\text{reactor}}F_0^{(n)}(\mathbf{r})\,d\mathbf{r} = \nu\int_{\text{reactor}}F_0^{(n)}(\mathbf{r})\,d\mathbf{r} = 1 \quad (8.441)$$

where the appropriate form for $F_0^{(n)}(\mathbf{r})$ is to be used in each region when performing the integration. Equation (8.441) states that one neutron is introduced into the reactor to start the generation. At the start of the next generation, we have n_0 neutrons, given by

$$n_0 = \sum_{i=1}^{N+1}\int_{\text{reactor}}\Sigma_{f_i}^{(n)}\bar{\phi}_{i_0}^{(n)}(\mathbf{r})\Delta u_i\,d\mathbf{r} \quad (8.442)$$

By definition, the multiplication constant of the system is the ratio of

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the number of neutrons in one generation to the number in the preceding generation; therefore,

$$k_0 = n_0 \tag{8.443}$$

Here the subscript zero is put on the multiplication constant to indicate that this is the estimate of the multiplication constant obtained from the use of the original source distribution $F_0^{(n)}(\mathbf{r})$. Of course, the number of neutrons in a second generation will, in general, depend on the spatial distribution in the original generation. The true multiplication constant for a reactor system will be obtained from that source distribution which repeats itself in the next generation. The calculation described above for the reactor fluxes may now be repeated, choosing a new source distribution $F_1^{(n)}(\mathbf{r})$ and normalization as follows:

$$F_1^{(n)}(\mathbf{r}) = \nu \sum_{i=1}^{N+1} \Sigma_{f_i}^{(n)} \bar{\phi}_{i_0}^{(n)}(\mathbf{r}) \Delta u_i \tag{8.444}$$

$$\nu \int_{\text{reactor}} F_1^{(n)}(\mathbf{r})\, d\mathbf{r} = 1 \tag{8.445}$$

As a result of this reactor calculation, the fluxes $\bar{\phi}_{i_1}^{(n)}(\mathbf{r})$ are obtained, and the multiplication constant may be computed by

$$k_1 = n_1 \tag{8.446}$$

$$n_1 = \nu \sum_{i=1}^{N+1} \int_{\text{reactor}} \Sigma_{f_i}^{(n)} \bar{\phi}_{i_1}^{(n)}(\mathbf{r}) \Delta u_i\, d\mathbf{r} \tag{8.447}$$

This process may be repeated until the values for the multiplication constant in succeeding iterations coincide to the desired accuracy. In general, it is found that the multiplication constant has converged upon a final value before that stage in the calculation at which the source distribution has converged acceptably.

As shown in the treatment of the multigroup equations using the definitions of Eq. (8.399), there is no approximation involved in the use of average cross sections for a group so long as the flux-weighted averages of Eq. (8.399) are used.[1] Of course at the outset of a calculation the lethargy dependence of the flux is not known; thus, unless some estimate of the behavior of the flux is available from previous calculations on similar systems, it is necessary to begin the calculation with cross sections obtained by some arbitrarily chosen averaging process. The simplest and most often used is the direct lethargy average

$$\Sigma_i^{(n)} \equiv \frac{\int_{u_{i-1}}^{u_i} \Sigma^{(n)}(u)\phi^{(n)}(\mathbf{r},u)\, du}{\int_{u_{i-1}}^{u_i} \phi^{(n)}(\mathbf{r},u)\, du} \simeq \frac{\int_{u_{i-1}}^{u_i} \Sigma^{(n)}(u)\, du}{\int_{u_{i-1}}^{u_i} du} \tag{8.448}$$

[1] Except in so far as the *spatial* dependence of the average cross section is neglected.

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Once a first estimate of the fluxes is obtained it is appropriate to perform the flux-weighted averages; however, this is a laborious, time-consuming operation and is usually avoided by choosing the lethargy intervals so that the flux has only slight variation over any one interval. This not only calls for small lethargy intervals, but, since only a limited number of intervals may be used in a practical calculation, it also calls for the use of more intervals in lethargy ranges over which the flux may be expected to be more rapidly varying. Such ranges are, of course, those over which the cross sections themselves vary most rapidly. (Note that the method of obtaining the multigroup equations implies that the macroscopic cross section, using the microscopic cross sections for each material present, should be obtained as a function of lethargy, and *then* averaged over an interval.)

There remains the discussion of the spatial integration of the multigroup equations and the treatment of the boundary conditions. Since there are numerous satisfactory methods of solving the spatial problem, no attempt will be made to give an exhaustive treatment here. However, there are a few interesting points in the special problem at hand which may be brought out by a brief description of one method which has been successfully used in the treatment of spherically symmetric systems. It is convenient (but not necessary) to divide the *radial* distance into uniform segments of length Δr in each region such that all medium-medium interfaces occur at integral multiples of Δr. Then the differential Eqs. (8.433) are converted into difference equations in terms of the values of the functions at various radial points.

If the following definitions are used:

$$\begin{aligned} \bar{\phi}_i^{(n)}(r) &\equiv \frac{W_i^{(n)}(r)}{r} \\ q^{(n)}(r,u_{i-1}) &\equiv \frac{V_{i-1}^{(n)}(r)}{r} \\ F^{(n)}(r) &\equiv \frac{H^{(n)}(r)}{r} \end{aligned} \tag{8.449}$$

Eq. (8.433) may be written in the form

$$\frac{d^2}{dr^2} W_i^{(n)}(r) - a_i^{(n)} W_i^{(n)}(r) = -C_i^{(n)}(r) \tag{8.450}$$

In (8.450) the following definitions have been used:

$$\begin{aligned} a_i^{(n)} &\equiv \frac{1}{\bar{D}_i^{(n)} \Delta u_i} [\bar{\Sigma}_{a_i}^{(n)} + 2(\xi\Sigma_t)_{u_i}^{(n)}] \qquad \text{and} \qquad a_{N+1}^{(n)} \equiv \frac{\Sigma_a^{\text{th}}}{D^{\text{th}}} \\ C_i^{(n)}(r) &\equiv \left[1 + \frac{(\xi\Sigma_t)_{u_i}^{(n)}}{(\xi\Sigma_t)_{u_{i-1}}^{(n)}}\right] \frac{V_{i-1}^{(n)}(r)}{\bar{D}_i^{(n)} \Delta u_i} + \frac{\nu\bar{\delta}_i H^{(n)}(r)}{\bar{D}_i^{(n)}} \\ C_{N+1}^{(n)}(r) &\equiv \frac{V_N^{(n)}(r)}{D^{\text{th}}} \end{aligned} \tag{8.451}$$

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Let the values of these functions at the chosen space points be denoted by a subscript; thus

$$W_i^{(n)}(m\,\Delta r) \equiv [W_i^{(n)}]_m \qquad V_i^{(n)}(m\,\Delta r) \equiv [V_i^{(n)}]_m \qquad H^{(n)}(m\,\Delta r) \equiv [H^{(n)}]_m \tag{8.452}$$

The second derivative of (8.450) may be approximated by (for example)

$$\frac{d^2}{dr^2} W_i^{(n)}(r) = \frac{1}{(\Delta r)^2} \{[W_i^{(n)}]_{m+1} - 2[W_i^{(n)}]_m + [W_i^{(n)}]_{m-1}\} \tag{8.453}$$

Then the difference equation may be written in the form

$$[W_i^{(n)}]_{m+1} = k_i^{(n)}[W_i^{(n)}]_m - [W_i^{(n)}]_{m-1} - [S_i^{(n)}]_m \tag{8.454}$$

where

$$k_i^{(n)} \equiv 2 + a_i^{(n)}(\Delta r)^2 \qquad [S_i^{(n)}]_m \equiv (\Delta r)^2[C_i^{(n)}]_m \tag{8.455}$$

Note that, from the definitions above and the assumption of (8.432*d*),

$$[W_i^{(n)}]_m = \frac{1}{2}\left\{\frac{[V_i^{(n)}]_m}{(\xi\Sigma_t)_{u_i}} + \frac{[V_{i-1}^{(n)}]_m}{(\xi\Sigma_t)_{u_{i-1}}}\right\} \tag{8.456}$$

In (8.454) the $k_i^{(n)}$ and $[S_i^{(n)}]_m$ are known from the calculated constants, the assumed source distribution and the results of the calculation of previous groups. The calculation of the $[W_i^{(n)}]_m$ may proceed, for example, from the center of the innermost region where it is assumed that $\bar{\phi}_i^{(1)}(0)$ is not singular; thus we have

$$\begin{aligned} [W_i^{(1)}]_0 &= 0 \\ [W_i^{(1)}]_{-1} &= -[W_i^{(1)}]_1 \\ [W_i^{(1)}]_2 &= k_i^{(1)}[W_i^{(1)}]_1 - [S_i^{(1)}]_1 \qquad \text{etc.} \end{aligned} \tag{8.457}$$

Thus Eq. (8.454) is used as a recursion formula to generate the W's of higher index m. Eventually an interface between two regions is reached, and it is necessary to invoke the conditions of continuity of flux and current in order to cross the boundary and proceed with the recursion formula (8.454) as applied to the second region. Let the interface be at radial position $R \equiv B_* \Delta r$; then, using the definitions above, we have

$$[W_i^{(1)}]_{B_*} = [W_i^{(2)}]_{B_*} \tag{8.458}$$

$$\bar{D}_i^{(1)}\left\{B_*\Delta r\left[\frac{d}{dr} W_i^{(1)}(r)\right]_{B_*\Delta r} - [W_i^{(1)}]_{B_*}\right\} = \bar{D}_i^{(2)}\left\{B_*\Delta r\left[\frac{d}{dr} W_i^{(2)}(r)\right]_{B_*\Delta r} - [W_i^{(2)}]_{B_*}\right\} \tag{8.459}$$

The application of the current condition (8.459) involves a choice for the representation of the first derivative in terms of the difference quantities W_m. As has been pointed out by the KAPL group,[1] it is very useful to make this choice in such a way that detailed neutron balance is pre-

[1] *Ibid.*

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served in the group. For example, let all integrals of continuous functions over any volume of the system be approximated by a summation of difference quantities as follows:

$$\int_{r_1=m_1\Delta r}^{r_2=m_2\Delta r} r\psi(r)\,dr = (\Delta r)^2\left\{\sum_{j=m_1}^{m_2} j\psi_j - \left(\frac{m_2}{2}\psi_{m_2} + \frac{m_1}{2}\psi_{m_1} + \frac{1}{6}\psi_{m_2} - \frac{1}{6}\psi_{m_1}\right)\right\} \tag{8.460}$$

where $$\psi_j \equiv \psi(j\Delta r)$$

This approximation may be used for the current at an interface by recognizing that

$$\begin{aligned} J_i^{(1)}(R)4\pi R^2 &= -\bar{D}_i^{(1)}\Delta u_i \int_{\text{surface}} \nabla\bar{\phi}_i^{(1)}(r)\cdot d\mathbf{S} \\ &= -\bar{D}_i^{(1)}\Delta u_i \int_{\text{volume}} \nabla^2\bar{\phi}_i^{(1)}(r)\,d\mathbf{r} \\ &= -\bar{D}_i^{(1)}\Delta u_i 4\pi \int_0^R r\left[\frac{d^2W_i^{(1)}}{dr^2}\right]dr \end{aligned} \tag{8.461}$$

If we use (8.453) for the second derivative in the final integral, it may be shown that the current condition (8.459) takes the form

$$\begin{aligned} \bar{D}_i^{(1)}&\left\{\frac{1}{6}\left([W_i^{(1)}]_{B_*+1} + [W_i^{(1)}]_{B_*-1}\right) + \frac{2}{3}[W_i^{(1)}]_{B_*}\right. \\ &\left. - \frac{B_*}{2}\left([W_i^{(1)}]_{B_*+1} - [W_i^{(1)}]_{B_*-1}\right)\right\} = \bar{D}_i^{(2)}\left\{\frac{1}{6}\left([W_i^{(2)}]_{B_*+1} + [W_i^{(2)}]_{B_*-1}\right)\right. \\ &\left. + \frac{2}{3}[W_i^{(2)}]_{B_*} - \frac{B_*}{2}\left([W_i^{(2)}]_{B_*+1} - [W_i^{(2)}]_{B_*-1}\right)\right\} \end{aligned} \tag{8.462}$$

For consistency the source term must be normalized using the form (8.460); that is, the integral in the condition expressed by Eq. (8.441) is to be replaced by the corresponding summation as obtained by applying (8.460). The advantage of this method is, then, realized at the end of a group calculation. Integrations of a continuous-group equation such as (8.433) over a given region or over the entire system give an equality whose physical content may be stated in the form

$$\begin{aligned} &\begin{bmatrix}\text{Total neutrons degraded}\\ \text{from previous group}\end{bmatrix} + \begin{bmatrix}\text{total neutrons thrown}\\ \text{into group from sources}\end{bmatrix} \\ &= \begin{bmatrix}\text{total neutrons lost}\\ \text{by leakage from}\\ \text{region (or system)}\end{bmatrix} + \begin{bmatrix}\text{total neutrons}\\ \text{lost by}\\ \text{absorption}\end{bmatrix} + \begin{bmatrix}\text{total neutrons}\\ \text{degraded into}\\ \text{next group}\end{bmatrix} \end{aligned} \tag{8.463}$$

With the consistent use of Eqs. (8.460) and (8.462), the relationship (8.463) may be written in terms of summations over constants and the $[W_i^{(*)}]_m$; because of the care taken in obtaining the choice (8.462), this

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equality holds rigorously if the calculations are absolutely accurate. Thus such summation equalities (for a given region or for the entire system) may be used as a valuable check on the numerical accuracy of the calculation.

Eventually, the process of using the recursion formula (8.454) in each region and crossing boundaries using (8.458) and (8.462)[1] reaches the very outside boundary of the system $R_0 \equiv B_0 \Delta r$; here the condition must be applied that

$$[W_i^{(f)}]_{B_0} = 0 \tag{8.464}$$

where index f denotes the final, outermost region. The resultant value of $[W_i^{(f)}]_{B_0}$ may be seen to depend on the choice for $[W_i^{(1)}]_1$ of Eq. (8.457) which is otherwise unspecified. Any general choice of $[W_i^{(1)}]_1$ will not result in a zero value for $[W_i^{(f)}]_{B_0}$. One simple way around this difficulty is to define two functions $\mathfrak{A}$ and $\mathfrak{B}$ such that

$$\begin{aligned} [\mathfrak{A}_i^{(n)}]_{m+1} &= k_i^{(n)}[\mathfrak{A}_i^{(n)}]_m - [\mathfrak{A}_i^{(n)}]_{m-1} \\ [\mathfrak{A}_i^{(1)}]_0 &= 0 \qquad [\mathfrak{A}_i^{(1)}]_1 = 1 \end{aligned} \tag{8.465}$$

$$\begin{aligned} [\mathfrak{B}_i^{(n)}]_{m+1} &= k_i^{(n)}[\mathfrak{B}_i^{(n)}]_m - [\mathfrak{B}_i^{(n)}]_{m-1} - [S_i^{(n)}]_m \\ [\mathfrak{B}_i^{(1)}]_0 &= 0 \qquad [\mathfrak{B}_i^{(1)}]_1 = 1 \end{aligned} \tag{8.466}$$

Thus $\mathfrak{B}$ is a particular solution of Eq. (8.454), and $\mathfrak{A}$ is a solution of the homogeneous part of Eq. (8.454). Both $\mathfrak{A}$ and $\mathfrak{B}$ are required to meet the same boundary conditions as W. The function W is then expressible as a linear combination of $\mathfrak{A}$ and $\mathfrak{B}$; thus

$$[W_i^{(n)}]_m = [\mathfrak{A}_i^{(n)}]_m + c[\mathfrak{B}_i^{(n)}]_m \tag{8.467}$$

so that once $\mathfrak{A}$ and $\mathfrak{B}$ are known at every point W may be calculated at every point if the constant c is known. The constant c is determined by the outside boundary condition

$$[\mathfrak{A}_i^{(f)}]_{B_0} + c[\mathfrak{B}_i^{(f)}]_{B_0} = 0 \tag{8.468}$$

While the scheme just presented for handling the outside boundary condition has been used successfully, it is not the most convenient or the most accurate method. The presentation of improved methods is, however, deemed outside the scope of the present discussion; the KAPL reference gives some details of such an improvement.

The results of a multigroup calculation give very detailed information

[1] Note that (8.462) requires the calculation of values of the function $W^{(n)}$ at points *outside* the nth region (e.g., $[W_i^{(1)}]_{B_0+1}$); this is a convenient mathematical device, and these values are not regarded as having physical meaning. Further, it should be noted that one of these is never really calculated; the quantity $[W_i^{(2)}]_{B_0-1}$ is related to $[W_i^{(2)}]_{B_0}$ and to $[W_i^{(2)}]_{B_0+1}$ through the recursion relationship of Eq. (8.454) for $n = 2$ and so does not need to be computed explicitly.

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about the reactor system. Besides the multiplication constant, the distribution of flux in energy and space is given, from which may be computed, for example, the distribution of neutron absorptions or fissions in energy and space or the current of neutrons of various energies emerging from the outside of the reactor system. Of course, the information input

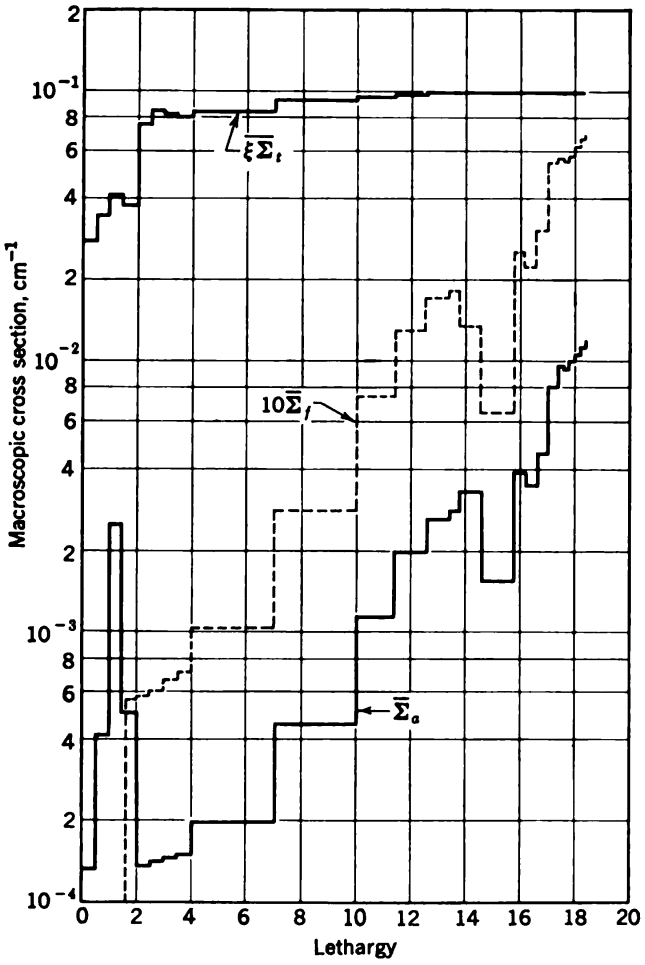

FIG. 8.31 Average cross sections in the core used in multigroup calculation.

is also very detailed. However, once the multigroup equations are coded for a fast computing machine, the major effort required for each system is the collection of nuclear data and the computation of the average cross sections for the various groups. Even this chore has been eliminated at many of the major computing facilities wherein "libraries" of subroutines are now available. These routine codes not only provide up-to-date nuclear cross-section data of all the elements from the thermal to the Mev range, but also incorporate various energy-averaging procedures for rapid preparation of group constants.

Various special modifications of the multigroup equations have been

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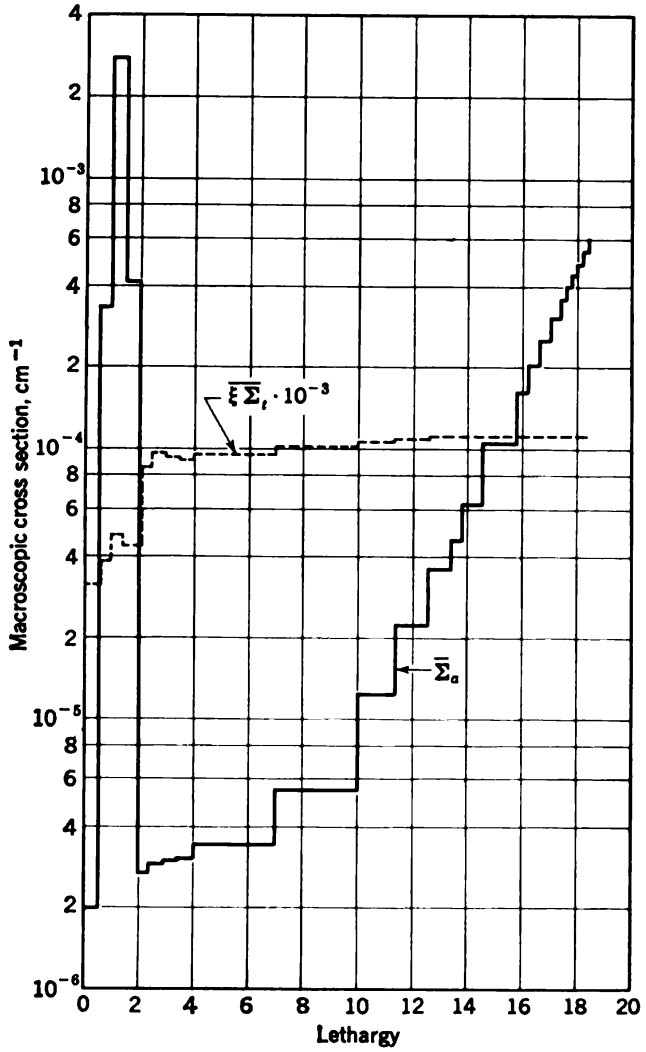

FIG. 8.32 Average cross sections in the reflector used in multigroup calculation.

TABLE 8.9 NUCLEAR DENSITIES IN CORE AND REFLECTOR
(In units of 10^{24} nuclei/cm³)

Material	Core	Reflector
Sodium	0.0003946	0.0002365
Carbon	0.0006835	0.0004096
Zirconium	0.0004274	0.000256
Fluorine	0.0005713	0.0002365
Oxygen	0.05827	0.06577
Beryllium	0.05742	0.06526
Uranium (235)	0.00004418	
Stainless steel	0.01546	
Inconel	.....	0.00127

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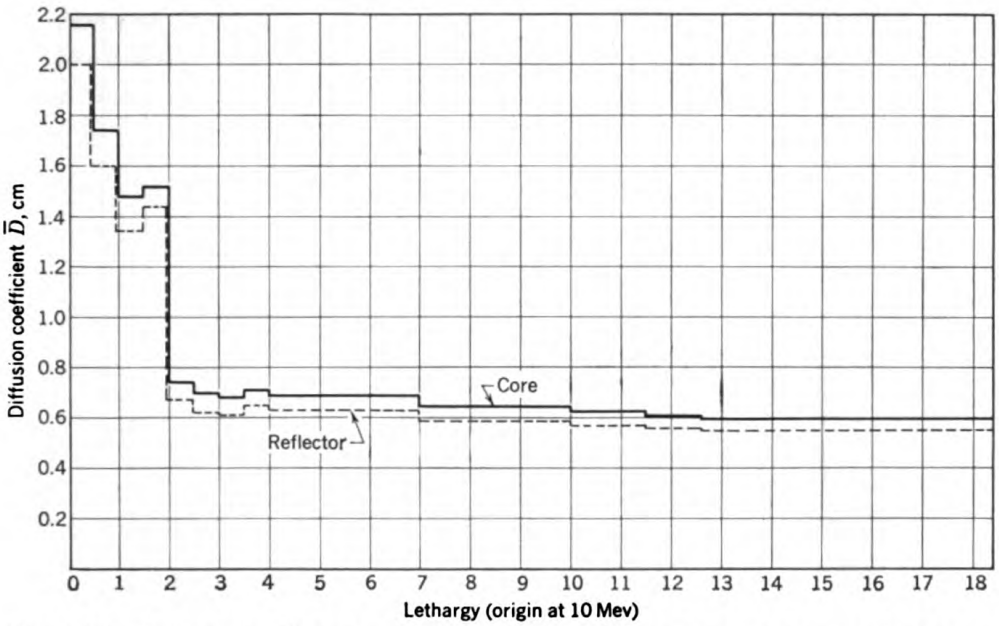

FIG. 8.33 Diffusion coefficients in the core and reflector used in multigroup calculation

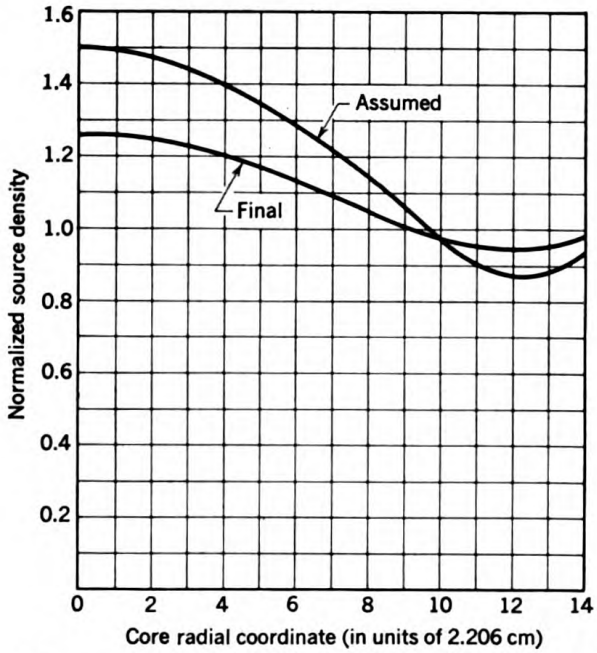

FIG. 8.34 Assumed and resultant source distributions over the core.

considered; three of these are: (1) the extension of the method to cylindrical geometries, (2) the modification to give improved results for hydrogenous systems, and (3) the improvement of the equations to include more information about the angular distribution of the neutron flux so as to gain an improvement in the treatment of the slowing-down process.

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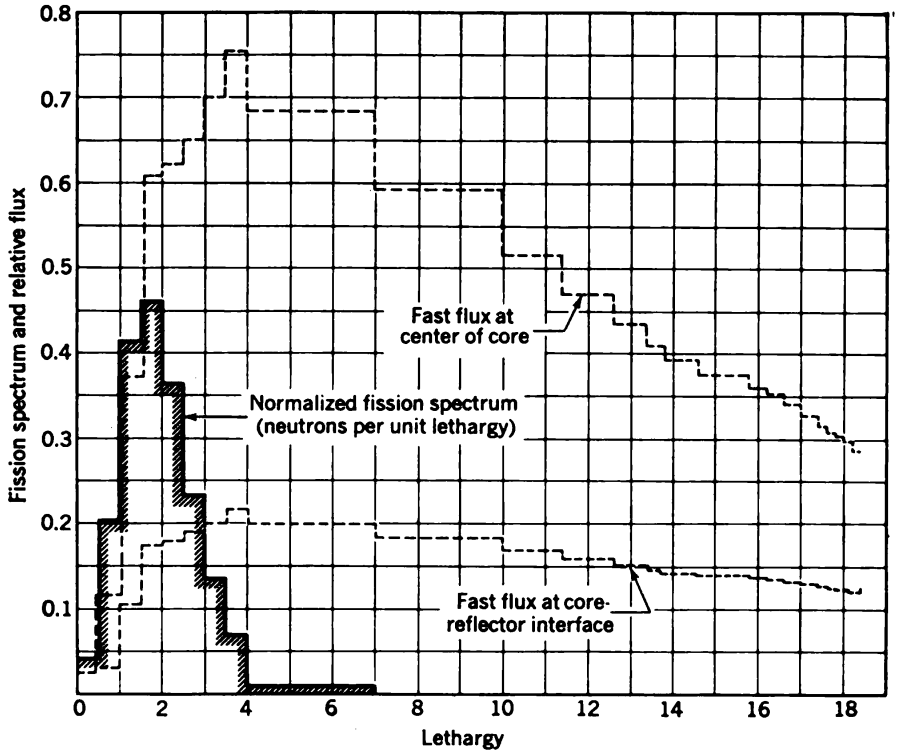

FIG. 8.35 Fission spectrum and average fast-flux distribution in lethargy.

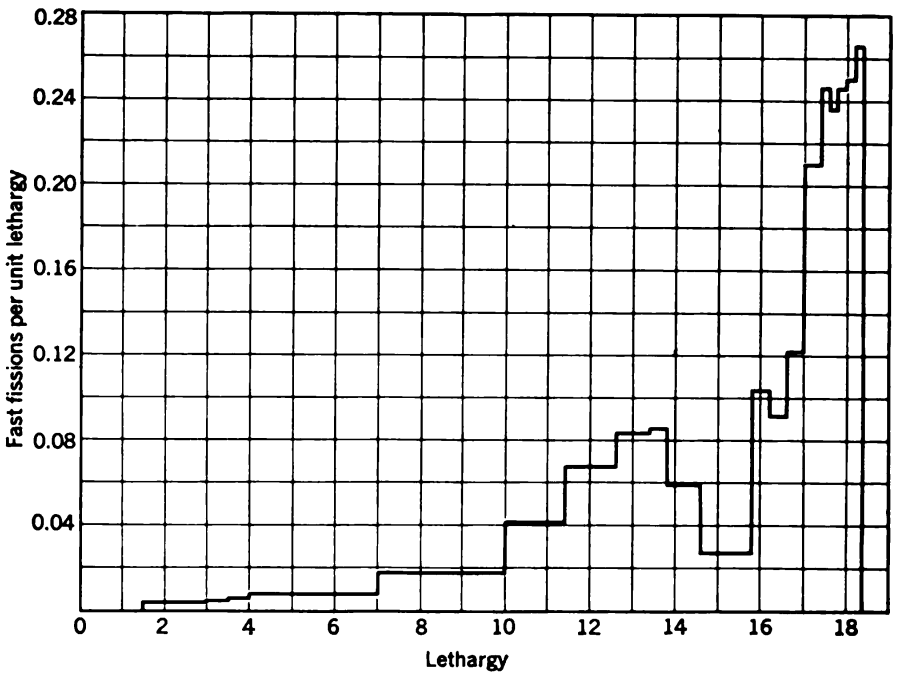

FIG. 8.36 Relative number of fissions, per unit lethargy, due to neutrons of given lethargy.

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The average cross sections and results for a typical multigroup calculation are shown in Figs. 8.31 through 8.39. The particular assembly chosen is a spherically symmetric system consisting of core and reflector. The fuel is U^{235} and the principal moderating material in both the core

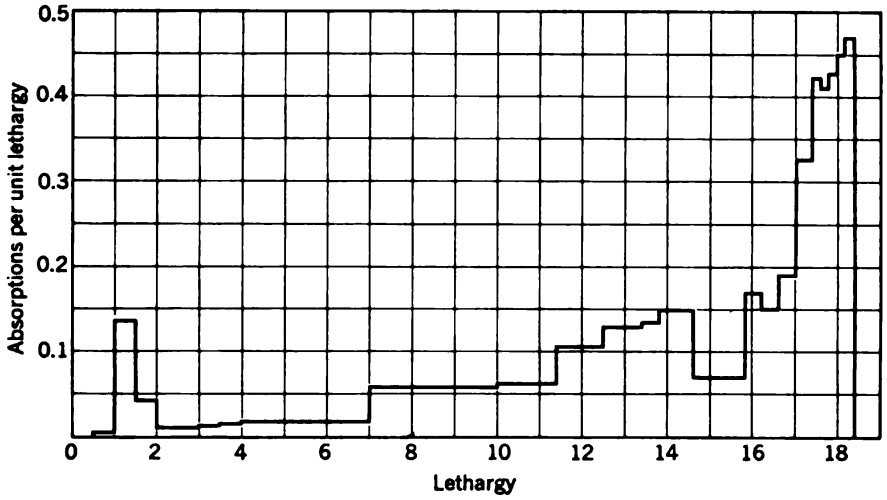

FIG. 8.37 Relative number of neutrons absorbed per unit lethargy.

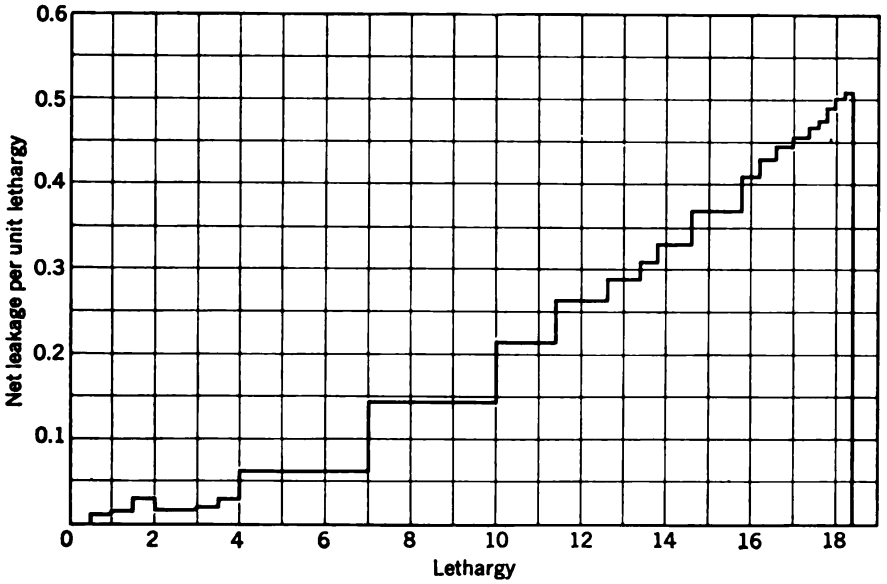

FIG. 8.38 Relative number of neutrons escaping from outer surface of reflector per unit lethargy.

and the reflector is BeO. Various coolant and structural materials are also present in both regions; the appropriate nuclear densities are listed in Table 8.9. The core is 30.88 cm in radius, and the reflector is 30.88 cm thick. With the zero of lethargy selected to correspond to 10 Mev, the fast range is divided into 25 lethargy intervals of various widths. The

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thermal base is chosen at a lethargy of 18.4 corresponding to 0.1021 ev or 1672°F. Figures 8.31, 8.32, and 8.33 show the average values for the cross sections and diffusion coefficients chosen in each group for the core and the reflector. The thermal cross sections are shown as points when they differ from the value of the immediately preceding group. The normalized source distribution in the core which was assumed to start

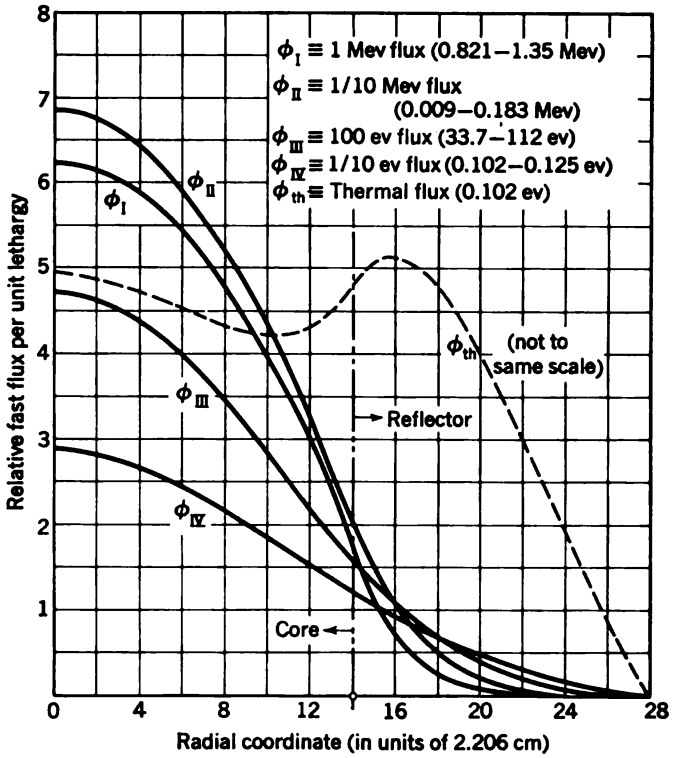

FIG. 8.39 Spatial distribution of average flux in four fast groups and in thermal group.

the calculation is shown in Fig. 8.34, along with the resultant neutron production distribution after one complete calculation. (In this case the calculation was not iterated to convergence of the source distribution.) The originally chosen form was based on previous experience with similar assemblies. Figure 8.35 shows the lethargy distribution assumed for the source neutrons [the χ_i of Eq. (8.402)] along with the average neutron flux in each group at two special space points, the center of the reactor, and the core-reflector interface. The thermal flux is not shown in Fig. 8.35 since the fast flux and the thermal flux are not dimensionally equivalent quantities.

For each neutron introduced into the assembly (in the distribution described in Figs. 8.34 and 8.35) the following final results are obtained:

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0.1563 neutron absorbed in the entire assembly as fast neutrons
0.4882 neutron absorbed in the entire assembly as thermal neutrons
0.0355 neutron escape from the assembly as fast neutrons
0.3200 neutron escape from the assembly as thermal neutrons
1.0000

and

0.08837 neutron enter into fission processes as fast neutrons
0.28139 neutron enter into fission processes as thermal neutrons

Thus, 76.1 per cent of all fissions are caused by thermal neutrons; 75.7 per cent of all neutrons absorbed are absorbed as thermal neutrons; 90.0 per cent of all neutrons escaping from the assembly escape as thermal neutrons. Figures 8.36, 8.37, and 8.38 show the distributions in lethargy of the fissions, absorptions, and escapes, respectively (all per unit lethargy), in the fast range. Figure 8.39 shows the spatial distribution through the core and reflector of the average fluxes in four particular fast groups plus the thermal flux distribution.

The assembly was found to have a multiplication constant $k = 0.924$.

PROBLEMS

8.1 Determine the critical equation for a one-velocity spherical reflected reactor which has the following specifications: (1) a homogeneous active core of radius a; (2) a spherical shell reflector of thickness $b - a$; the neutron properties of which are, in general, different from those of the core; (3) an infinitesimally thin spherical shell of absorbing material at the core-reflector interface; this absorber has the property that it transmits a fraction α of all neutrons incident on either face.

Show that the critical equation for this system reduces to the usual relation for the one-velocity reflected reactor when the transmittancy $\alpha \to 1$. Obtain an expression for $k(\alpha)$ and compute

$$\frac{\delta k}{k(\alpha)} \equiv \frac{k(1)}{k(\alpha)} - 1$$

for $\alpha = 0.95$ and 0.80 using $a = 50$ cm, $b - a = 25$ cm, $\varkappa_R = 0.02$ cm^{-1}, $D_R = D_C = 1$ cm, and $L_C^2 = 128$ cm^2.

8.2 Derive the complete expression for criticality in a spherical reflected reactor which has the following properties:

Active core: (1) Uniform distribution of fissionable material and moderator throughout a sphere of radius a; (2) neutron cross sections are energy dependent, except for the condition given in (3); (3) diffusion coefficient $D_C(u) \equiv D_C$, which is energy independent; (4) neutrons from fission appear monoenergetically (at $u = 0$). *Reflector:* (5) spherical shell of inner radius a, outer radius b; (6) contains no fuel; (7) neutron cross sections of all materials are energy independent; (8) neutrons suffer no change in energy from elastic-scattering collisions with nuclei of reflector material.

In both the core and the reflector regions of this reactor, the neutrons enter a diffusion process at $u = u_{th}$. The coupling relation $q(\mathbf{r},u) = \xi\Sigma_t(u)\phi(\mathbf{r},u)$ may be assumed to hold in the fast range.

8.3 A reactor consisting of a homogeneous core contained in a spherical pressure shell is operating at steady state. This reactor is to be analyzed on the basis of the

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following model: (1) The active core is a homogeneous sphere of radius R. (2) The neutron-reflecting properties of the pressure shell can be described by means of the relation

$$\phi'(R,u)/\phi(R,u) = -a$$

where $\phi(\mathbf{r},u)$ is the neutron flux in the core and a some constant independent of lethargy (which can be determined from the albedo of the shell material). (3) The Fermi age model applies. (4) The fissionable material has a zero fission cross section at all energies except thermal energy. (5) One of the components of the circulating fuel has an $(n,2n)$ reaction cross section which is different from zero for all $u < u_{th}$. (6) All neutrons produced from fission and $(n,2n)$ reactions appear at $u = 0$.

a. Obtain expressions for the slowing-down density $q(\mathbf{r},u)$, the fast flux $\phi(\mathbf{r},u)$, and the thermal flux $\phi_{th}(\mathbf{r})$. Use the relations

$$q(\mathbf{r},u) = \xi\Sigma_t(u)\phi(\mathbf{r},u) \qquad \text{with} \begin{cases} \Sigma_t(u) \equiv \Sigma_a(u) + \Sigma_s(u) \\ \Sigma_a(u) \equiv \Sigma_f(u) + \Sigma_\gamma(u) + \Sigma_n(u) \end{cases}$$

where Σ_f and Σ_γ have the usual definitions and Σ_n denotes the $(n,2n)$ reaction cross section of the circulating fuel.

b. Find the critical equation for this system.

c. Derive an expression for the multiplication k in this reactor.

d. Evaluate any arbitrary constant which remains in the results of part (*a*) in terms of P, the total power output of the reactor. Note that the energy release during a fission reaction will be, in general, different from that released from the $(n,2n)$ reaction. Also, the energy released by the $(n,2n)$ reaction may be dependent on the energy of the incident neutron.

8.4 *a.* Consider a homogeneous, bare spherical reactor consisting of the materials described in Prob. 4.8. Compute the critical radius R_C of this reactor for $N_M/N_F = 10{,}000$ on the assumption that the resonance escape and fast nonleakage probabilities are unity, that is, the one-velocity model applies.

b. On the basis of the information given in part (*a*), compute and plot N_M/N_F for criticality as a function of reflector thickness T by using the above sphere of radius R_C as the active core. For these calculations assume that the same material is used in the reflector as in the moderator of the core and that $D_C \simeq D_R$.

c. What fraction of the decrease in fuel mass obtainable with an infinite reflector is realized with a reflector of one diffusion-length thickness?

d. Plot the ratio of the decrease in fuel mass to the reflector mass as a function of reflector thickness.

e. Plot, as a function of reflector thickness, the fraction of neutrons produced per second in the core which are (1) absorbed in the core, (2) absorbed in the reflector, and (3) escape from the reactor.

8.5 Compute k for a reflected spherical reactor which has an active core radius $R = 37.2$ cm and a spherical shell reflector of thickness 20 cm. For this calculation use: (1) the material composition described for the reactor in Prob. 4.8, in the case $N_M/N_F = 15{,}500$; (2) nuclear cross sections for fuel and moderator given in Prob. 4.8; (3) the two-group method.

Assume that the moderator and fuel are uniformly distributed in the reactor core and that the reflector consists of pure moderator. Compare the value of k obtained from this calculation with the value obtained for this reactor on the basis of a one-velocity model (as computed in Prob. 8.4).

8.6 A homogeneous right cylinder of beryllium containing some U^{235} has been left exposed in an unattended area of a laboratory. The cylinder is standing upright on

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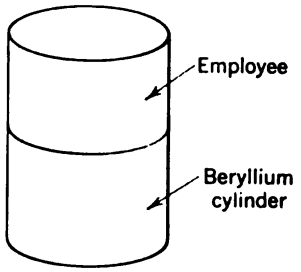
one of its flat surfaces. In its present state, it is not critical. The radius of the cylinder is 20 cm and its height 40 cm. The macroscopic cross section of the U^{235} it contains is $\Sigma_a^{(25)} = 0.01297$ cm^{-1}. One day a new employee of the laboratory who had been working nearby took a moment's rest and sat on the cylinder. The resulting configuration (in first approximation) is shown at the right. What are the consequences of this experiment? For the analysis of this problem use the data:

Beryllium: $L = 23.6$ cm
$D = 0.70$ cm
$\Sigma_a = 0.0013$ cm^{-1}
$\rho = 1.84$ g/cm^3
Water: $L = 2.85$ cm
$D = 0.16$ cm



and the following idealized model: (1) One-velocity diffusion theory applies in all regions. (2) All surfaces and diffusion media, which may be near the cylinder (such as floor, etc.) may be ignored, except the presence of the employee. (3) Employee is composed of pure water.

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CHAPTER 9

REACTOR KINETICS

In this treatment the problems of reactor kinetics are grouped into three categories. Each category deals with a basically different type of nonstationary neutron phenomenon. To the first category we assign those problems which involve rapid transients in the neutron population arising from various disturbances. These disturbances are conveniently specified as changes in the multiplication constant or the reactivity. The study of these problems is based on the methods and results of the one-velocity diffusion and the continuous-slowing-down diffusion theory models. In the first few analyses the effect of the delayed-neutron groups is omitted so as to focus attention on other physical features of the problems. The delayed-neutron groups are introduced in subsequent, more general, treatments, and various analytical models are discussed for describing them. Finally, several of the analytical tools developed in these studies are applied to the analysis of the pulsed neutron-beam experiment previously mentioned in Sec. 5.5.

The kinetics problems of the second category are concerned with the influence of the reactor temperature on the time-dependent behavior of the neutron flux and on the power generation. The phenomena of interest here are again short-term effects, and the importance of the delayed-neutron groups in establishing dynamically stable systems is discussed. The analytical models used in these treatments draw principally from the so-called *Stein* model.[1]

The problems of the third category deal with phenomena that cause long-term variations in the neutron population. These include the production and decay of the principal fission fragments that appear as poisons in the reactor. The effect of these substances upon the reactivity of the system is examined and their practical significance noted. This treatment is concluded with a study of the use of burnout poisons for controlling these long-term changes in reactivity.

Nearly all the physical situations considered in these analyses refer to bare reactors. This restriction allows solutions to be obtained by the method of separation of variables. Thus the kinetic features of a specific

[1] J. M. Stein, "Simplified Reactor Theory Lecture Series," Lecture III, Kinetic Equations, Oak Ridge School of Reactor Technology, ORNL CF 51-5-113, June 4, 1951.

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problem can be examined independently of the system geometry. A further restriction which is made is that the transient problems of the first category arise from nuclear changes which are introduced *uniformly* over the entire reactor. Transients in the flux (and power and temperature) due to local changes are best treated by numerical methods or by the application of perturbation theory.

9.1 Time-dependent Behavior of the Neutron Flux with Delayed Neutrons Neglected

a. Statement of General Problem in Terms of the Fermi Age Model. In the present treatment the most general neutron-balance relations developed for a nonstationary system are given in terms of the Fermi age model. Although this formulation allows some improvement over the very elementary models used in Secs. 3.2 and 5.4a, it is, nevertheless, still a somewhat limited tool. Its greatest usefulness is realized in the study of bare homogeneous systems. Closed-form analytical solutions for multiregion systems are not possible with this model because of the difficulty in satisfying boundary conditions which involve simultaneously a dependence on space, lethargy, and time. This difficulty was previously encountered in the application of the continuous-slowing-down diffusion model to a bare system at steady state [see discussion following Eq. (6.56)]. The only way in which this model can be applied to the study of nonstationary multiregion systems is through the use of numerical methods such as the one described in Sec. 8.8.

Apart from this limitation, the principal restriction in the use of the Fermi age model is due to the large number of conditions which must be satisfied by the physical system if the model is to be valid (see Sec. 7.3i). However, even with this shortcoming, the model is still very useful for preliminary studies and for gaining insight into the essential features of many reactor problems. The Fermi age or continuous-slowing-down diffusion theory, then, offers some generality without introducing excessive computational difficulties.

The appropriate time-dependent differential equations for the combined-slowing-down diffusion theory may be obtained by recognizing that the time rate of change of the neutron density is given by the difference between the rates of disappearance and appearance of neutrons from sinks and sources. Thus we write

$$\frac{\partial}{\partial t} n(\mathbf{r},u,t) = S(\mathbf{r},u,t) + D(u,t)\,\nabla^2\phi(\mathbf{r},u,t) - \Sigma_a(u,t)\,\phi(\mathbf{r},u,t) - \frac{\partial}{\partial u} q(\mathbf{r},u,t) \qquad \text{for } u > u_{\text{th}} \tag{9.1}$$

$$\frac{\partial}{\partial t} n_{\text{th}}(\mathbf{r},t) = S_{\text{th}}(\mathbf{r},t) + D_{\text{th}}(t)\,\nabla^2\phi_{\text{th}}(\mathbf{r},t) - \Sigma_a^{\text{th}}(t)\,\phi_{\text{th}}(\mathbf{r},t) \qquad \text{for } u = u_{\text{th}} \tag{9.2}$$

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The coupling equations for this set are taken to be

$$n(\mathbf{r},u,t) \equiv \frac{1}{v(u)}\,\phi(\mathbf{r},u,t) \qquad n_{\text{th}}(\mathbf{r},t) \equiv \frac{1}{v_{\text{th}}}\,\phi_{\text{th}}(\mathbf{r},t) \tag{9.3}$$

and

$$q(\mathbf{r},u,t) = \xi\Sigma_t(u,t)\,\phi(\mathbf{r},u,t) \tag{9.4}$$

The various functions which appear in these relations are defined by

$n(\mathbf{r},u,t) \equiv$ number of neutrons per unit volume per unit lethargy around $(\mathbf{r},u)$ at time t

$\phi(\mathbf{r},u,t) \equiv$ neutron flux per unit lethargy around $\mathbf{r}$ at time t

$S(\mathbf{r},u,t) \equiv$ number of neutrons produced by sources per unit volume per unit lethargy per unit time around $(\mathbf{r},u)$ at time t

$q(\mathbf{r},u,t) \equiv$ number of neutrons slowing down past lethargy u per unit volume per unit time around $(\mathbf{r},u)$ at time t

$n_{\text{th}}(\mathbf{r},t) \equiv$ number of neutrons of thermal speed per unit volume around $\mathbf{r}$ at time t

$\phi_{\text{th}}(\mathbf{r},t) \equiv$ thermal-neutron flux around $\mathbf{r}$ at time t

$S_{\text{th}}(\mathbf{r},t) \equiv$ number of thermal neutrons produced by sources per unit volume per unit time around $\mathbf{r}$ at time t

The set of equations (9.1) through (9.4) define the distribution of neutrons in space, time, and lethargy for a *homogeneous* system (note that all neutron constants are independent of $\mathbf{r}$, but are, in general, functions of u and t).

The solutions to the above equations must satisfy an appropriate set of boundary conditions. For these we choose:

(1) The flux (or neutron density) must be symmetric about the central axes of the reactor; or the solutions must have no singularities at points not occupied by localized sources.

(2) The flux must be continuous through interfaces between dissimilar media.

(3) The neutron net current

$$\mathbf{J}(\mathbf{r},u,t) \equiv -D(u,t)\,\nabla\phi(\mathbf{r},u,t) \tag{9.5}$$

must be continuous through interfaces.

(4) The flux must vanish at the extrapolated outer surfaces of the system (or the return current from the surrounding space must be zero).

b. One-velocity Bare Reactor. Two special cases of the one-velocity time-dependent problem have already been discussed in connection with other methods of reactor analysis. The results from these calculations are summarized below as an introduction to the more general one-velocity problem. In the first of these cases we examined the problem of the approach to steady-state operation in a multiplying slab (Sec. 5.4a). For the thermal system the appropriate differential equation is (9.2),

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where the source term is now [cf. Eq. (5.128)]

$$S_{\text{th}}(\mathbf{r},t) \equiv \nu\Sigma_f^{\text{th}}\phi_{\text{th}}(\mathbf{r},t) \tag{9.6}$$

The general solution for $\phi_{\text{th}}(\mathbf{r},t)$ is given by Eq. (5.144).

$$\phi_{\text{th}}(x,t) = G_0 \sum_{n=0}^{\infty} (A_n e^{-D v \lambda_n^2 t} \cos \alpha_n x + C_n e^{-D v \omega_n^2 t} \sin \gamma_n x) \tag{9.7}$$

where the α_n and γ_n are given by (5.137) and the λ_n and ω_n by (5.139). The coefficients A_n and C_n are obtained from (5.143).

The second special one-velocity problem previously considered dealt with the spreading out of neutrons through a nonmultiplying infinite medium after an initial burst at $t = 0$ from a plane source at $x = 0$ (Secs. 6.2a and b). In this case, Eq. (9.2) is used everywhere except at $x = 0$, and the source condition is given by [cf. Eq. (6.7)]

$$S_{\text{th}}(x,t) \equiv 0 \qquad \text{with } \lim_{t \to 0^+} \int_{-\infty}^{\infty} n(x,t)\, dx = q_0 \tag{9.8}$$

The solution is given in (6.32); in the present notation, this is

$$n_{\text{th}}(x,t) = \frac{q_0 e^{-v_{\text{th}}\Sigma_a^{\text{th}} t}\, e^{-(x^2/4D_{\text{th}}v_{\text{th}}t)}}{(4\pi D_{\text{th}} v_{\text{th}} t)^{\frac{1}{2}}} \tag{9.9}$$

As a somewhat more general statement of the one-velocity problem consider the time-dependent behavior of the thermal-neutron flux in a bare homogeneous multiplying medium which is operating as a chain reactor at steady state up until time $t = 0$. At $t = 0$ some uniform change (over the entire reactor) is made in the concentrations of the materials in the system so that the reactor is no longer critical. If we assume that no changes are made in the fast cross sections or in the size of the reactor, and that the slowing-down time is zero (with no fast fissions) but with fast-neutron losses by escape and capture, then the appropriate equation is (9.2), with the source $S_{\text{th}} = \nu\Sigma_f^{\text{th}} p_{\text{th}} g_{\text{th}} \phi_{\text{th}}$; thus,

$$-D_{\text{th}}\nabla^2\phi_{\text{th}}(\mathbf{r},t) + \Sigma_a^{\text{th}}\phi_{\text{th}}(\mathbf{r},t) = \nu\Sigma_f^{\text{th}} p_{\text{th}} g_{\text{th}} \phi_{\text{th}}(\mathbf{r},t) - \frac{1}{v_{\text{th}}}\frac{\partial}{\partial t}\phi_{\text{th}}(\mathbf{r},t) \tag{9.10}$$

where p_{th} is the resonance-escape probability to thermal, and the fast nonleakage probability g_{th} is given by

$$g_{\text{th}} \equiv e^{-B^2\tau_{\text{th}}} \qquad \text{with } B^2 \equiv \alpha_0^2 = \left(\frac{\pi}{2\tilde{a}}\right)^2 \qquad \text{for a slab reactor} \tag{9.11}$$

Note that Eq. (9.10) implies that all neutrons are *prompt*.

The solution $\phi_{\text{th}}(\mathbf{r},t)$ to (9.10) must satisfy the conditions

$$\begin{aligned} &(1)\ \phi_{\text{th}}(\mathbf{r},0) \equiv \phi_0(\mathbf{r}) \\ &(2)\ \phi_{\text{th}}(\mathbf{r},t) \text{ is nonsingular} \\ &(3)\ \phi_{\text{th}}(\tilde{\mathbf{R}},t) = 0 \end{aligned} \tag{9.12}$$

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where $\tilde{R}$ denotes the extrapolated outer surface of the reactor. The initial flux distribution $\phi_0(\mathbf{r})$ satisfies the equation

$$\nabla^2\phi_0(\mathbf{r}) + B^2\phi_0(\mathbf{r}) = 0 \tag{9.13}$$

where B is given in (9.11) for the slab (see also Table 5.3).

The solution to (9.10) may be obtained by the method of separation of variables used in the case of the slab problem of Sec. 5.4a [cf. Eq. (5.128)]; therefore, take

$$\phi_{\text{th}}(\mathbf{r},t) \equiv \phi_0(\mathbf{r})\, G(t) \tag{9.14}$$

Clearly, all conditions of (9.12) are met by this relation since $\phi_0(\mathbf{r})$ is chosen to meet conditions (2) and (3). However, we require also

$$G(0) \equiv 1 \tag{9.15}$$

The substitution of (9.14) into (9.10), using also (9.13), yields the equation

$$\frac{1}{v_{\text{th}}\Sigma_a^{\text{th}}(1 + L^2B^2)}\frac{d}{dt}G(t) = \left[\frac{\nu\Sigma_f^{\text{th}}p_{\text{th}}g_{\text{th}}}{\Sigma_a^{\text{th}}(1 + L^2B^2)} - 1\right]G(t) \tag{9.16}$$

with $L^2 \equiv D_{\text{th}}/\Sigma_a^{\text{th}}$. The first term at the right may be recognized as the effective multiplication constant of the system *after* the change is made; that is,

$$k = \frac{\nu\Sigma_f^{\text{th}}p_{\text{th}}g_{\text{th}}}{\Sigma_a^{\text{th}}(1 + L^2B^2)} \tag{9.17}$$

Note that the cross sections which appear in this relation were derived from the noncritical system of Eq. (9.10); therefore, by definition $k \neq 1$.

The factor on the left-hand side of Eq. (9.16) may be recognized as the average neutron lifetime in the one-velocity reactor (cf. Sec. 5.4g). It follows immediately from (5.223) that

$$\frac{1}{v_{\text{th}}\Sigma_a^{\text{th}}(1 + L^2B^2)} \equiv l \equiv l_0 p_{\text{NL}} \tag{9.18}$$

where l_0 is the life of a thermal neutron in an infinite medium [see (5.221)] and p_{NL} is the nonleakage probability for thermal neutrons [see (5.219)]. Further, if we call

$$\delta k \equiv k - 1 \tag{9.19}\star$$

then, with the above identifications, (9.16) reduces to

$$l\frac{d}{dt}G(t) = \delta k\, G(t) \tag{9.20}$$

Note that these results depend upon the assumption that no change is made in the dimensions of the bare reactor. The solution to (9.20) is seen to be

$$G(t) = e^{\delta k\, t/l} \tag{9.21}$$

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where we have used (9.15). The thermal flux is given by

$$\phi_{\text{th}}(\mathbf{r},t) = \phi_0(\mathbf{r})e^{\delta k\, t/l} \tag{9.22}\star$$

Thus we find on the basis of this simple model that the flux rises and falls exponentially in time, depending on whether the change made at $t = 0$ results in a k which is greater or less than unity, respectively.

The *period* of the reactor is defined as the time required for the flux to rise to a value e times its value at $t = 0$ (in the case of an increase in k). By the above model

$$\text{The period} \equiv T \equiv \frac{l}{\delta k} \tag{9.23}\star$$

It should be borne in mind that this result is based on the assumption that there are no delayed neutrons in the system. Thus all neutrons from fission appear "immediately." The expression (9.23) should be compared with the more general relation obtained in Sec. 9.3.

c. Fermi Age Model of Multivelocity Bare Reactor. The general approach developed for the one-velocity bare reactor may be applied to the multivelocity system with the aid of the Fermi age model. Although a complete analytical solution is possible for the multivelocity bare reactor, the present treatment is confined to a study of the time behavior at long times after the system has been disturbed from an initial steady-state condition. Thus the analysis which follows does not include the transients which appear immediately after the perturbation has been introduced but gives only the asymptotic time variations in the neutron population. Further, this calculation is limited by the assumptions that all fission neutrons are prompt and that the changes which disturb the reactor from steady state are introduced uniformly over the entire system.

The specific details of the model used to describe the multivelocity bare reactor are:

(1) The differential equations (9.1) and (9.2) describe the fast and thermal fluxes, respectively.

(2) All neutrons born from fission appear (without delay) monoenergetically at $u = 0$, and fissions are due only to thermal neutrons; therefore, the fast-neutron source is

$$S(\mathbf{r},u,t) \equiv \nu\Sigma_f^{\text{th}}\phi_{\text{th}}(\mathbf{r},t)\ \delta(u) \tag{9.24}$$

(3) The thermal sources are due entirely to the slowing down of fast neutrons; thus,

$$S_{\text{th}}(\mathbf{r},t) \equiv q(\mathbf{r},u_{\text{th}},t) \tag{9.25}$$

(4) The boundary conditions are

$$\phi(\mathbf{r},u,t) \text{ and } \phi_{\text{th}}(\mathbf{r},t) \text{ are everywhere nonsingular} \tag{9.26}$$

$$\phi(\tilde{\mathbf{R}},u,t) \text{ and } \phi_{\text{th}}(\tilde{\mathbf{R}},t) \text{ are zero} \tag{9.27}$$

where $\tilde{\mathbf{R}}$ denotes the extrapolated outer boundary of the reactor.

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(5) The initial conditions are

$$\phi(\mathbf{r},u,0) = \phi_0(\mathbf{r},u) \tag{9.28}$$
$$\phi_{\text{th}}(\mathbf{r},0) = \phi_0(\mathbf{r}) \tag{9.29}$$

where $\phi_0(\mathbf{r},u)$ and $\phi_0(\mathbf{r})$ denote the fast and thermal fluxes, respectively, in the bare reactor as it operated at steady state prior to $t = 0$.

(6) At $t = 0$, a change is made in the *thermal* cross sections so that the reactor is no longer critical, and these new cross sections remain fixed for all $t > 0$.

The appropriate differential equations indicated in (1) may be written in the present case

$$\begin{aligned} -\frac{1}{v(u)\xi\Sigma_t(u)}\frac{\partial}{\partial t}q(\mathbf{r},u,t) &= -\frac{D(u)}{\xi\Sigma_t(u)}\nabla^2 q(\mathbf{r},u,t) + \frac{\Sigma_a(u)}{\xi\Sigma_t(u)}q(\mathbf{r},u,t) + \frac{\partial}{\partial u}q(\mathbf{r},u,t) \\ q(\mathbf{r},0,t) &= \nu\Sigma_f^{\text{th}}\phi_{\text{th}}(\mathbf{r},t) \\ -\frac{1}{v_{\text{th}}}\frac{\partial}{\partial t}\phi_{\text{th}}(\mathbf{r},t) &= -D_{\text{th}}\nabla^2\phi_{\text{th}}(\mathbf{r},t) + \Sigma_a^{\text{th}}\phi_{\text{th}}(\mathbf{r},t) - q(\mathbf{r},u_{\text{th}},t) \end{aligned} \tag{9.30}$$

We know from the analysis of the one-velocity bare reactor that the solutions may be written as the product of a space function and an exponential in time. Let us look, then, for solutions to the above equations in the form [see also (6.59) and (6.60)]

$$\left.\begin{aligned} q(\mathbf{r},u,t) &\equiv F(\mathbf{r})\,\mathcal{Q}(u)e^{\mu t} \\ \phi_{\text{th}}(\mathbf{r},t) &\equiv F(\mathbf{r})\,\psi_{\text{th}}e^{\mu t} \end{aligned}\right\} \quad \text{where } \nabla^2 F(\mathbf{r}) + B^2F(\mathbf{r}) = 0 \tag{9.31}$$

and $B^2 = (\pi/\tilde{R})^2$, etc., chosen to meet conditions (9.27). The substitution of Eqs. (9.31) into (9.30) yields

$$\begin{aligned} -\frac{\mu}{v(u)\xi\Sigma_t(u)}\mathcal{Q}(u) &= [\alpha(u)B^2 + \beta(u)]\,\mathcal{Q}(u) + \frac{d}{du}\mathcal{Q}(u) && (a) \\ \mathcal{Q}(0) &= \nu\Sigma_f^{\text{th}}\psi_{\text{th}} && (b) \\ -\frac{\mu}{v_{\text{th}}}\psi_{\text{th}} &= (D_{\text{th}}B^2 + \Sigma_a^{\text{th}})\psi_{\text{th}} - \mathcal{Q}(u_{\text{th}}) && (c) \end{aligned} \tag{9.32}$$

with $\alpha(u)$ and $\beta(u)$ given by (6.58). The solution to (9.32a), which satisfies the condition (9.32b), is seen to be

$$\mathcal{Q}(u) = \nu\Sigma_f^{\text{th}}\psi_{\text{th}} \exp\left\{-\int_0^u \left[\alpha(u')B^2 + \beta(u') + \frac{\mu}{\xi v(u')\,\Sigma_t(u')}\right]du'\right\} \tag{9.33}$$

If we use this result to compute $\mathcal{Q}(u_{\text{th}})$, (9.32$c$) takes the form

$$\begin{aligned} -\frac{\mu\psi_{\text{th}}}{v_{\text{th}}} &= (D_{\text{th}}B^2 + \Sigma_a^{\text{th}})\psi_{\text{th}} \\ &\quad - \nu\Sigma_f^{\text{th}}\psi_{\text{th}} \exp\left\{-\int_0^u \left[\alpha(u')B^2 + \beta(u') + \frac{\mu}{\xi v(u')\,\Sigma_t(u')}\right]du'\right\} \end{aligned} \tag{9.34}$$

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This result may be simplified if we introduce the following identifications[1] [cf. Eq. (9.18)]:

$$l_F = \int_0^{u_{th}} \frac{du}{\xi v(u)\,\Sigma_t(u)} \qquad l_{th} \equiv \frac{1}{v_{th}(\Sigma_a^{th} + D_{th}B^2)} \tag{9.35}\star$$

and recognize that the first term of the exponential is simply g_{th}, the fast nonleakage probability to thermal, and that the second term is p_{th}, the resonance-escape probability to thermal. Thus, (9.34) may be written

$$\frac{ke^{-\mu l_F}}{1 + \mu l_{th}} = 1 \tag{9.36}\star$$

where we have defined

$$k \equiv \frac{\nu\Sigma_f^{th} p_{th} g_{th}}{\Sigma_a^{th} + D_{th}B^2} \tag{9.37}$$

Equation (9.36) may be regarded as the defining relation for μ and must be satisfied if the forms (9.31) are to be solutions to the set (9.30).

It should be borne in mind that the solution obtained in this way applies only asymptotically, that is, at times long after the initial perturbation has occurred at $t = 0$; however, the result given may be a good approximation very near $t = 0$. It is evident that a complete solution has not been developed since (9.33) does not satisfy the initial condition on the reactor. Thus, if the nuclear properties of the reactor at steady state (for $t < 0$) are denoted by the superscript (0), then we find that (9.33) does not satisfy the expression for the slowing-down density prior to the perturbation:

$$q(\mathbf{r},u,0) = \nu\Sigma_f^{(0)}\psi_{th}^{(0)}F(\mathbf{r}) \exp\left[- \int_0^u \frac{B^2D^{(0)}(u') + \Sigma_a^{(0)}(u')}{\xi\Sigma_t^{(0)}(u')}\,du' \right] \tag{9.38}$$

We may conclude, therefore, that the solutions (9.31) describe only the long-term behavior of the system and omit entirely the transients which arise immediately following the introduction of the changes in the cross sections.

The separability assumed in Eqs. (9.31) is not rigorously valid. For example, the number of neutrons slowing down into the thermal group will not change appreciably until a time of the order of l_F after the change is made in the thermal cross sections; thus $q(\mathbf{r},u_{th},t)$ cannot rise as $e^{\mu t}$. The problem treated in this section could be as well represented on the basis of the time-dependent thermal equation (9.30) with the source term taken as

$$q(\mathbf{r},u_{th},t) = \nu\Sigma_f^{th}(t - l_F)\,\phi_{th}(\mathbf{r},t - l_F)p_{th}e^{-B^2\tau_{th}}$$

Solutions may be obtained for the thermal flux as a function of time

[1] The expression for l_F is only an approximation. The proper definition is given in Eq. (9.40).

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under the conditions (1) through (6) of this section using this form of the source function; these will show different analytic forms in each time interval of length l_F after $t = 0$. Asymptotically, the exponential rise of Eqs. (9.31) may be observed with a time constant $1/\mu$ to be obtained from an equation very similar to (9.36). More general problems may be treated by observing that Eq. (9.1) implies a delay time in the fast range, i.e., the slowing-down density at time t and lethargy u is to be obtained by repeated application of the form

$$q(u,t) = q(u - \Delta u, t - \Delta t) - q(u - \Delta u, t - \Delta t) \frac{\Sigma_a(u,t)}{\xi \Sigma_s(u,t)} \Delta u$$

if Δt is small and related to Δu by

$$\Delta u = \xi \Sigma_s(u,t)\, v(u)\, \Delta t$$

It should be further noted that an important part of the actual physics of the time-dependent behavior in the fast range is not included in the basic equations of this section, (9.1) through (9.4). Neutrons introduced at a particular time at a lethargy u will slow down past a higher lethargy $(u + \Delta u)$ at *different* subsequent times. Equation (9.1) actually implies a continuous-slowing-down rate, representing the average behavior; the spreading out in energy and time cannot be handled on this basis.

The expression for the life of a thermal neutron l_{th} given in (9.35) is precisely the relation obtained previously in (9.18). The function l_F, on the other hand, is an entirely new quantity and needs to be examined in detail. It is evident from the form of this function that l_F may be identified as the *slowing-down time* for fast neutrons. This identification is easily demonstrated by making a direct calculation of the time required by a neutron to slow down through a lethargy interval Δu; thus [cf. Eq. (4.47)]

$$\begin{bmatrix}\text{Average time}\\ \text{to slow down}\\ \text{through } \Delta u\end{bmatrix} = \begin{bmatrix}\text{average number of}\\ \text{collisions needed to}\\ \text{slow down through } \Delta u\end{bmatrix} \begin{bmatrix}\text{average time between}\\ \text{collisions at lethargies}\\ \text{around the interval } \Delta u\end{bmatrix}$$
$$\equiv \frac{\Delta u}{\xi} \left[\frac{\lambda_s(u)}{v(u)}\right] \tag{9.39}$$

To compute the time required to slow down from $u = 0$ to $u = u_{th}$ we need only to sum the slowing-down times through all the intervals along the way. Then, if we call l_F the average time required to slow down to thermal, we have

$$l_F = \int_0^{u_{th}} \frac{du}{\xi v(u)\, \Sigma_s(u)} \tag{9.40}$$

which is approximately the relation given in (9.35), if $\Sigma_s(u) \simeq \Sigma_t(u)$.

As a simple illustration of the use of the general result (9.36), consider

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the case when $\mu l_F \ll 1$. Equation (9.36) may then be written in the approximate form

$$\frac{k}{1 + \mu(l_{th} + l_F)} = 1 \qquad \text{where } \mu = \frac{\delta k}{l_{th} + l_F} \tag{9.41}$$

so that, for example, the time behavior of the thermal flux is given by

$$\phi_{th}(\mathbf{r},t) = F(\mathbf{r})\psi_{th} \exp\left(\frac{\delta k\, t}{l_{th} + l_F}\right) \tag{9.42}$$

In comparison to (9.22) it may be seen that in this approximation the effect of including the time-dependent behavior in the fast range is to add the average slowing-down time to the average thermal lifetime.

d. One-velocity Reflected Reactor. The computational difficulties encountered in determining the time behavior of neutrons in multiregion media were mentioned previously, and it was pointed out that problems of this type require the use of numerical (multigroup) methods. This very general approach is not treated here; we consider instead a simple description of a reflected reactor based on the one-velocity model which will display some of the essential physical features of the general problem.

The reactor to be examined consists of a multiplying core surrounded by a single nonmultiplying reflector. It is assumed that this system is critical with the cross sections and neutron parameters listed below:

Core	*Reflector*
Diffusion coefficients: $D_C^{(0)}$	$D_R^{(0)}$
Cross sections: $\Sigma_f^{(0)}, \Sigma_a^{C(0)}$	$\Sigma_a^{R(0)}$
Buckling: $B_0^2 \equiv \frac{1}{D_C^{(0)}}[\nu\Sigma_f^{(0)} - \Sigma_a^{C(0)}]$	Diffusion length: $L_0^2 \equiv \frac{1}{\kappa_0^2} \equiv \frac{D_R^{(0)}}{\Sigma_a^{R(0)}}$

Now suppose that an amount of absorption $\delta\Sigma_a$ is added uniformly throughout the core and reflector at time $t = 0$; the one-velocity flux $\phi(\mathbf{r},t)$ must now satisfy the following time-dependent differential equations:

Core:

$$-D_C^{(0)}\nabla^2\phi_C(\mathbf{r},t) + [\Sigma_a^{C(0)} + \delta\Sigma_a]\phi_C(\mathbf{r},t) = -\frac{1}{v}\frac{\partial}{\partial t}\phi_C(\mathbf{r},t) + \nu\Sigma_f^{(0)}\phi_C(\mathbf{r},t)$$

Reflector: (9.43)

$$-D_R^{(0)}\nabla^2\phi_R(\mathbf{r},t) + [\Sigma_a^{R(0)} + \delta\Sigma_a]\phi_R(\mathbf{r},t) = -\frac{1}{v}\frac{\partial}{\partial t}\phi_R(\mathbf{r},t)$$

and the boundary conditions

(1) $\phi(\mathbf{r},t)$ is nonsingular (2) $\phi_C(\mathbf{R},t) = \phi_R(\mathbf{R},t)$
(3) $D_C^{(0)}\nabla\phi_C(\mathbf{R},t) = D_R^{(0)}\nabla\phi_R(\mathbf{R},t)$ (4) $\phi_R(\tilde{\mathbf{R}}',t) = 0$ (9.44)

where $\mathbf{R}$ denotes the core-reflector interface and $\tilde{\mathbf{R}}'$ the extrapolated outer boundary of the reflector.

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As solutions to Eqs. (9.43) try

$$\phi_C(\mathbf{r},t) \equiv F(\mathbf{r})e^{-v\,\delta\Sigma_a t} \qquad \phi_R(\mathbf{r},t) \equiv G(\mathbf{r})e^{-v\,\delta\Sigma_a t} \tag{9.45}$$

where

$$\nabla^2 F(\mathbf{r}) + B_0^2 F(\mathbf{r}) = 0 \qquad \nabla^2 G(\mathbf{r}) - \varkappa_0^2 G(\mathbf{r}) = 0 \tag{9.46}$$

and

$$\begin{array}{ll} (1)\ F \text{ and } G \text{ are nonsingular} & (2)\ F(\mathbf{R}) = G(\mathbf{R}) \\ (3)\ D_C^{(0)}\nabla F(\mathbf{R}) = D_R^{(0)}\nabla G(\mathbf{R}) & (4)\ G(\tilde{\mathbf{R}}') = 0 \end{array} \tag{9.47}$$

Thus all the conditions (9.44) are met, and $\phi_C(\mathbf{r},t)$ and $\phi_R(\mathbf{r},t)$ reduce to the proper space forms at $t = 0$. Direct substitution of (9.45) into (9.43) shows that the assumed solutions for the fluxes satisfy the required differential equations.

We determine next the value of ν_c which would be required to maintain the system in the steady state after the extra absorption has been added, i.e., the value of ν_c which satisfies the following differential equation and boundary conditions:

$$\begin{aligned} -D_C^{(0)}\nabla^2\phi_C(\mathbf{r}) + [\Sigma_a^{C(0)} + \delta\Sigma_a]\phi_C(\mathbf{r}) &= \nu_c\Sigma_f^{(0)}\phi_C(\mathbf{r}) \\ -D_R^{(0)}\nabla^2\phi_R(\mathbf{r}) + [\Sigma_a^{R(0)} + \delta\Sigma_a]\phi_R(\mathbf{r}) &= 0 \end{aligned} \tag{9.48}$$

$$\begin{array}{ll} (1)\ \phi(\mathbf{r}) \text{ is nonsingular} & (2)\ \phi_C(\mathbf{R}) = \phi_R(\mathbf{R}) \\ (3)\ D_C^{(0)}\nabla\phi_C(\mathbf{R}) = D_R^{(0)}\nabla\phi_R(\mathbf{R}) & (4)\ \phi_R(\tilde{\mathbf{R}}') = 0 \end{array} \tag{9.49}$$

Equations (9.48) and (9.49) lead to the criticality condition which will be of the type given in (8.11) for a spherical reflected reactor

$$BR\cot BR - 1 = -\frac{D_R^{(0)}}{D_C^{(0)}}[\varkappa R\coth\varkappa(\tilde{R}' - R) - 1] \tag{9.50}$$

where

$$B^2 \equiv \frac{1}{D_C^{(0)}}[\nu_c\Sigma_f^{(0)} - \Sigma_a^{C(0)} - \delta\Sigma_a] \qquad \varkappa^2 \equiv \frac{1}{D_R^{(0)}}[\Sigma_a^{R(0)} + \delta\Sigma_a] \tag{9.51}$$

Thus, for known reactor dimensions and original core and reflector cross sections, and a given value of $\delta\Sigma_a$, a value of ν_c may be computed from (9.50) using (9.51). Then $\delta\Sigma_a$ may be related to δk, which is given by

$$\delta k = \frac{\nu}{\nu_c} - 1 \tag{9.52}$$

According to Eqs. (9.45) (which are now seen to be acceptable solutions to the problem), the flux throughout the reactor in this special case falls exponentially, preserving the space form of the flux which existed during the steady-state operation (for $t < 0$).

9.2 Reactor Parameters by Pulsed Neutron Beam

a. The Experiment. The measurement of the fundamental reactor parameters such as the diffusion coefficient and the diffusion length, as

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well as the determination of the critical size and the fast nonleakage probabilities for a specific reactor geometry and internal arrangement of nuclear components, is an essential step in the design and development of every basically new system. Unfortunately this phase of a project is invariably extremely expensive and time consuming. This is especially true in the case of the experimental verification of the critical size estimates of a complex heterogeneous structure. Measurements of this type are customarily carried out with the aid of exponential-pile and criticality experiments. The purpose of the exponential-pile experiment is the determination of the critical fuel concentration for a specified arrangement of fuel and moderator from measurements of the thermal-flux distributions in a subcritical assembly of the actual system. Since such a system cannot support a steady-state neutron population on the basis of its fission sources alone, a steady-state distribution is achieved by adding a steady external neutron source. The criticality experiment, on the other hand, utilizes a test system which is designed and operated as a critical assembly. The desired nuclear information (such as the critical fuel concentration) is obtained by direct measurement and test. As a rule, the test system is a crude mock-up which serves primarily to simulate the essential nuclear features and geometry of the actual reactor. Thus many important engineering features of the reactor may be omitted, but from the viewpoint of the neutrons, the test system is nearly indistinguishable from the reactor. Even then, the complete experiment can be quite costly in time and facilities.

It is clear that although the exponential pile and criticality experiments would be required ultimately in any large-scale reactor development program, it would be desirable to obtain some preliminary experimental verification of reactor calculations by means of other more modest tests. One experiment which appears to be eminently suited to this purpose is based on the pulsed neutron-beam technique. This technique has been applied by several investigators to the determination of the thermal-diffusion coefficient and macroscopic absorption cross sections of reactor materials.[1–4] More recently, it has been used by E. C. Campbell and P. H. Stelson in the study of short-lived isomers and for the measurement of reactor parameters of multiplying media.[5] The experiment con-

[1] G. von Dardel and N. G. Siöstrand, *Phys. Rev.*, **96**, 1245 (1954).

[2] F. R. Scott, D. B. Thomson, and W. Wright, *Phys. Rev.*, **95**, 582 (1954).

[3] A. V. Antonov et al., "A Study of Neutron Diffusion in Beryllium, Graphite and Water by the Impulse Method," *Paper* 661, *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, 1955, vol. 5, pp. 3–12.

[4] R. Ramanna et al., "On the Determination of Diffusion and Slowing-down Constants of Ordinary Water and Beryllium Oxide Using a Pulsed Neutron Source," *Paper* 872, *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, 1955, vol. 5, pp. 24–27.

[5] Oak Ridge National Laboratory, Physics Division, Semi-annual Progress Report for Period Ending Mar. 10, 1956, ORNL Report 2076, pp. 37–39.

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sists essentially of irradiating a sample of the reactor material with a very short burst of neutrons and measuring the decay constant of the fundamental mode excited in the specimen. The reactor parameters of interest may then be deduced from a knowledge of the dependence of the decay constant on the shape and size of the specimen (i.e., the buckling). This experiment is especially useful for determining the thermal-neutron properties of the material. However, it is not necessary in this case to provide a pulsed beam of thermal neutrons. In fact, it would be preferable to expose the specimen to a burst of fast neutrons since in this way it

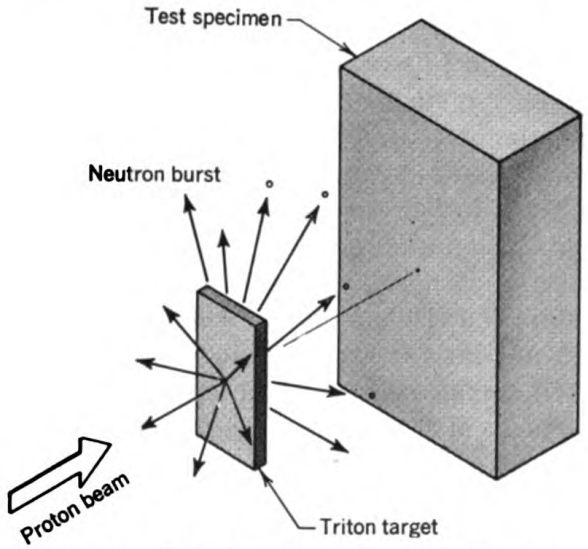

FIG. 9.1 Pulsed neutron-beam experiment.

would be possible to obtain also some information about the slowing-down properties of the system in question. The pulsed beam of fast neutrons can be obtained by means of certain charged-particle reactions. Two useful examples are the deuteron-deuteron and proton-triton reactions:[1]

$$\begin{aligned} {}_1H^2 + {}_1H^2 &\rightarrow {}_2He^3 + {}_0n^1 \\ {}_1H^2 + {}_1H^3 &\rightarrow {}_2He^4 + {}_0n^1 \end{aligned} \tag{9.53}$$

Briefly, the experimental procedure consists of bombarding a target of, say, triton with a beam of protons from an accelerator (see Fig. 9.1). The ensuing reaction, as indicated in the above equation, produces high-energy neutrons (in the order of 1 Mev). If the proton beam is applied intermittently, the target will yield bursts of fast neutrons, and some of these in turn will produce neutron-flux transients in a test specimen placed in the vicinity of the target. The time variations in neutron density at various stations within the specimen may then be measured

[1] See also E. Fermi, "Nuclear Physics," rev. ed., pp. 180–181, course notes compiled by Jay Orear, A. H. Rosenfeld, and R. A. Schluter, University of Chicago Press, Chicago, 1950.

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by means of suitable counting devices, and from these one obtains the decay rates of the spatial modes of the neutron-flux distribution. The determination of the various reactor parameters from a knowledge of the decay constant for the fundamental mode is demonstrated in the work which follows.

b. Flux Transients in Subcritical Multiplying Media Irradiated by Neutron Burst. As the first step in the analysis of the pulsed neutron-beam experiment we derive the expressions for the neutron flux in the specimen as a general function of space and time. Since our primary purpose is to obtain the relationship between the decay constant of the fundamental mode and the reactor parameters, we will assume that certain preliminary information about the neutron slowing-down process in the specimen is already available. In this way we can focus attention on the study of the space and time distribution of the thermal neutrons in the specimen. For this purpose we introduce the following model of the neutron processes in the system: (1) A short burst of fast neutrons occurs either outside or within the test specimen at $t = 0$. (2) These fast neutrons give rise, ultimately, to a thermal-flux distribution ϕ_0. (3) It is assumed that the flight time from the fast-neutron source to the specimen is negligibly small, and (4) that the slowing-down time within the specimen may also be neglected so that the distribution ϕ_0 appears effectively at $t = 0$. (5) The leakage and capture of fast neutrons during slowing down is taken into account, but fast fissions are ignored.

Thus our immediate problem is to determine the thermal-flux distribution $\phi(\mathbf{r},t)$ in an arbitrary geometry, given the distribution $\phi_0(\mathbf{r})$ at $t = 0$. For simplicity, both analytical and experimental, we assume that the specimen is a bare homogeneous system and of some elementary shape such as a parallelepiped, cylinder, or sphere. The differential equation for such a system which contains fissionable material is given by [cf. Eq. (9.10)]

$$-D\nabla^2\phi(\mathbf{r},t) + \Sigma_a\phi(\mathbf{r},t) = \nu\Sigma_f p_{\text{th}} g_* \phi(\mathbf{r},t) - \frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},t) \tag{9.54}$$

where g_* denotes the fast nonleakage probability. The solution to this equation must satisfy the conditions

$$\phi(\tilde{\mathbf{R}},t) = 0 \qquad \phi(\mathbf{r},0) \equiv \phi_0(\mathbf{r}) \tag{9.55}$$

where $\tilde{\mathbf{R}}$ represents the extrapolated outer surface of the specimen. It is convenient to write (9.54) in the form

$$\nabla^2\phi(\mathbf{r},t) + B^2\phi(\mathbf{r},t) = \frac{1}{Dv}\frac{\partial}{\partial t}\phi(\mathbf{r},t) \tag{9.56}$$

with

$$B^2 \equiv \frac{1}{D}(\nu\Sigma_f p_{\text{th}} g_* - \Sigma_a) = \frac{\nu\Sigma_f p_{\text{th}} g_*}{D} - \varkappa^2 \tag{9.57}$$

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The solution may be obtained by the separation of variables. Thus take $\phi(\mathbf{r},t) \equiv F(\mathbf{r})\,G(t)$. Equation (9.57) may then be written

$$\frac{\nabla^2 F}{F} + B^2 = \frac{1}{Dv}\frac{G'}{G} \equiv -\rho^2 \tag{9.58}$$

where ρ^2 is some constant. The solution is given by

$$\begin{aligned} G(t) &= G(0)e^{-Dv\rho^2 t} && (a) \\ \nabla^2 F(\mathbf{r}) + (B^2 + \rho^2)F(\mathbf{r}) &= 0 && (b) \end{aligned} \tag{9.59}$$

The solutions to the wave equation (9.59*b*) were developed previously in Secs. 5.4a, b, and e. For our present purposes it is convenient to write these solutions in the form

$$\nabla^2 F_{mns} + \alpha_{mns}^2 F_{mns} = 0 \qquad \text{with } F_{mns}(\tilde{\mathbf{R}}) = 0 \tag{9.60}$$

where the F_{mns} are the various eigenfunctions (modes) of (9.59*b*), and the α_{mns} are the corresponding eigenvalues. If the results of Sec. 5.4a are extended to the case of the parallelepiped, it is easy to show that

$$F_{mns}(\mathbf{r}) = \cos \alpha_n x \cos \beta_m y \cos \gamma_s z \tag{9.61}$$

where

$$\alpha_n \equiv \frac{\pi}{2\tilde{a}}(2n+1) \qquad \beta_m \equiv \frac{\pi}{2\tilde{b}}(2m+1) \qquad \gamma_s \equiv \frac{\pi}{2\tilde{c}}(2s+1) \tag{9.62}$$

and $2a$, $2b$, and $2c$ are the x, y, and z dimensions, respectively, of the medium. Note that in this case

$$\alpha_{mns}^2 \equiv \alpha_n^2 + \beta_m^2 + \gamma_s^2 \tag{9.63}$$

and in general, ρ^2 takes on the values

$$\rho^2 \to \rho_{mns}^2 \equiv \alpha_{mns}^2 - B^2 \tag{9.64}$$

The general solution to (9.56) is obtained by summing the contributions of all the modes [cf. Eq. (5.141)]. If we call

$$\phi_{mns}(\mathbf{r},t) \equiv G(0)\,F_{mns}(\mathbf{r})e^{-Dv(\alpha^2_{mns}-B^2)t} \tag{9.65}$$

then ϕ is given by the triply infinite series (m, n, and $s = 0, 1, \ldots$)

$$\phi(\mathbf{r},t) = \sum_{m,n,s} a_{mns}\phi_{mns}(\mathbf{r},t) \tag{9.66}$$

where $G(0)$ has been absorbed into the arbitrary coefficients a_{mns}. These are of course obtained from the initial condition. Thus, from (9.55)

$$\phi(\mathbf{r},0) \equiv \phi_0(\mathbf{r}) = \sum_{m,n,s} a_{mns}F_{mns}(\mathbf{r}) \tag{9.67}$$

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In the usual way, we compute

$$a_{mns} = \int_V \phi_0(\mathbf{r})\, F_{mns}(\mathbf{r})\, d\mathbf{r} \tag{9.68}$$

the integration to extend over the entire volume of the specimen. Finally, we can write the general solution in the form

$$\phi(\mathbf{r},t) = \sum_{m,n,s} \exp\left[-Dv\left(\alpha_{mns}^2 + \varkappa^2 - \frac{\nu\Sigma_f p_{\text{th}} g_*}{D}\right) t\right] \times F_{mns}(\mathbf{r}) \int_V \phi_0(\mathbf{r}')\, F_{mns}(\mathbf{r}')\, d\mathbf{r}' \tag{9.69}$$

where we have introduced the definition for B^2 from (9.57).

c. Reactor Parameters from the Decay Constant. At long times after each neutron burst from the target plate, the time behavior of the thermal flux in the specimen is well represented by the fundamental mode $F_{000}(\mathbf{r})$. Thus the expression (9.69) reduces to the form

$$\phi(\mathbf{r},t) \sim a_{000} \exp\left[-(v\Sigma_a + vD\alpha_{000}^2 - v\nu\Sigma_f p_{\text{th}} g_*)t\right] F_{000}(\mathbf{r}) \tag{9.70}$$

The quantity α_{000} is the eigenvalue corresponding to the fundamental mode, and for the case of the parallelepiped is given by [see Eq. (9.63)]

$$\alpha_{000}^2 = \left(\frac{\pi}{2\tilde{a}}\right)^2 + \left(\frac{\pi}{2\tilde{b}}\right)^2 + \left(\frac{\pi}{2\tilde{c}}\right)^2 \tag{9.71}$$

The *decay constant* of the relation (9.70) is the coefficient of t in the exponential factor. We give this the symbol λ_{000}^2:

$$\lambda_{000}^2 \equiv v\Sigma_a + vD\alpha_{000}^2 - v\nu\Sigma_f p_{\text{th}} g_* \tag{9.72}$$

In the case of the multiplying medium it will be convenient to separate the fuel and nonfuel components of the system. Then, we write, in the usual notation,

$$\lambda_{000}^2 = v\Sigma_a^{(O)} + vD\alpha_{000}^2 + v\Sigma_a^{(F)}(1 - \eta p_{\text{th}} g_*) \tag{9.73}\star$$

It is evident from this relation that a knowledge of the decay constant λ_{000} as a function of the *buckling* α_{000} can yield considerable information about the various parameters which appear in (9.73). The principal task of the experimenter, then, is to obtain measurements of λ_{000} for various α_{000}. This can be accomplished, however, only if the higher modes have virtually disappeared so that the relation (9.70) is valid.[1] Thus the first step in making the measurements is to record the neutron density, or some quantity proportional to it, as a function of time. When this function is plotted on semilog graph paper, the curve reduces to a

[1] The interaction due to higher modes can be reduced appreciably if the detector is placed at a nodal point of the first higher mode.

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straight line over that portion of the time scale wherein the approximation (9.70) is satisfied. An example of such a curve is shown in Fig. 9.2. These data were obtained by Campbell and Stelson from measurements performed on a 5-in. cylinder of pure water.[1] The decay of the

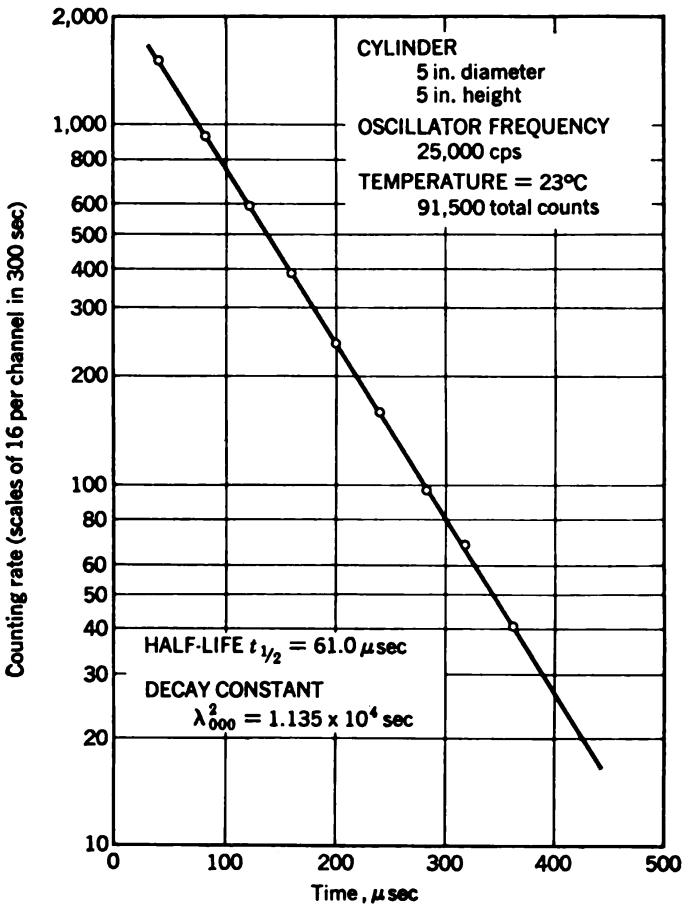

FIG. 9.2 Decay curve of thermal neutrons diffusing out of H_2O cylinder as detected with $Li^6I(Eu)$ scintillation counter.

higher spatial modes mentioned above has also been studied, and a typical set of thermal-flux distributions at various time instances is given in Fig. 9.3. These measurements were made on a rectangular block of beryllium. The curves in the figure show the shapes of the flux distribution at successive 100-μsec time intervals. The presence of the higher modes, which decay more rapidly than the fundamental, can be seen from the gradual shift to the symmetric fundamental mode. From these results it appears that the fundamental mode is established within about one millisecond after the initial burst.

It is evident from the relation (9.70) that the decay constant of the

[1] Oak Ridge National Laboratory, Physics Division, Semi-Annual Progress Report for Period Ending Mar. 10, 1956, ORNL Report 2076, p. 36.

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fundamental mode is simply the slope of the decay curve, and measurements of λ_{000}^2 may be readily obtained from data of the type given in Fig. 9.2. The second and final step in the evaluation of the reactor parameters, then, is the measurement of λ_{000}^2 for various α_{000}^2, and this may be accomplished by determining the decay curves for specimens of

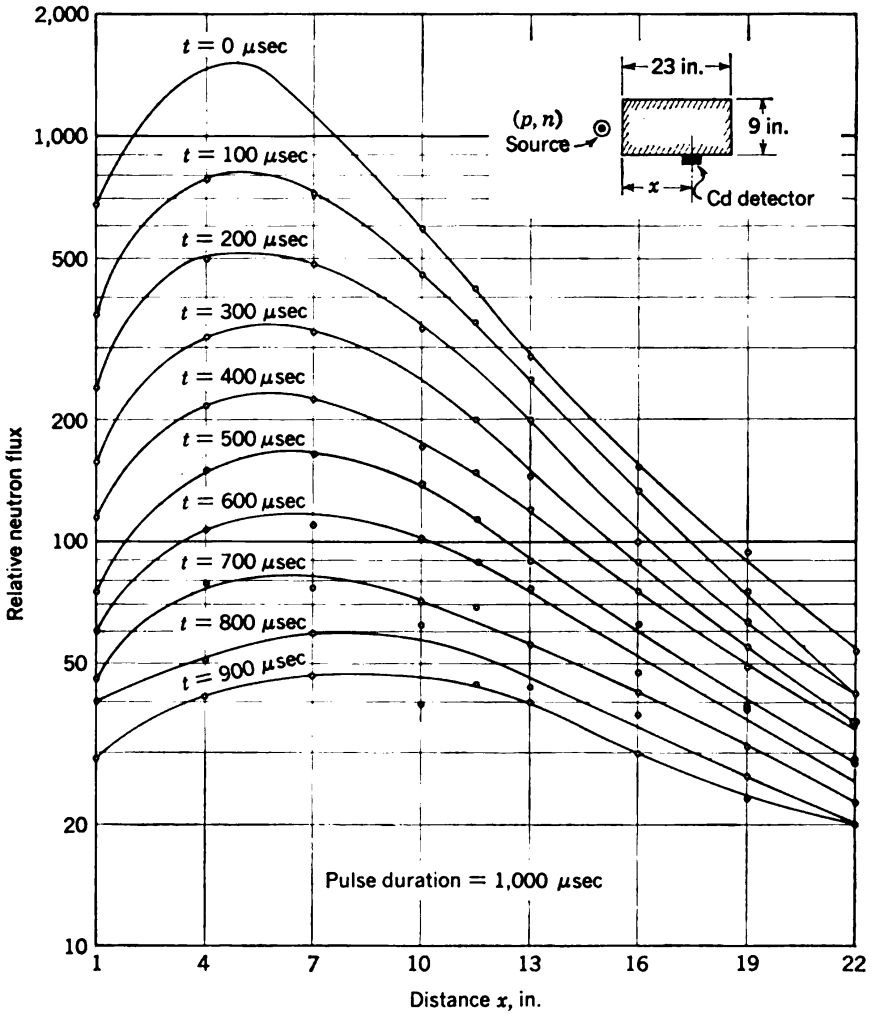

FIG. 9.3 Time-space distribution of thermal neutrons diffusing out of a beryllium block (9 by 11¼ by 23 in.) following 1,000-μsec bursts of fast neutrons.

various size (and/or shape). When λ_{000}^2 is plotted as a function of the buckling α_{000}^2, the resulting curve will be of the form shown in Fig. 9.4. These data are then easily interpreted in terms of the reactor parameters. Consider first the case in which the specimen contains no fissionable material:

(1) Nonmultiplying media: $\Sigma_a^{(F)} \equiv 0$

$$\lambda_{000}^2 = v[\Sigma_a^{(O)} + D\alpha_{000}^2] \tag{9.74}$$

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which follows immediately from (9.73). This function is shown in the figure. It is clear that the $\alpha_{000}^2 = 0$ intercept yields the absorption cross section of the material of the specimen, and the slope of the curve, the diffusion coefficient.

When the specimen contains fissionable material, the λ_{000}^2 curve is no longer a linear function of α_{000}^2, but has the general shape indicated in the figure. A specific case of this type is represented in Fig. 9.5, which

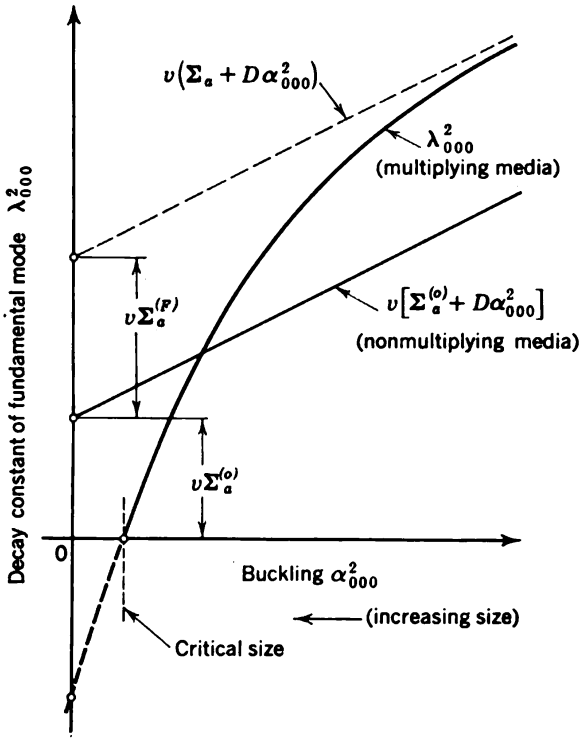

FIG. 9.4 Decay constant of thermal neutrons in a multiplying medium.

presents the decay constant for thermal neutrons in a cylinder of U^{235}-H_2O solution.[1] The interpretation of the results for the general multiplying medium is as follows. Consider the case when

$$(2)\ \lambda_{000}^2 \equiv 0 \tag{9.75}$$

This case obviously corresponds to the critical system, since the time-dependent function in the expression (9.70) now vanishes. Thus a second important fact to be obtained from the pulsed neutron-beam experiment is the critical dimensions of a given geometry. Another fact to be learned from the experiment is the diffusion coefficient of the mixture and the absorption cross section of the fuel, since for small specimens

$$(3)\ \alpha_{000}^2 \to \infty \qquad \lambda_{000}^2 \sim v(\Sigma_a + D\alpha_{000}^2) \tag{9.76}$$

[1] From the work of Campbell and Stelson reported in ORNL Report 2076, p. 38.

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This function is represented by the broken-line curve in Fig. 9.4. Note that the $\lambda_{000}^2 = 0$ intercept yields the total absorption cross section of the mixture, and the slope of the curve, as before, the diffusion coefficient. With these results for D, $\Sigma_a^{(F)}$, and $\Sigma_a^{(O)}$ the g_* corresponding to each value

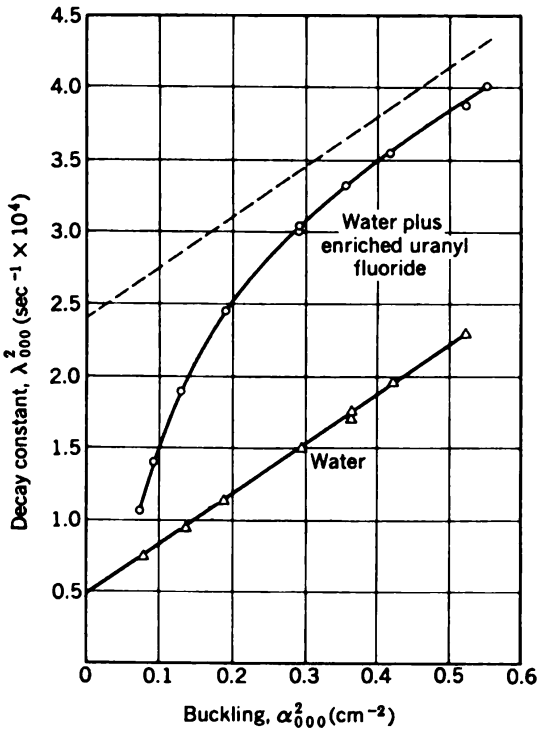

FIG. 9.5 Decay constant of thermal neutrons diffusing out of cylinders of U^{235}-H_2O solution as function of buckling.

of α_{000} is now obtained directly from the known values of λ_{000}^2. Thus for the intermediate range,

(4) $(\alpha_{000})_{\text{critical}} \leq \alpha_{000} < \infty$
λ_{000}^2 yields the product $p_{\text{th}} g_*$ (9.77)

If the resonance-escape probability has been obtained by some other means, then a knowledge of λ_{000}^2 yields g_*. It is interesting to note that, for very large specimens, the value of g_* obtained in this way should agree well with the Fermi age expression for the fast nonleakage probability, that is,

(5) For small α_{000}

$$g_* \simeq g_{\text{th}} \equiv e^{-\alpha_{000}^2 \tau_{\text{th}}} \tag{9.78}$$

It is perhaps evident from these observations that the relatively modest and inexpensive pulsed neutron-beam experiment can yield a great deal of information about a particular reactor material. The outstanding

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feature of this experiment is, of course, the fact that this wealth of data may be obtained without the use of a complete reactor facility.

9.3 Effect of Delayed Neutrons

a. Delayed-neutron Groups. Most of the neutrons produced by the fission process appear, effectively, at the instant of fission. However, a small fraction of the neutrons produced in fission is emitted by daughter nuclei from the primary fission fragments when these nuclei subsequently undergo beta decay. Neutrons produced by this mechanism do not appear at the instant of fission but at some later time. The delay period is principally determined by the decay constant of the preceding beta emission, since the neutron is produced immediately after the beta decay. These tardy neutrons are conveniently classified by groups, each group being defined by the characteristic delay time between the instant of fission and the appearance of the neutrons. Tables 9.1 and 9.2 list some of the properties of the principal delayed-neutron groups for various fissionable nuclei.[1]

The presence of the delayed neutrons is not very important to the steady-state operation of a stationary-fuel reactor. In such a reactor at any time instant the beta emitters are located essentially at the point of the original fissions, and these nuclei are present in concentrations which reflect the rate of fissioning at some previous time. However, since the flux and fuel concentrations are constant in time (over times of the order of 100 sec or longer), the number of delayed neutrons being produced per unit time at any instant is given by

$$\begin{matrix}\text{Number of delayed neutrons}\\ \text{produced per unit time per}\\ \text{unit volume in the steady state}\end{matrix} = \sum_{i=1}^{N} \lambda_i C_i(\mathbf{r}) \tag{9.79}$$

where $C_i(\mathbf{r}) \equiv$ concentration of beta emitters of the ith type which are precursors of delayed-neutron emitters
$\lambda_i \equiv$ decay constant for the neutron emitter of the ith type
$N \equiv$ total number of delayed groups

In the steady state each emitter which decays must, on the average, be replaced by an emitter of the same type from fission; that is,

$$\lambda_i C_i(\mathbf{r}) = \mathfrak{f}_i \Sigma_f \phi(\mathbf{r}) \tag{9.80}$$

where $\mathfrak{f}_i$ is the fraction of fissions which yield precursors of the ith type. Then the total number of neutrons (prompt and delayed) produced per

[1] See also G. R. Keepin, "Delayed Neutrons," in R. A. Charpie et al. (eds.), Progress in Nuclear Energy, Series I, Physics and Mathematics, vol. 1, pp. 191–225, Pergamon Press, New York, 1956.

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TABLE 9.1 DELAYED NEUTRONS FROM FISSION OF U^{235}, U^{233}, AND Pu^{239}

Group i	Mean[a] energy, Mev	U^{235}				U^{233}		Pu^{239}			
		$\bar{E}^{b}_{neut}$ = thermal[c]		$\bar{E}_{neut}$ = fission spectrum[d]		$\bar{E}_{neut}$ = thermal[e]		$\bar{E}_{neut}$ = thermal[f]		$\bar{E}_{neut}$ = fission spectrum[d]	
		Half-life, sec	β_i	Half-life, sec	Neutrons per fission	Half-life, sec	β_i	Half-life, sec	β_i	Half-life, sec	Neutrons per fission
1	0.250	55.6	0.00025	54.3	0.00062	55.6	0.00023	55.0	0.00070	53.7	0.00025
2	0.560	22.0	0.00166	21.7	0.00363	22.0	0.00073	24.4	0.00683	22.9	0.00178
3	0.430	4.51	0.00213	5.87	0.00332	4.51	0.00108	4.5	0.00748	6.11	0.00129
4	0.620	1.52	0.00241	2.18	0.00708	1.52	0.00078	1.05	0.00822	2.14	0.00253
5	0.420	0.43	0.00085	0.46	0.00233	0.43	0.00026	.....	.....	0.40	0.0008
6	.....	0.05	0.00025	0.13	0.00031	0.05	.....	.....	.....	0.15	
Total....	.....	.....	0.00755	.....	0.0173	.....	0.00307	.....	0.0232	.....	0.0067

[a] Based on U^{235} measurements by D. J. Hughes, J. Dabbs, A. Cahn, and D. Hall, *Phys. Rev.*, **73**, 111(1948).

[b] $\bar{E}_{neut}$ denotes average energy of neutrons producing fission.

[c] From Hughes, Dabbs, Cahn, and Hall, *op. cit.*

[d] G. R. Keepin and T. F. Wimett, Delayed Neutrons, *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, 1955, vol. 4, P/831, pp. 162–170, 1956.

[e] A. Cahn, J. Dabbs, and D. J. Hughes, "Delayed Neutrons from Fission of U^{233}," Declassified Document CP-3147, 1945; the β_i/β values are from this report and the value of β from Keepin and Wimett, *op. cit.*, using $\nu = 2.54$ (delayed neutrons per fission $= \nu\beta$).

[f] W. Redman and D. Saxon, *Phys. Rev.*, **72**, 570 (1947).

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unit time is

$$\text{Total neutrons produced per unit volume per unit time} = \sum_{i=1}^{N} \lambda_i C_i(\mathbf{r}) + (1 - \beta)\nu\Sigma_f\phi(\mathbf{r}) \quad (9.81)$$

where β is the fraction of neutrons from fission which are delayed:

$$\beta = \sum_{i=1}^{N} \beta_i$$

and β_i is the fraction of neutrons from fission which are produced by the

TABLE 9.2 DELAYED NEUTRONS[a] FROM FAST FISSIONS[b] OF Th^{232} AND U^{238}

Group i	Th^{232}		U^{238}	
	Half-life, sec	Neutrons per fission	Half-life, sec	Neutrons per fission
1	54.0	0.00208	53.0	0.00035
2	22.0	0.00838	22.0	0.00445
3	6.2	0.01083	5.30	0.00695
4	2.1	0.02890	2.00	0.01796
5	0.52	0.01068	0.51	0.01048
6	0.18	0.00221	0.18	0.00383
Total....	.....	0.063	.....	0.044

[a] G. R. Keepin and T. F. Wimett, "Delayed Neutrons," *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, 1955, vol. 4, P/831, pp. 162–170, 1956.

[b] Fissions produced by neutrons from fission spectrum.

ith precursor. The substitution of (9.80) into (9.81) yields, for the steady state,

$$\begin{aligned}\text{Total neutrons produced per unit volume per unit time} &= \sum_{i=1}^{N} \mathfrak{f}_i\Sigma_f\phi(\mathbf{r}) + (1 - \beta)\nu\Sigma_f\phi(\mathbf{r}) \\ &= \nu\Sigma_f\phi(\mathbf{r})\left[\frac{1}{\nu}\sum_{i=1}^{N} \mathfrak{f}_i + (1 - \beta)\right] = \nu\Sigma_f\phi(\mathbf{r})\end{aligned} \quad (9.82)$$

The last step in the equation follows from the fact that each precursor decay yields one neutron, so that

$$\frac{1}{\nu}\sum_{i=1}^{N} \mathfrak{f}_i = \frac{\mathfrak{f}}{\nu} = \beta \quad (9.83)$$

We may conclude from this calculation that in the steady state the total number of neutrons produced from fission is not altered by the dis-

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tribution of the delayed neutrons; however, it is true that the energy spectrum of the fission neutrons is dependent upon the properties of the delayed neutrons. Thus, if the average energy of neutrons from precursors is different from the average energy of prompt neutrons, an accurate calculation must take this effect into account. Actually, there is some difference in the average energies of the prompt neutrons and the delays (see Table 9.1), but the difference is not significant from the standpoint of the calculation of the fast leakage and absorptions in a thermal reactor. In the analyses which follow, we shall disregard this effect.

b. Application to One-velocity Bare Reactor. The presence of the delayed neutrons has an important effect in determining the neutron-flux distribution in a nonstationary system. The essential features of this behavior are amply demonstrated by means of the one-velocity bare reactor. In this case the appropriate time-dependent differential equations for the thermal flux and the concentration of precursors are

$$-D\nabla^2\phi(\mathbf{r},t) + \Sigma_a\phi(\mathbf{r},t) = -\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},t) + (1-\beta)\nu\Sigma_f p_{\text{th}} g_{\text{th}}\phi(\mathbf{r},t) + p_{\text{th}} g_{\text{th}} \sum_{i=1}^{N} \lambda_i C_i(\mathbf{r},t) \quad (9.84)\star$$

$$\frac{\partial}{\partial t} C_i(\mathbf{r},t) = -\lambda_i C_i(\mathbf{r},t) + \beta_i \Sigma_f \phi(\mathbf{r},t) \qquad i = 1, 2, \ldots, N \quad (9.85)\star$$

where all the neutron constants which appear refer to the thermal neutrons. The second and third terms on the right side of (9.84) account for, respectively, the source neutrons which are prompt and those which are released from the delayed-neutron precursors. The precursors are represented by the concentrations:

$$C_i(\mathbf{r},t) \equiv \frac{\text{concentration of delayed-neutron precursors}}{\text{of the } i\text{th type at space point } \mathbf{r} \text{ and time } t} \quad (9.86)$$

The quantities p_{th} and g_{th} in (9.84) denote the resonance-escape and fast nonleakage probabilities to thermal in the usual way [(4.260) and (6.79)]. Finally, note that since the reactor described by (9.84) is not critical the multiplication constant will be different from unity and may be obtained from the relation

$$k = \frac{\nu\Sigma_f p_{\text{th}} g_{\text{th}}}{\Sigma_a(1 + L^2B^2)} \qquad \text{with } B^2 = \left(\frac{\pi}{\tilde{R}}\right)^2 \text{ for a sphere, etc.} \quad (9.87)$$

In the system under consideration (a noncirculating-fuel bare thermal reactor), it is physically reasonable to suppose that the spatial variation of the concentration of delayed-neutron precursors is proportional to that of the neutron flux and that this mode persists even though the magni-

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tude of the flux changes in time. Thus we assume

$$\phi(\mathbf{r},t) \equiv F(\mathbf{r})\,\phi(t) \qquad C_i(\mathbf{r},t) \equiv F(\mathbf{r})\,C_i(t) \tag{9.88}$$

where $\nabla^2 F + B^2 F = 0$ and $F(\tilde{\mathbf{R}}) = 0$. The substitution of these relations into (9.84) and (9.85) yields

$$(DB^2 + \Sigma_a)\phi(t) = -\frac{1}{v}\,\phi(t) + (1-\beta)\nu\Sigma_f p_{th} g_{th}\phi(t) + p_{th} g_{th} \sum_{i=1}^{N} \lambda_i C_i(t)$$

$$\dot{C}_i(t) = -\lambda_i C_i(t) + \mathfrak{f}_i \Sigma_f \phi(t) \qquad i = 1, 2, \ldots, N \tag{9.89}$$

This set may be further reduced by introducing the definitions

$$G_i(t) \equiv \frac{C_i(t)}{\nu\Sigma_f} \qquad \text{and} \qquad \mathfrak{f}_i \equiv \nu\beta_i \tag{9.90}$$

along with the relation (9.87); thus

$$-[(1-\beta)k - 1]\phi(t) = -l_{th}\phi(t) + k \sum_{i=1}^{N} \lambda_i G_i(t) \tag{9.91}$$

$$\dot{G}_i(t) = -\lambda_i G_i(t) + \beta_i \phi(t) \qquad i = 1, 2, \ldots, N \tag{9.92}$$

where l_{th} is given by (9.35).

We attempt a solution of Eqs. (9.91) and (9.92) in the form

$$\phi(t) \equiv \sum_{j=1}^{N'} \varphi_j e^{\alpha_j t} \qquad \text{and} \qquad G_i(t) \equiv \sum_{j=1}^{N'} \Gamma_{ij} e^{\alpha_j t} \tag{9.93}$$

The number N' is to be chosen so that these relations are solutions to the differential equations (9.91) and (9.92). The substitution of (9.93) into (9.92) yields

$$\sum_{j=1}^{N'} \Gamma_{ij}(\alpha_j + \lambda_i) e^{\alpha_j t} = \sum_{j=1}^{N'} \varphi_j \beta_i e^{\alpha_j t} \qquad i = 1, 2, \ldots, N \tag{9.94}$$

Since the functions $e^{\alpha_j t}$ are linearly independent, (9.94) must be met by requiring

$$\Gamma_{ij}(\alpha_j + \lambda_i) \equiv \varphi_j \beta_i \tag{9.95}$$

By a similar set of operations the substitution of (9.93) into (9.91) yields, with the aid of (9.95),

$$[(1-\beta)k - 1] - l_{th}\alpha_j = -k \sum_{i=1}^{N} \frac{\lambda_i \beta_i}{\alpha_j + \lambda_i} \qquad j = 1, 2, \ldots, N' \tag{9.96}$$

Thus, Eqs. (9.93) are acceptable solutions of (9.91) and (9.92) provided

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that the quantities α_j are chosen to be the solutions (N' in number) of the following characteristic equation for α:

$$[(1 - \beta)k - 1] - l_{\text{th}}\alpha = -k \sum_{i=1}^{N} \frac{\lambda_i \beta_i}{\alpha + \lambda_i} \tag{9.97}$$

This equation is conveniently written in terms of the reactivity ω previously defined by (6.143):

$$\omega = \frac{l_{\text{th}}\alpha}{1 + l_{\text{th}}\alpha} + \frac{1}{1 + l_{\text{th}}\alpha} \sum_{i=1}^{N} \frac{\alpha \beta_i}{\alpha + \lambda_i} \equiv F(\alpha) \tag{9.98}\star$$

We have used here the definition $\beta \equiv \sum_{i=1}^{N} \beta_i$. Equation (9.98) is a condition on the solutions (9.93); that is, if the indicated functions are to

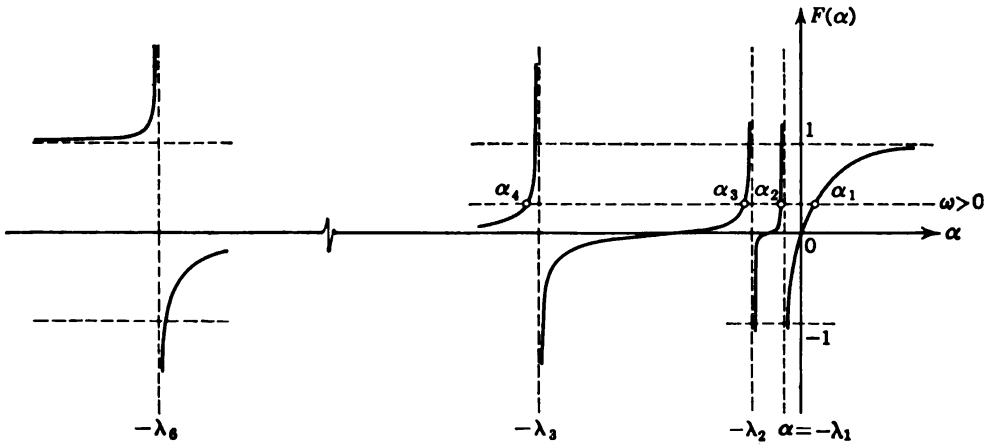

FIG. 9.6 Characteristic function $F(\alpha)$.

satisfy the relations (9.91) and (9.92), then the α_j to be used are the solutions to Eq. (9.98). Thus given the neutron lifetime l_{th} and the delayed-neutron properties λ_i and β_i, (9.98) yields the set α_j. Figure 9.6 shows a sketch of the function $F(\alpha)$, given by the right-hand side of (9.98).

The situation under discussion here is one where, over the period of time in which Eqs. (9.93) apply [and therefore (9.98)], the multiplication constant of the system (thus also the reactivity) is constant in time (and, in general, different from unity). The reactivity is actually a dimensionless number, but on the basis of (9.98) it is sometimes measured in *inhours*. An inhour (a unit specific to a given reactor) is defined as that amount of reactivity which will allow, as a solution to (9.98),

$$\alpha_1 = \frac{1}{3{,}600} \text{ sec}^{-1}$$

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It is easy to verify that the function $F(\alpha)$ has the following properties:

$$\begin{aligned} F(0) &= 0 \qquad & \lim_{\epsilon \to 0} F(-\lambda_i \pm \epsilon) &= \mp \infty \\ \lim_{\alpha \to \pm \infty} F(\alpha) &= 1 \qquad & \lim_{\epsilon \to 0} F\left(-\frac{1}{l_{th}} \pm \epsilon\right) &= \mp \infty \end{aligned} \tag{9.99}$$

From these facts it may be seen that for a given ω (positive or negative) there are $(N + 1)$ solutions for α of Eq. (9.98). Here N is the number of delayed precursor types. If we denote the roots α_j in the order shown in Fig. 9.6 we find that α_1 is positive if ω is positive; if ω is negative, α_1 is also negative, but it is also, then, the smallest root in absolute magnitude; i.e.,

$$|\alpha_1| < |\alpha_2| < \cdots |\alpha_{N+1}| \qquad \text{for } \omega < 0 \tag{9.100}$$

Evidently in this case the roots α_m (where $m = 2, 3, \ldots, N + 1$) are all negative.

In a specific problem the general solutions (9.93) will be required to satisfy certain initial conditions on the system. For example, the reactivity ω may be introduced at $t = 0$, at which instant

$$\phi(0) = \phi_0 \qquad G_i(0) = G_{i_0} \tag{9.101}$$

The problem is to determine the functions $\phi(t)$ and $G_i(t)$ for all subsequent time ($t > 0$). A solution is now possible since the system has been completely specified. Note that the conditions (9.101) along with the Eqs. (9.95) provide a sufficient number of relations from which to compute the coefficients φ_j and Γ_{ij}, since the α_j are now known; thus,

$$\sum_{j=1}^{N+1} \varphi_j = \phi_0 \qquad \Gamma_{ij}(\alpha_j + \lambda_i) = \varphi_j \beta_i$$

with (9.102)

$$\sum_{j=1}^{N+1} \Gamma_{ij} = G_{i_0} \qquad \begin{cases} i = 1, 2, \ldots, N \\ j = 1, 2, \ldots, N, N + 1 \end{cases}$$

give $(N + 1)^2$ equations for $(N + 1)^2$ unknowns.

The time-dependent behavior of the flux is, then, quite complicated in the immediate time interval after the addition of the reactivity ω; however, after a sufficient time the terms of (9.93) involving $e^{\alpha_j t}$ (for $j > 1$) will be negligible in comparison with the first term $e^{\alpha_1 t}$. It follows, therefore, that after some time

$$\phi(t) \sim e^{\alpha_1 t} \quad . \quad G_i(t) \sim e^{\alpha_1 t} \tag{9.103}$$

An approximate solution for the root α_1 may be readily obtained when the reactivity ω is small and positive. In that case we may expand $F(\alpha)$

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in a Taylor series about $\alpha = 0$; thus, for small ω,

$$\omega \simeq F(0) + F'(0)\alpha_1 + \cdots \tag{9.104}$$

It is easily shown that

$$\alpha_1 = \frac{\omega}{l_{th} + \sum_{i=1}^{N} \frac{\beta_i}{\lambda_i}} \tag{9.105}$$

We find, then, that after sufficiently long time the effective reactor period would be

$$T \simeq \left(l_{th} + \sum_{i=1}^{N} \frac{\beta_i}{\lambda_i}\right) \frac{1}{\delta k} \tag{9.106}\star$$

We have used here the approximation $\delta k \simeq \omega$, which is valid for small reactivities. By comparing (9.106) with (9.23), which gives the reactor period in the absence of delayed neutrons, it may be seen that the effect of the delays is to increase the effective mean lifetime of the neutrons. A result similar to (9.106) may be obtained heuristically by calculating an average lifetime for all neutrons; thus,

$$\bar{l}_{th} = (1 - \beta)l_{th} + \beta_1\left(l_{th} + \frac{1}{\lambda_1}\right) + \beta_2\left(l_{th} + \frac{1}{\lambda_2}\right) + \cdots + \beta_N\left(l_{th} + \frac{1}{\lambda_N}\right) = l_{th} + \sum_{i=1}^{N} \frac{\beta_i}{\lambda_i} \tag{9.107}$$

since the prompt neutrons [of which there are $(1 - \beta)$] have lifetime l_{th}, while the delayed neutrons in each group have the lifetime l_{th}, as thermal neutrons plus the average delay time before they appear in the system, that is, $1/\lambda$. Of course, (9.106) holds only for very small reactivity; in general, the form of $1/\alpha_1$ will have a more complicated dependence on the parameters involved. The coefficient in the exponent of the leading term after long time is usually referred to as the reciprocal of the *stable reactor period;* that is,

$$\text{Stable reactor period} \equiv T = \frac{1}{\alpha_1} \tag{9.108}$$

where α_1 is to be obtained as the appropriate solution of (9.98).

c. Time Dependence with One Group of Delayed Neutrons. The complexity of the flux time behavior [as given by (9.93)], which results from the large number of important delay groups, tends to conceal the physical picture of the time-dependent behavior of the reactor. It is possible, however, to display the essential features of the system by introducing a simplified model based on the concept of a single group of delayed

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neutrons. The foregoing results may be applied directly to this model by setting $N \equiv 1$. The properties of this single group are represented by the decay constant λ and the fraction of *all* neutrons delayed by β. These quantities may be chosen to make the results of the calculation as nearly correct quantitatively as possible. For this purpose, it is convenient to choose

$$\beta = 0.00755 \tag{9.109}$$

for U^{235} (see also Table 9.1 for values of β for other fissionable materials). In the present notation λ is given by

$$\frac{\beta}{\lambda} = \sum_{i=1}^{N} \frac{\beta_i}{\lambda_i} \tag{9.110}$$

where β_i and λ_i are the delay fractions and decay constants of the actual delay groups.

The appropriate relations for the one delay-group model are easily obtained from the more general results of Sec. 9.3b. These are

$$\phi(t) = \varphi_1 e^{\alpha_1 t} + \varphi_2 e^{\alpha_2 t} \qquad G(t) = \Gamma_1 e^{\alpha_1 t} + \Gamma_2 e^{\alpha_2 t} \tag{9.111}$$

and

$$\omega = \frac{\alpha}{1 + l_{th}\alpha}\left(l_{th} + \frac{\beta}{\alpha + \lambda}\right) \tag{9.112}$$

If we denote the initial conditions ϕ_0 and G_0, then from (9.102)

$$\begin{aligned} \varphi_1 + \varphi_2 &= \phi_0 & \Gamma_1 + \Gamma_2 &= G_0 \\ \Gamma_1(\alpha_1 + \lambda) &= \varphi_1\beta & \Gamma_2(\alpha_2 + \lambda) &= \varphi_2\beta \end{aligned} \tag{9.113}$$

The roots α_1 and α_2 are readily computed from (9.112), which is now a quadratic equation in α; these are

$$\alpha = -\frac{\omega(1 + l_{th}\lambda) - (\beta + l_{th}\lambda)}{2l_{th}(\omega - 1)} \pm \frac{1}{2}\left\{\left[\frac{\omega(1 + l_{th}\lambda) - (\beta + l_{th}\lambda)}{l_{th}(\omega - 1)}\right]^2 - \frac{4\lambda\omega}{l_{th}(\omega - 1)}\right\}^{\frac{1}{2}} \tag{9.114}$$

where α_1 is identified as the positive root and α_2 as the negative.

The particular solution for the coefficients φ_1, φ_2, Γ_1, and Γ_2 depends upon the initial values of the flux ϕ_0 and the concentration of the single precursor G_0. These quantities, in turn, are dependent upon the state of the reactor prior to $t = 0$. [In any case, the applicability of the present results is always limited to systems in which the separation of time and space functions, as in Eqs. (9.88), may be assumed.] For the special case wherein the reactor was at steady state with flux ϕ_0 prior to $t = 0$, we may obtain a relation between ϕ_0 and G_0 by noting the balance for precursors in the steady-state condition as given by (9.80). In the case of

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a single group of delays, this relation yields

$$\lambda G_0 = \beta \phi_0 \tag{9.115}$$

With this condition the coefficients are found to be, from (9.113),

$$\varphi_1 = \frac{\alpha_2(\alpha_1 + \lambda)\phi_0}{\lambda(\alpha_2 - \alpha_1)} \qquad \varphi_2 = \frac{\alpha_1(\alpha_2 + \lambda)\phi_0}{\lambda(\alpha_1 - \alpha_2)} \tag{9.116}$$

In the case that the reactivity change ω is positive, an appropriate set of approximate expressions for α_1, the reciprocal of the stable reactor period, is found to be

$$\begin{aligned} \alpha_1 &\simeq \frac{\omega}{l_{\text{th}} + \beta/\lambda} && \omega \ll \beta \\ \alpha_1 &\simeq \frac{\omega}{l_{\text{th}}(1 - \omega)} && \omega \gg \beta \end{aligned} \tag{9.117}$$

These results are obtained from Eq. (9.114) by neglecting ω as compared to β and unity for the case when $\omega \ll \beta$, and by neglecting β as compared to ω when $\omega \gg \beta$. It is of interest to compare these relations with the reactor period calculated for the system in which the delayed neutrons were neglected, (9.23),

$$\frac{1}{T} = \frac{\omega}{l_{\text{th}}(1 - \omega)}$$

For large reactivities, the stable reactor period is very nearly the reactor period computed without regard to delayed neutrons. The critical point for the reactivity is at $\omega \equiv \beta$, as may be seen from Eq. (9.91); in the one delay-group model, this equation has the form

$$l_{\text{th}}\dot{\phi}(t) = \left(\frac{\omega - \beta}{1 - \omega}\right)\phi(t) + \frac{\lambda}{1 - \omega} G(t) \tag{9.118}$$

When $\omega = \beta$, this relation reduces to

$$l_{\text{th}}\dot{\phi}(t) = \frac{\lambda}{1 - \omega} G(t) \tag{9.119}$$

Thus the flux is changing in time only because of the appearance of the delayed neutrons; the reactor is critical on the basis of the prompt neutrons alone. It is customary to refer to a reactor system in which the added reactivity is just equal to the fraction of neutrons which are delayed as *prompt critical.*

Some idea of the magnitudes of the various parameters and variables involved in a transient system may be obtained from the following numerical example based on the one delay-group model. Let:

$$\begin{aligned} \beta &= 0.00755 & \lambda &= 0.076 \text{ sec}^{-1} \\ \omega &= 0.001 & l_{\text{th}} &= 0.001 \text{ sec} \end{aligned}$$

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In this case the roots of the characteristic equation may be approximated by the expressions

$$\alpha_1 \simeq \frac{\omega\lambda}{\beta - \omega} \qquad \alpha_2 \simeq \frac{\omega - \beta}{l_{th}} \tag{9.120}$$

which may be derived from Eq. (9.114) by neglecting λl_{th} relative to unity. With these relations and the set (9.116), we obtain

$$\alpha_1 = 0.0117 \text{ sec}^{-1} \qquad \alpha_2 = 6.55 \text{ sec}^{-1}$$
$$\varphi_1 = 1.132\phi_0 \qquad \varphi_2 = -0.132\phi_0$$

The corresponding expression for the flux $\phi(t)$ is

$$\frac{\phi(t)}{\phi_0} = 1.132e^{0.0117t} - 0.132e^{-6.55t} \tag{9.121}$$

where t is given in seconds. For this problem we have assumed that the reactor was in steady state (flux level ϕ_0) up to time $t = 0$, when the reactivity 0.001 was introduced. Figure 9.7 shows the resulting time

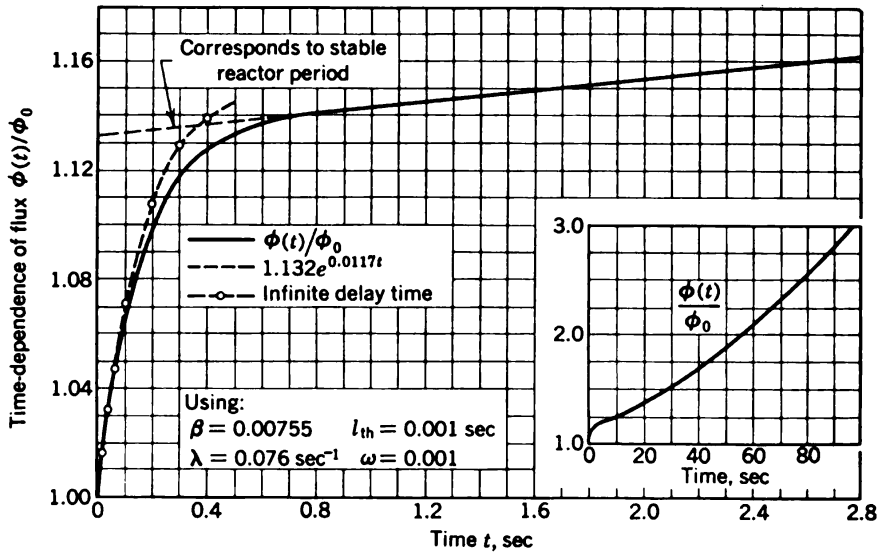

FIG. 9.7 Flux transients due to addition of positive reactivity.

behavior of the flux according to (9.121). It is of particular interest to note that, within about 0.5 sec after the introduction of the reactivity, the flux is rising essentially with the stable reactor period; that is (see broken-line curve in figure),

$$\frac{\phi(t)}{\phi_0} \simeq 1.132e^{0.0117t} \qquad \text{for } t > 0.5 \text{ sec}$$

The insert in the figure shows the long-term behavior of the flux.

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d. Infinite Delay Time. Another approximation of the delayed neutrons, which is sometimes used, is that of the "infinite delay time." In this approximation it is assumed that the delayed neutrons, which appear after the time when the reactivity was introduced, appear at a constant rate taken to be the rate at which they appeared in the reactor prior to time $t = 0$. The differential equation for the time-dependent behavior of the flux based on the one delay-group model and infinite delay time is obtained from (9.118) by setting the precursor term equal to its value prior to $t = 0$. Since the present approach to the delayed-neutron problem will be most useful when dealing with transients in a system formerly at steady state, the precursor concentration to be used in (9.118) is the steady-state value from (9.115); thus

$$l_{\text{th}}\dot{\phi}(t) = \left(\frac{\omega - \beta}{1 - \omega}\right)\phi(t) + \frac{\beta\phi_0}{1 - \omega} \tag{9.122}$$

The solution to this equation is

$$\phi(t) = \phi_0\left\{\frac{\omega}{\omega - \beta}\exp\left[\frac{\omega - \beta}{l_{\text{th}}(1 - \omega)}\,t\right] - \frac{\beta}{\omega - \beta}\right\} \tag{9.123}$$

This model would have some application in the early stages of the flux rise after $t = 0$, and before the concentration of the delayed-neutron precursors has been built up significantly over its steady-state value. For example, Eq. (9.122) is a good approximation to (9.121) when the transient effects, represented by the second term, are significantly larger than those due to the stable reactor period, i.e., when

$$\left|\frac{\varphi_1\alpha_1 t}{\varphi_2 e^{\alpha_2 t}}\right| \ll 1 \tag{9.124}$$

which follows from the approximation [cf. Eq. (9.121)]

$$\frac{\phi(t)}{\phi_0} \simeq \varphi_1 + \varphi_1\alpha_1 t + \varphi_2 e^{\alpha_2 t} \qquad \text{for } \alpha_1 t \ll 1$$

The solution for a system with infinite delay time (9.123) has been applied to the foregoing numerical example, and the results for $\phi(t)/\phi_0$ are shown in Fig. 9.7. Evidently this model gives a reasonably good representation of the transients in this example even up to 0.3 sec after the addition of the excess reactivity.

9.4 Reactor Kinetics with Temperature Dependence

a. General Equations for Stationary-fuel Reactor. In the preceding sections the primary objective was to establish the time-dependent relations for the thermal (one-velocity) flux and to study various approxi-

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mate models for describing the delayed neutron groups. The purpose of the present section is to explore with the aid of these relations some of the elementary kinetics problems of stationary-fuel reactors taking into account the effects of temperature. The influence of temperature upon the reactor kinetics stems from the temperature sensitivity of the nuclear characteristics of the system. Some of these effects were discussed in Sec. 6.5.

For the purposes of the present study these effects are described by means of a single parameter, the temperature coefficient of the reactor. The dependence of the kinetic behavior on the temperature coefficient is due to the coupling between the flux level and the power generation. It is clear that if for some reason there appears an increase in the flux level this will result in a proportionate increase in the fission rate. Therefore, an increase in flux gives rise to an increase in the power generation, and this in turn disturbs the energy balance of the system. As a consequence there must be an adjustment in the temperature level of the reactor. Since the nuclear characteristics of the reactor are temperature sensitive, a change in the flux level leads to a change in reactivity. Thus, a realistic treatment of reactor kinetics must necessarily include this coupling between the flux and the energy content of the system.

The temperature-dependent kinetics of stationary-fuel reactors is treated on the basis of the one-velocity model. The appropriate relations for the time behavior of the thermal flux are given by (9.84) and (9.85). In addition to these we require a third equation which states the energy balance for the system. If $H(\mathbf{r},t)$ denotes the energy content per unit volume at the space point $\mathbf{r}$ and at time t, then the time rate of change of H must be given by the net energy gain per unit time and volume from the fission reactions and the reactor cooling. The mathematical statement of this balance condition is[1]

$$\frac{\partial}{\partial t} H(\mathbf{r},t) = \varepsilon \Sigma_f \phi(\mathbf{r},t) - \mathcal{P}(\mathbf{r},t) \tag{9.125}$$

where $\varepsilon \equiv$ average energy released per fission
$\Sigma_f \equiv$ macroscopic fission cross section
$\phi(\mathbf{r},t) \equiv$ thermal flux (9.126)
$\mathcal{P}(\mathbf{r},t) \equiv$ energy loss per unit volume per unit time due to external cooling (radiation, convection, and/or conduction)

It is convenient to express the energy content in terms of the reactor temperature T by introducing a heat capacity c. The following relation is assumed:

$$H(\mathbf{r},t) \equiv c[T(\mathbf{r},t) - T_0(\mathbf{r},t)] \equiv c\Theta(\mathbf{r},t) \tag{9.127}$$

[1] Properly, the cross sections should also be written as time-dependent functions, but for the present treatment we assume that these time variations are negligible in comparison to the rapid transients in the other functions (e.g., in ϕ).

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where $c \equiv$ heat capacity of the reactor system (energy per unit volume per unit temperature)

$T_0(\mathbf{r},t) \equiv$ some convenient reference temperature

The immediate objective is to obtain a solution of the reactor-kinetics equations [(9.84) and (9.85)] which also satisfy the energy-balance relation (9.125). As in the previous problems in time dependence, we look for a solution which is separable in time and space. This treatment is limited, therefore, to bare systems and to reflected systems which meet the requirements set forth in Sec. 9.1d. On this basis we can use the relations (9.88) for the flux and the precursor concentrations. In addition, we select

$$\begin{aligned} \Theta(\mathbf{r},t) &\equiv F(\mathbf{r})\,\Theta(t) && (a) \\ \mathcal{P}(\mathbf{r},t) &\equiv F(\mathbf{r})\,\mathcal{P}(t) && (b) \end{aligned} \tag{9.128}$$

The substitution of these relations, along with (9.88), into (9.84), (9.85), and (9.125), yields

$$\begin{aligned} l_{\text{th}}\dot{\phi}(t) &= [(1-\beta)\delta k(t) - \beta]\phi(t) \\ &\quad + \frac{p_{\text{th}} g_{\text{th}}}{\Sigma_a + DB^2} \sum_i \lambda_i C_i(t) && (a) \\ \dot{C}_i(t) &= -\lambda_i C_i(t) + \nu\beta_i \Sigma_f \phi(t) && (b) \\ c\dot{\Theta}(t) &= -\mathcal{P}(t) + \varepsilon\Sigma_f \phi(t) && (c) \end{aligned} \tag{9.129}$$

The first two equations are essentially the set (9.89), the only difference being that we have included here the time dependence of the change in multiplication $\delta k(t)$. This dependence has been introduced since in the present treatment the reactivity will be a temperature-sensitive function, and the temperature in turn is coupled to the time-dependent neutron relations.

In the analyses which follow, attention is focused on the transients which arise when a given system initially at steady state ($t < 0$) is perturbed in some prescribed fashion. At $t = 0$, a disturbance is introduced which affects the energy balance of the system, and the problem will be to determine, in particular, the time behavior of the flux, temperature, and power for all subsequent time ($t > 0$). The steady-state values of the flux and precursor concentrations were previously indicated in (9.80) [see also (9.101)]. The steady-state requirements on the power removal can be obtained directly from (9.125); thus,

$$\mathcal{P}(\mathbf{r},0) = \varepsilon\Sigma_f \phi(\mathbf{r},0) \tag{9.130}$$

If we use for determining $\mathcal{P}(\mathbf{r},0)$ the expression (9.128b), and for $\phi(\mathbf{r},0)$, (9.88), we obtain the condition

$$\mathcal{P}(0) = \varepsilon\Sigma_f \phi(0) \tag{9.131}$$

which states that the steady-state value of the cooling (heat removal) rate is proportional to the flux level. It is convenient to introduce a

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new function $P(t)$ which is related to $\phi(t)$ in the same way that $\mathcal{P}(0)$ is related to $\phi(0)$. The function $P(t)$ will describe, then, the time behavior of the reactor power generation when the system is disturbed from its equilibrium. Therefore let

$$P(t) \equiv \varepsilon\Sigma_f\phi(t) \tag{9.132}$$

Note that this function has the units of energy per unit time per unit volume [see Eqs. (9.88) and (9.128)].

The transients in the reactor power production will be of special interest in the present study; therefore we write the kinetic equations (9.129) in terms of the function $P(t)$.

$$\begin{aligned} l_{th}\dot{P}(t) &= [(1-\beta)\delta k(t) - \beta]P(t) \\ &\quad + \frac{\varepsilon\Sigma_f p_{th} g_{th}}{\Sigma_a + DB^2}\sum_i \lambda_i C_i(t) \qquad (a) \\ \dot{C}_i(t) &= -\lambda_i C_i(t) + \frac{\nu}{\varepsilon}\beta_i P(t) \qquad (b) \\ c\dot{\Theta}(t) &= P(t) - \mathcal{P}(t) \qquad (c) \end{aligned} \tag{9.133}\star$$

In its present form this set is not complete since the temperature (and time) dependence of the multiplication $k(t)$ has not been specified. For the present case we take

$$k = k(T) = k[T(t)] \tag{9.134}$$

where the temperature dependence is assumed to be

$$k(t) \equiv 1 + \delta k(t) = (1 + \delta k_0)\{1 - \gamma[T(t) - T_0(t)]\} \tag{9.135}$$

with $\delta k_0 \equiv$ a step change in k applied to the reactor at $t = 0$

$\gamma \equiv$ a constant

Experience shows that this linear dependence based on the constant γ is a reasonable assumption if $\delta T/T \ll 1$. An approximate expression for $\delta k(t)$ can be obtained from this equation if the product $\gamma\delta k_0$, which is small in comparison to unity, is neglected:

$$\delta k(t) \simeq \delta k_0 - \gamma[T(t) - T_0(t)] \equiv \delta k_0 - \gamma\Theta(t) \tag{9.136}$$

where we have used the definition of Θ given in (9.127). It is evident from the above relation that the temperature coefficient of the reactor is dependent on the sign of the constant γ. The temperature coefficient of reactivity is given by (6.144), and for the present case we obtain, using (9.136),

$$\frac{\partial\omega}{\partial T} = \frac{1}{k^2}\frac{\partial k}{\partial T} = -\frac{(1 + \delta k_0)\gamma}{[1 + \delta k(T)]^2} \simeq -\frac{\gamma}{[1 + \delta k(T)]^2} \tag{9.137}$$

Thus if $\gamma > 0$, the temperature coefficient is negative. In this case if a system is given a step change in multiplication, $\delta k_0 > 0$ at $t = 0$, then

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the reactivity is initially positive and the system will "accelerate"; that is, the flux level will rise. However, as the flux rises, the power output and temperature will also rise. But because of the negative temperature coefficient [see (9.136)], the reactivity will decrease from its initial value and thereby retard the acceleration so as to stabilize the system. Clearly, a negative temperature coefficient implies an inherently stable chain-reacting system.

It will be helpful in exploring certain kinetics problems to use in place of the basic set of equations (9.133) a somewhat simplified version based on the assumption of *infinite delay time* (see Sec. 9.3d). In these cases we assume that at any time $t > 0$ the delayed neutrons which appear are those which were produced at the flux level existing before $t = 0$. This approximation has some practical value for analyzing situations wherein the power disturbances in the reactor occur as rapid oscillations and also for studying the time behavior of the system at very short time intervals after the initial disturbance is introduced.

The steady-state production of delayed neutrons may be obtained by computing the steady-state concentration of the precursors. From (9.133*b*), we have

$$\lambda_i C_i(0) = \frac{\nu\beta_i}{\varepsilon} P(0) \tag{9.138}$$

This expression is used as a first estimate for the precursor concentrations in (9.133*a*); then, the set (9.133) reduces to

$$\begin{aligned} l_{th}\dot{P}(t) &= [(1 - \beta)\delta k(t) - \beta]P(t) + [1 + \delta k(t)]\beta P(0) && (a) \\ c\dot{\Theta}(t) &= P(t) - \mathcal{P}(t) && (b) \\ \delta k(t) &= \delta k_0 - \gamma\Theta(t) && (c) \end{aligned} \tag{9.139}\star$$

This set will serve as the fundamental relations to be used in the study of reactor power kinetics problems which include the effect of delayed neutrons. Note that in the present, somewhat generalized formulation there are three unknown functions $P(t)$, $\Theta(t)$, and $\mathcal{P}(t)$. In order to obtain a complete solution, it is necessary to specify one of these functions. It is most convenient to give the cooling rate $\mathcal{P}(t)$ (the power removed per unit volume from the reactor). With this information and the following set of initial conditions, the formulation of the problem is complete:

$$\begin{aligned} P(0) &= \mathcal{P}(0) && && (a) \\ \dot{\Theta}(0) &= 0 && && (b) \\ \Theta(0) &= 0 && T(0) = T_0 && (c) \end{aligned} \tag{9.140}\star$$

In using the relations (9.139) and (9.140) it should be borne in mind that the temperature coefficient of the system is given by γ, since, from (9.135),

$$\gamma = -\frac{1}{1 + \delta k_0}\frac{\partial k}{\partial T} = -\frac{1}{k(T_0)}\frac{\partial k}{\partial T} \simeq -\frac{\partial k}{\partial T} \tag{9.141}$$

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The approximation is introduced because in most cases of interest $\delta k_0 \ll 1$. The derivative $\partial k/\partial T$ may be computed by the methods outlined in Sec. 6.5.

b. Constant Power Removal without Delayed Neutrons. As a first example of the application of Eqs. (9.139), consider a system which has the following characteristics: (1) constant power removal, $\mathcal{P}(t) \equiv \mathcal{P}_0$; (2) no delayed neutrons, $\beta \equiv 0$. Assumption (2) is introduced only as a simplification, but assumption (1) has some merit from the standpoint of a practical application. In many problems the situations of interest will be those which involve small oscillations of the flux (power) level, but, because of the large heat capacity of the reactor, these small (and relatively rapid) oscillations will have a negligible effect on the power-removal rate. Thus the assumption of constant power removal is entirely reasonable and should be useful for obtaining first estimates of the dynamic response of actual systems.

The appropriate differential equations for this system are

$$\dot{P}(t) = \frac{\delta k(t)\, P(t)}{l_{th}} \tag{9.142}$$

$$c\dot{\Theta}(t) = P(t) - \mathcal{P}_0 \tag{9.143}$$

It should be recalled that $P(t)$ is proportional to $\phi(t)$, which we previously found to be of exponential form; therefore, we take, as a possible solution [cf. Eq. (9.140*a*)]

$$P(t) = \mathcal{P}_0 e^{w(t)} \tag{9.144}$$

and look for a function $w(t)$ such that Eqs. (9.142) and (9.143) are satisfied along with the requirement that $w(0) \equiv 0$ [cf. assumption (1)]. The substitution of (9.144) into (9.142) and (9.143) yields the nonlinear set

$$c\dot{\Theta}(t) = \mathcal{P}_0[e^{w(t)} - 1] \tag{9.145}$$

$$\dot{w}(t) = [\delta k_0 - \gamma\Theta(t)]\frac{1}{l_{th}} \tag{9.146}$$

with the initial conditions

$$w(0) = 0 \qquad \Theta(0) = 0 \tag{9.147}$$

A general solution of Eqs. (9.145) and (9.146) will not be presented here, a qualitative study being entirely adequate for this analysis.

We first reduce the pair of equations to a transcendental relation for $\dot{w}(t)$ by differentiating (9.146) and applying (9.145):

$$\ddot{w}(t) = -\frac{\gamma}{l_{th}}\dot{\Theta}(t) = -\frac{\gamma\mathcal{P}_0}{l_{th}c}[e^{w(t)} - 1]$$

The integration of this equation yields

$$\frac{1}{2}[\dot{w}(t)]^2 = -\frac{\gamma\mathcal{P}_0}{l_{th}c}[e^{w(t)} - w(t)] + C$$

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The constant of integration is determined by using the initial conditions and (9.146), with the result that

$$C = \frac{\gamma \mathcal{P}_0}{l_{th} c} + \frac{1}{2}\left(\frac{\delta k_0}{l_{th}}\right)^2$$

Thus
$$\frac{1}{2}[\dot{w}(t)]^2 = -\frac{\gamma \mathcal{P}_0}{l_{th} c}[e^{w(t)} - w(t) - 1] + \frac{1}{2}\left(\frac{\delta k_0}{l_{th}}\right)^2 \tag{9.148}$$

The solution of this nonlinear differential equation yields the function $w(t)$, and this result may be used to obtain $\Theta(t)$ from (9.145). The resulting functions would be found to have the form shown in Fig. 9.8.

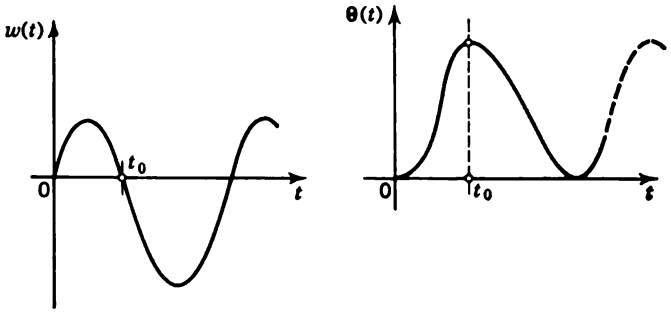

FIG. 9.8 Constant power removal without delayed neutrons.

These functions represent the solutions for the case $\gamma > 0$, which, as previously noted, implies a negative temperature coefficient for the system. This case describes a stable system that oscillates about the steady-state condition existing prior to $t = 0$. The case $\gamma < 0$ yields a divergent behavior with time, the functions $w(t)$ and $\Theta(t)$ increasing monotonically.

Some additional information can be obtained about the solution to this problem without a detailed calculation of the actual functions. In particular, we can estimate the maximum temperature rise experienced by the reactor, for a given rise in reactivity, δk_0. The peak value of the function $\Theta(t)$ may be computed from the condition $\dot{\Theta}(t) = 0$. This condition requires that $w(t) = 0$ [see (9.145)]; thus we must determine the roots of this equation. The slope of the function $w(t)$ at its zeros is obtained directly from (9.148); these are

$$\dot{w} = \pm\left(\frac{\delta k_0}{l_{th}}\right)$$

It is evident from Fig. 9.8 that the positive root corresponds to $t = 0$ (and the corresponding cyclic values of t) and that the negative root corresponds to the position of Θ_{max}. If t_0 denotes the value of t at which the maximum temperature values occur, then the magnitude of the temperature rise is obtained from (9.146):

$$\Theta_{max} \equiv \Theta(t_0) = 2\left(\frac{\delta k_0}{\gamma}\right) \tag{9.149}$$

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This result is generally valid, its only limitations are that the power removal be constant and that the delayed-neutron contributions be negligible.

c. Linearized Solution for Constant Power Removal. It is of interest to examine the problem of constant power removal under more restrictive conditions. A practical situation which can be easily examined in detail is the case of small oscillations. For this problem the study would be limited to small disturbances δk_0 on the steady-state operation of the reactor. Under these circumstances the problem can be linearized, and closed-form solutions can be readily obtained. We assume, therefore, that the excursions from steady state, which are represented by the function $w(t)$, are small [see (9.144)]. In this case we may approximate the power production by the relation

$$P(t) \simeq \mathcal{P}_0[1 + w(t)] \tag{9.150}$$

The differential equations for this problem are

$$\dot{\Theta}(t) \simeq \frac{\mathcal{P}_0 w(t)}{c} \qquad \dot{w}(t) \simeq \frac{1}{l_{\text{th}}}[\delta k_0 - \gamma\Theta(t)] \tag{9.151}$$

with the initial conditions (9.147) as before. This set of simultaneous equations may be solved by the method of Laplace transforms. If $W(s)$ and $\mathfrak{I}(s)$ are the Laplace transforms of $w(t)$ and $\Theta(t)$, respectively [see (6.22)], then the Laplace transforms of Eqs. (9.151) are

$$s\mathfrak{I}(s) = \frac{\mathcal{P}_0 W(s)}{c} \qquad sW(s) = \frac{1}{l_{\text{th}}}\left[\frac{\delta k_0}{s} - \gamma\mathfrak{I}(s)\right] \tag{9.152}$$

which have the solutions

$$W(s) = \frac{\delta k_0}{l_{\text{th}}(s^2 + \gamma\mathcal{P}_0/l_{\text{th}}c)} \qquad \mathfrak{I}(s) = \frac{\mathcal{P}_0\delta k_0}{cl_{\text{th}}s(s^2 + \gamma\mathcal{P}_0/l_{\text{th}}c)} \tag{9.153}$$

The inversion of these equations yields

$$\begin{aligned} w(t) &= \delta k_0\left(\frac{c}{\gamma l_{\text{th}}\mathcal{P}_0}\right)^{\frac{1}{2}} \sin\left(\frac{\gamma\mathcal{P}_0}{l_{\text{th}}c}\right)^{\frac{1}{2}} t \\ \Theta(t) &= 2\left(\frac{\delta k_0}{\gamma}\right)\sin^2\left(\frac{\gamma\mathcal{P}_0}{l_{\text{th}}c}\right)^{\frac{1}{2}}\frac{t}{2} \end{aligned} \tag{9.154}$$

These functions, along with $P(t)$, are shown in Fig. 9.9. Note that the maximum temperature rise is, indeed, $2(\delta k_0/\gamma)$, as previously predicted. The solutions (9.154) are for the case $\gamma > 0$. If γ had been taken to be negative, the appropriate solutions would be

$$\begin{aligned} w(t) &= \delta k_0\left(\frac{c}{\gamma l_{\text{th}}\mathcal{P}_0}\right)\sinh\left(\frac{\gamma\mathcal{P}_0}{l_{\text{th}}c}\right)^{\frac{1}{2}} t \\ \Theta(t) &= \frac{\delta k_0}{\gamma}\left[\cosh\left(\frac{\gamma\mathcal{P}_0}{l_{\text{th}}c}\right)^{\frac{1}{2}} t - 1\right] \end{aligned}$$

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where the absolute magnitude of γ is implied. Clearly, these functions increase monotonically with time, although these solutions could apply only for short times after $t = 0$.

d. Constant Power Removal with Delayed Neutrons. The effect of the delayed neutrons upon the power kinetics of a reactor may be demonstrated by means of the following example. Consider a bare homogeneous thermal reactor with constant power removal operating at steady

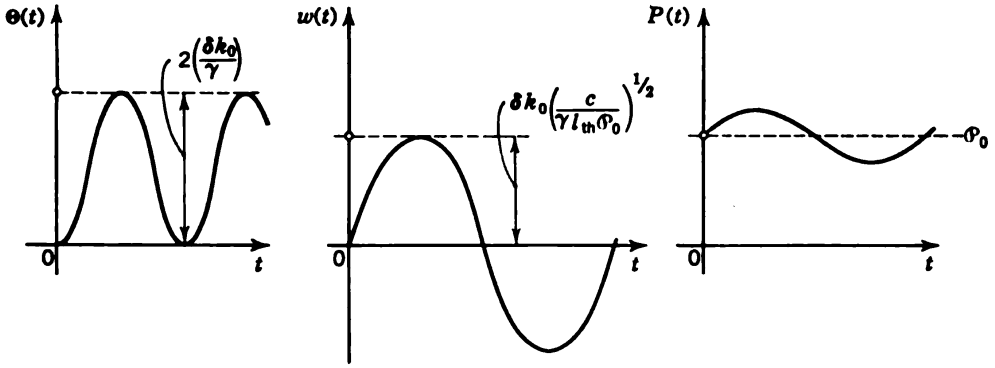

FIG. 9.9 Linearized solutions for constant power removal and no delayed neutrons.

state up until time $t = 0$. At $t = 0$ some excess multiplication δk_0 is added uniformly to the system. The problem is to determine the temperature and power output of the system for all subsequent time. For this calculation we introduce the following model:

$$\left.\begin{array}{l}(1)\ \text{One group of delayed neutrons with infinite delay time}\\ (2)\ \mathcal{P}(t) \equiv \mathcal{P}_0 \equiv P(0)\\ (3)\ \delta k(t) = \delta k_0 - \gamma\Theta(t)\end{array}\right\} \tag{9.155}$$

The appropriate kinetics equations for this system are given by (9.139). If we bear in mind that $\beta \ll 1$, then for sufficiently small δk_0 and γ, $\beta\delta k_0$ and $\beta\gamma$ are also appreciably less than unity. In this case (9.139) reduces to the set [where we have used also (9.155)]

$$\dot{P}(t) = [\delta k(t) - \beta]\frac{P(t)}{l_{\text{th}}} + \frac{\beta\mathcal{P}_0}{l_{\text{th}}} \tag{9.156}$$

$$\dot{\Theta}(t) = \frac{1}{c}[P(t) - \mathcal{P}_0] \tag{9.157}$$

with the initial conditions

$$P(0) = \mathcal{P}_0 \qquad \Theta(0) = \dot{\Theta}(0) = 0 \tag{9.158}$$

As in the preceding examples, we seek a solution for the reactor power which can be written in the form (9.144). In terms of the power func-

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tion $w(t)$, Eqs. (9.156) and (9.157) take the form

$$\dot{w}(t) = \frac{1}{l_{th}} [\delta k_0 - \gamma\Theta(t) - \beta] + \frac{\beta e^{-w(t)}}{l_{th}} \tag{9.159}$$

$$\dot{\Theta}(t) = \frac{\mathcal{P}_0}{c} [e^{w(t)} - 1] \tag{9.160}$$

The essential features of the present example may be amply demonstrated by means of the linearized solution. Therefore, using the approximation $e^w \simeq 1 + w$ [see also (9.150)], the above relations reduce to

$$\dot{w}(t) \simeq \frac{1}{l_{th}} [\delta k_0 - \gamma\Theta(t) - \beta w(t)] \tag{9.161}$$

$$\dot{\Theta}(t) \simeq pw(t) \tag{9.162}$$

where we have introduced the parameter

$$p \equiv \frac{\mathcal{P}_0}{c} \tag{9.163}$$

The pair (9.161) and (9.162) may be solved by differentiating (9.161) and applying (9.162) to eliminate the Θ term. The resulting expression for w is

$$\ddot{w}(t) + \frac{\beta}{l_{th}} \dot{w}(t) + \frac{\gamma p}{l_{th}} w(t) = 0 \tag{9.164}$$

The initial conditions for this equation are

$$w(0) = 0 \qquad \dot{w}(0) = \frac{\delta k_0}{l_{th}} \tag{9.165}$$

where we have applied (9.158) and (9.161). It is of some interest to note that the linearized equation for this reactor problem has the form of the familiar equation for an oscillating system with viscous damping.[1] A direct comparison to a mechanical dynamics problem reveals that the temperature-coefficient parameter γ has the role of the "spring constant," i.e., it is a measure of the "stiffness" of the system. Thus a large temperature coefficient implies a rapidly responding system, and higher frequencies of oscillation, following a disturbance. Note that the power parameter p also appears in this term; since the heat capacity c is in the denominator of this quantity, it follows that systems with large heat capacities are "soft" systems, i.e., respond slowly to excitation and experience oscillations of lower frequencies. Finally, we find that the viscous damping term ($\dot{w}$) involves the parameter β. Thus the presence

[1] See, for example, Theodore v. Kármán and Maurice A. Biot, "Mathematical Methods in Engineering," pp. 130–132, McGraw-Hill Book Company, Inc., New York, 1940.

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of delayed neutrons in the system serves to damp oscillations arising from a disturbance. This role of the delayed groups was mentioned previously.

The solution to Eq. (9.164) with initial conditions (9.165) is easily obtained by the usual methods (see, for example, Sec. 9.4c). This result, along with the corresponding solution for $\Theta(t)$ from (9.162), is found to be

$$w(t) = \left(\frac{\delta k_0}{l_{th}}\right) \frac{e^{-\chi t} \sin \psi t}{\psi}$$
$$\Theta(t) = \frac{\delta k_0}{\gamma}\left[1 - e^{-\chi t}\left(\frac{\chi}{\psi} \sin \psi t + \cos \psi t\right)\right] \tag{9.166}$$

where
$$\chi \equiv \frac{\beta}{2l_{th}} \qquad \psi^2 \equiv \frac{p\gamma}{l_{th}} - \chi^2 \tag{9.167}$$

The expressions (9.166) correspond to the case for $\psi^2 > 0$. Since the

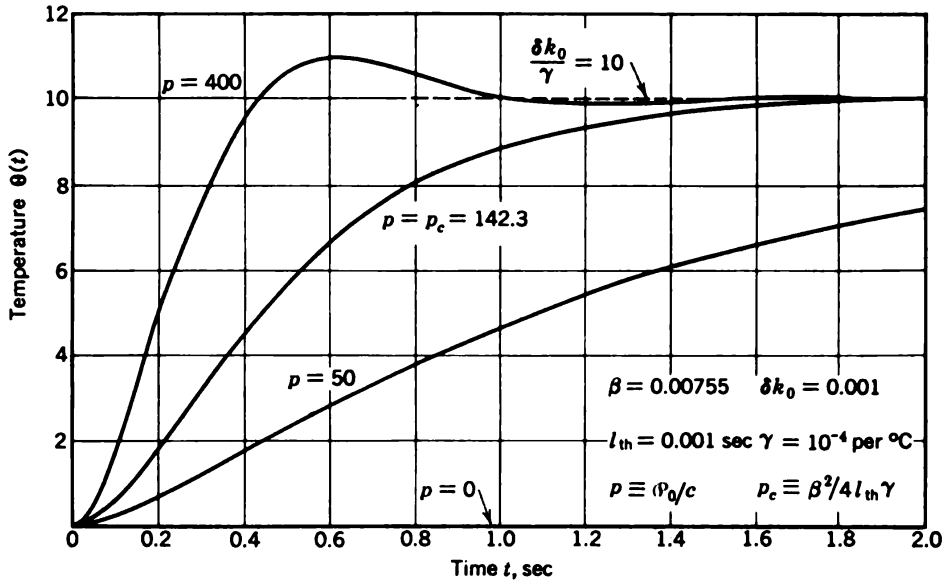

FIG. 9.10 Temperature for constant power removal and one of group delayed neutrons with infinite delay time.

parameter p may take on all values $p \geq 0$, ψ may be less than, equal to, or greater than zero. For each of these cases the solutions (9.166) have a particular form. The general results are well demonstrated by means of a numerical example. Figures 9.10 and 9.11 show the temperature difference Θ and the power function $w(t)$ for various values of p obtained for the case

$$\beta = 0.00755 \qquad \delta k_0 = 0.001$$
$$l_{th} = 0.001 \text{ sec} \qquad \gamma = 10^{-4}/°\text{C} \tag{9.168}$$

A discussion of these results follows. Consider first the case $\psi^2 > 0$. This corresponds to

(1) $$p > p_c$$

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where p_c is the value of p which yields $\psi \equiv 0$, i.e.,

$$p_c \equiv \frac{\beta^2}{4l_{th}\gamma} \tag{9.169}$$

The solutions for this case have the form of damped oscillations. Thus the initial disturbance due to the introduction of δk_0 causes a

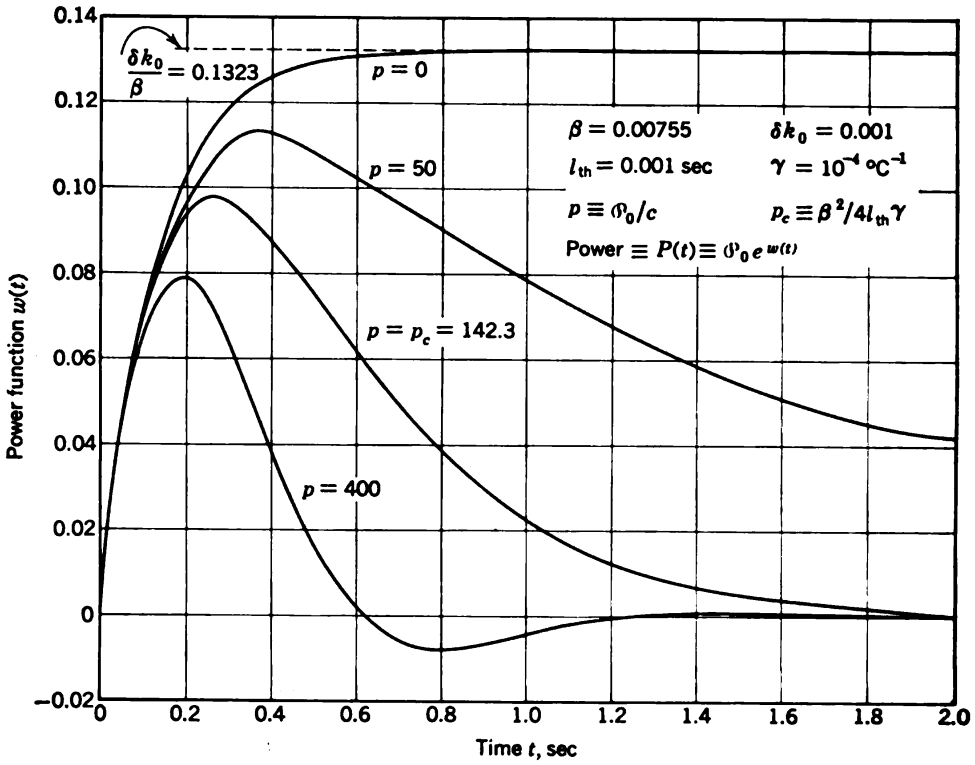

FIG. 9.11 Power function for constant power removal and one of group delayed neutrons with infinite delay time.

rapid change in both the temperature and power output of the reactor. The system stabilizes rapidly, however, and the oscillations decay to negligible magnitudes within a few cycles. It is obvious from (9.166) that the reactor power eventually stabilizes at $\mathcal{P}_0$, as is to be expected, and the temperature difference Θ at $\delta k_0/\gamma$; these arguments follow from the fact that

$$\lim_{t\to\infty} w(t) = 0 \qquad \lim_{t\to\infty} \Theta(t) = \frac{\delta k_0}{\gamma} \tag{9.170}$$

Since the limiting value of Θ is greater than zero, the system evidently stabilizes at a new higher (for positive δk_0) average temperature. This is a consequence of the negative temperature coefficient of the system. Note that the introduction of positive reactivity causes the power production to increase, this causes a rise in the temperature since the rate

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of power removal is constant; but, as the temperature rises, the reactivity of the system begins to decrease and very shortly thereafter becomes negative. The power level then begins to drop rapidly, and with it the temperature. But the system overcompensates, causing the temperature to drop low enough to produce a positive reactivity. There is then an upswing in both temperature and power. Eventually, the system reaches an equilibrium condition wherein the excess multiplication δk_0 is exactly compensated by an increase in the temperature level [given by (9.170)]. Another interesting feature of this system is the effect of the delayed neutrons. It was previously mentioned that delayed neutrons cause a damping of the oscillations in a system. More precisely, in the present model based on the infinite delay time, the reactor behaves like a subcritical assembly without delays which is maintained at steady state by a constant external source ($\nu\beta\mathcal{P}_0/\varepsilon$ neutrons per unit volume per unit time). By assumption, $w(t) \ll 1$; therefore the present results apply to very small changes in multiplication, and the functions (9.166) are valid only for very short time periods after $t = 0$ (cf. Fig. 9.7).

(2) $$p = p_c$$

In this case $\psi \equiv 0$, and Eqs. (9.166) reduce to the form

$$\begin{aligned} w(t) &= \left(\frac{\delta k_0}{l_{\text{th}}}\right) te^{-\chi t} \\ \Theta(t) &= \frac{\delta k_0}{\gamma}\left[1 - e^{-\chi t}(\chi t + 1)\right] \end{aligned} \tag{9.171}$$

The limiting values of these functions are seen to be the same as those given in (9.170). This case corresponds to a system with "critical damping." As the value of p decreases toward p_c, the magnitude of the oscillations in both temperature and power output diminishes until finally at $p = p_c$ the oscillations disappear altogether (see Figs. 9.10 and 9.11). It is helpful in understanding this behavior if one imagines the reduction in the parameter p to be due to an increase in the heat capacity of the system [see (9.163)]. Clearly, for a given disturbance δk_0, the larger the heat capacity, the less sensitive the system will be to changes in the power level. Thus large changes in power (i.e., reactivity) are more easily absorbed and the temperature rise required in order to accomplish this is correspondingly smaller. At $p = p_c$ the heat capacity of the system is sufficiently large to accommodate the change in reactivity without "overshooting."

(3) $$p < p_c$$

Under these circumstances $\psi^2 < 0$ and may be more conveniently written in the form

$$\psi \equiv iS \qquad \text{with } S^2 \equiv \chi^2 - \left(\frac{\gamma p}{l_{\text{th}}}\right) \tag{9.172}$$

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and χ defined as before by (9.167). The solutions for this case are

$$w(t) = \left(\frac{\delta k_0}{l_{th}}\right)\frac{e^{-\chi t}\sinh St}{S}$$
$$\Theta(t) = \frac{\delta k_0}{\gamma}\left[1 - e^{-\chi t}\left(\frac{\chi}{S}\sinh St + \cosh St\right)\right] \tag{9.173}$$

As in the preceding case, these solutions are nonoscillatory and the system is "overdamped." This corresponds to very large values of the heat capacity. In these cases the temperature and power are relatively loosely coupled functions and respond very slowly to reactivity changes (compare curves labeled $p = 50$ to those labeled $p = 142.3$ in Figs. 9.10 and 9.11).

(4) $p = 0$

This case is primarily of academic interest. On the basis of the present model, the solutions are found to be

$$w(t) = \frac{\delta k_0}{\beta}(1 - e^{-\beta t/l_{th}}) \tag{9.174}$$
$$\Theta(t) = 0$$

Since this case corresponds to a system with infinite heat capacity, it is to be expected that the temperature will be unaffected by changes in reactivity (i.e., power level). However, under these circumstances the negative temperature coefficient cannot be of aid in stabilizing the system and one would expect the power level to rise without limit. Since the result given above reaches a finite value it is clear that this elementary model breaks down when applied to this case; however, note that for short times after the addition of δk_0 (i.e., $\beta t/l_{th} \ll 1$) the power function has the correct form. In this case

$$w(t) \simeq \left(\frac{\delta k_0}{l_{th}}\right)t \qquad \text{and} \qquad P(t) \simeq P_0 \exp\left(\frac{\delta k_0 t}{l_{th}}\right) \tag{9.175}$$

Thus the power level rises exponentially.

The various cases discussed above are shown graphically in Figs. 9.10 and 9.11 for the data given in (9.168). Again it is pointed out that these results are meaningful only during the short time period following $t = 0$.

9.5 Kinetics of Circulating-fuel Reactors

The influence of the delayed neutrons and of the flux-temperature coupling on the dynamical characteristics of a stationary-fuel reactor was demonstrated in the preceding sections. These two factors are also essential to the dynamical behavior of circulating-fuel systems. In the present section we treat several aspects of the general problem and in particular show how the influence of the delayed neutrons and of the temperature coupling is altered by the circulation of the fuel. Since the

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general problem is very complex, no attempt is made here to provide a thorough study, and most of the discussion is based on relatively crude models of both the nuclear and fluid dynamical features of the system. Although this approach reduces considerably the analytical difficulties in the calculations, it also tends to render the final results less useful for computational purposes. However, these models do allow a clear exhibition of the essential features of the kinetics of circulating-fuel reactors; moreover, some of the analytical techniques developed in this treatment are sufficiently general that they may be applied to the analysis of more complicated problems. In any case the general conclusions and results obtained with the simplified models provide a correct qualitative description of these systems.

All the analyses which follow are based on two fundamental assumptions, namely, that the neutron population can be well described by means of the one-velocity model and that the circulation of the fuel can be represented by a slug flow.[1] The nuclear features of these systems, then, may be developed from the general results obtained in Sec. 9.4. The assumption of slug flow consists, essentially, of limiting the transport term in the fluid-flow relations to variations in the direction of flow only. Thus the flow of mass (and energy) around the fuel circuit occurs along parallel channels or tubes, and no communication (cross-flow) is allowed between adjacent channels. This assumption greatly simplifies the fluid dynamical aspects of the problem and thereby permits, with a minimum of effort, a clear display of the coupling between the fluid-flow and nuclear characteristics of the system.

In the present treatment the effect of the delayed neutrons and the effect of temperature-flux coupling on the reactor kinetics are handled separately along the general lines previously developed for the stationary-fuel systems. Although in general these two factors must be considered jointly, there are many practical situations wherein the two effects can be separated. The intent here is primarily to simplify the analysis so as to focus attention on one feature at a time.

a. Low-power Kinetics with Delayed Neutrons. The effect of delayed neutrons on the kinetic behavior of circulating-fuel reactors is studied with the aid of an analytical model which describes a *low-power* thermal reactor. In the present discussion the expression low power is used to denote systems wherein the power level is sufficiently low that small variations in it produce negligible changes in the temperature of the nuclear constituents of the reactor. Thus the energy balance of the system is not considered, and no coupling is allowed between the temperature and the neutron flux. For the present, then, we ignore entirely temperature coefficients of reactivity and focus attention on the role of the delayed neutrons. The specific objective is to derive the inhour

[1] The expression "slug flow" is used to denote a one-dimensional flow field.

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equation for the circulating-fuel system which may be written in the form of (9.98) obtained for the stationary-fuel reactor. The effect of the fuel circulation on the reactivity is then easily demonstrated. The analytical technique applied in this study follows the general scheme used by J. A. Fleck.[1] For the sake of simplicity, the calculation is specialized to the one-velocity case. Somewhat more general treatments based on the two-group model have been obtained by Fleck and are presented in the reference reports.

For the purposes of this analysis we assume that the reactor core is a bare right circular cylinder of radius ρ_0 and height h. The circulating-fuel circuit is a closed loop with the same cross-sectional area as the core. The length of the circuit external to the core is h_e, and the length of the complete circuit $(h + h_e)$ is $h_\star$. By definition, fissioning takes place only in the core region of the circuit. Although the assumption of a uniform cross-sectional area for the entire circuit may introduce some conceptual difficulties in establishing which portion of the circuit should properly be termed the "core," it leads to no loss in the generality of the results that are obtained since only the transit time ϑ_e through the external loop is essential to the calculation. It is clear that if a different average cross-sectional area were assigned to the external circuit so as better to represent the piping and heat exchangers, etc., in the loop, this would merely result in an average flow velocity through that portion different from that through the core. This effect is easily accounted for by adjusting the transit time ϑ_e.

In this model we further assume that the hydrodynamics of the system is adequately represented by the relations for slug flow and that the neutron physics may be described by means of the one-velocity model. Finally, we assume also that the fuel in the core is homogeneous and that the neutron macroscopic cross sections are time independent. On the basis of these assumptions we can write the following pair of differential equations to describe the circulating-fuel reactor:

$$-D\nabla^2\phi(\mathbf{r},t) + \Sigma_a\phi(\mathbf{r},t) = -\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},t) + (1-\beta)\nu\Sigma_f p_{\text{th}} g_{\text{th}}\phi(\mathbf{r},t) + p_{\text{th}} g_{\text{th}} \sum_i \lambda_i C_i(\mathbf{r},t) \quad (9.84)$$

$$\frac{\partial}{\partial t}C_i(\mathbf{r},t) + V\frac{\partial}{\partial z}C_i(\mathbf{r},t) = \nu\beta_i\Sigma_f\phi(\mathbf{r},t) - \lambda_i C_i(\mathbf{r},t) \qquad 0 \le z \le h \quad (9.176)$$

$$\frac{\partial}{\partial t}C_i(\mathbf{r},t) + V\frac{\partial}{\partial z}C_i(\mathbf{r},t) = -\lambda_i C_i(\mathbf{r},t) \qquad h \le z \le h_\star \quad (9.177)$$

[1] J. A. Fleck, Jr., "Theory of Low-power Kinetics of Circulating Fuel Reactors with Several Groups of Delayed Neutrons," Brookhaven National Laboratory, BNL 334 (T-57), April, 1955; "The Theory of Circulating Reactor Kinetics at Low Power," BNL 1933, Apr. 13, 1954; also, Kinetics of Circulating Reactors at Low Power, *Nucleonics*, **12**, 11 (1954).

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where V is the velocity of the circulating fuel and the coordinate z is the distance around the circuit measured from the inlet to the core. All the remaining symbols have the previously assigned definitions (see Sec. 9.3). The differential equation for the precursor concentration is obtained from the relation for a stationary-fuel system (9.85) by the addition of a transport (or convective) term. Note that Eqs. (9.176) and (9.177) are merely special forms of the usual continuity relation of fluid mechanics.[1]

We begin the calculation with a reduction of the flux equation to a simpler form appropriate to the present problem; therefore let us define

$$\phi(\mathbf{r},t) \equiv R(\rho)\,\varphi(z,t) \qquad C_i(\mathbf{r},t) \equiv R(\rho)\,S_i(z,t) \tag{9.178}$$

where ρ denotes the radial coordinate in the core. The substitution of these expressions into (9.84) yields

$$\begin{aligned} -D\left\{\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}R(\rho)\right]\varphi(z,t) + R(\rho)\frac{\partial^2}{\partial z^2}\varphi(z,t)\right\} + \Sigma_a R(\rho)\,\varphi(z,t) \\ = -\frac{R(\rho)}{v}\frac{\partial}{\partial t}\varphi(z,t) + (1-\beta)\nu\Sigma_f p_{\text{th}} g_{\text{th}} R(\rho)\,\varphi(z,t) \\ + p_{\text{th}} g_{\text{th}} R(\rho)\sum_i \lambda_i S_i(z,t) \end{aligned} \tag{9.179}$$

By introducing the relation

$$\frac{1}{\rho}\frac{d}{d\rho}\left[\rho\frac{d}{d\rho}R(\rho)\right] + B_\rho^2 R(\rho) = 0 \qquad R(\tilde{\rho}_0) = 0 \tag{9.180}$$

along with $l_0 \equiv 1/v\Sigma_a$, $L^2 \equiv D/\Sigma_a$, and $k_\infty \equiv \nu\Sigma_f p_{\text{th}}/\Sigma_a$, (9.179) may be further reduced to the form

$$\begin{aligned} l_0\frac{\partial}{\partial t}\varphi(z,t) = L^2\left[\frac{\partial^2}{\partial z^2}\varphi(z,t) - B_\rho^2\varphi(z,t)\right] + [k_\infty g_{\text{th}}(1-\beta) - 1]\,\varphi(z,t) \\ + \frac{k_\infty g_{\text{th}}}{\nu\Sigma_f}\sum_i \lambda_i S_i(z,t) \end{aligned} \tag{9.181}$$

If we call

$$B^2 \equiv B_\rho^2 + \mu^2 \qquad \text{where } \mu^2 \equiv \left(\frac{\pi}{h}\right)^2 \tag{9.182}$$

add and subtract $L^2B^2\varphi$ from the right-hand side of (9.181), and finally

[1] See, for example, L. Prandtl and O. G. Tietjens, "Fundamentals of Hydro-Aeromechanics," pp. 100–104, McGraw-Hill Book Company, Inc., New York, 1934; S. Goldstein, "Modern Developments in Fluid Mechanics," vol. I, chap. III, Oxford University Press, New York, 1938.

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divide through by $(1 + L^2B^2)$, (9.181) may be written

$$l_{\text{th}} \frac{\partial}{\partial t} \varphi(z,t) = \frac{p_L}{B^2}\left[\frac{\partial^2}{\partial z^2}\varphi(z,t) + \mu^2\varphi(z,t)\right] + [(1 - \beta)\,\delta k - \beta]\varphi(z,t) + \frac{k}{\nu\Sigma_f}\sum_i \lambda_i S_i(z,t) \quad (9.183)\star$$

where we have used $k = k_\infty g_{\text{th}}(1 + L^2B^2)^{-1}$, $k \equiv \delta k + 1$, Eq. (9.35) for l_{th}, and (5.220) for p_L.

We consider next the precursor equations (9.176) and (9.177). The application of (9.178) reduces these to

$$\frac{\partial}{\partial t} S_i(z,t) + V\frac{\partial}{\partial z} S_i(z,t) = \nu\Sigma_f\beta_i\varphi(z,t) - \lambda_i S_i(z,t) \qquad 0 \le z \le h \quad (9.184)$$

$$\frac{\partial}{\partial t} S_i(z,t) + V\frac{\partial}{\partial z} S_i(z,t) = -\lambda_i S_i(z,t) \qquad h \le z \le h_\star \quad (9.185)$$

These equations may be solved with the aid of the Laplace transform. If we take $S_i(z,0) \equiv 0$ and call

$$\mathfrak{L}\{S_i(z,t)\} \equiv \int_0^\infty S_i(z,t)e^{-st}\,dt \equiv \chi_i(z,s) \quad (9.186)$$

then the transforms of Eqs. (9.184) and (9.185) are

$$\begin{aligned} &\frac{d}{dz}\chi_i(z,s) + \left(\frac{s + \lambda_i}{V}\right)\chi_i(z,s) = \frac{\beta_i}{V}\Phi(z,s) \qquad 0 \le z \le h \quad (a) \\ &\frac{d}{dz}\chi_i(z,s) + \left(\frac{s + \lambda_i}{V}\right)\chi_i(z,s) = 0 \qquad h \le z \le h_\star \quad (b) \end{aligned} \quad (9.187)$$

where $\Phi(z,s)$ is the transform of $P(z,t)$ and $P(z,t) \equiv \nu\Sigma_f\varphi(z,t)$. The solutions to these equations are

$$\chi_i(z,s) = e^{-(s+\lambda_i)z/V}\left[\frac{\beta_i}{V}\int_0^z \Phi(y,s)e^{(s+\lambda_i)y/V}\,dy + \chi_i(0,s)\right] \qquad 0 \le z \le h \quad (9.188)$$

$$\chi_i(z,s) = \chi_i(h,s)e^{-(s+\lambda_i)(z/V-\vartheta)} \qquad h \le z \le h_\star \quad (9.189)$$

where

$$\vartheta \equiv \frac{h}{V} \quad (9.190)$$

is the transit time through the core. The constants of integration $\chi_i(0,s)$ and $\chi_i(h,s)$ are obtained by the application of continuity requirements. Thus Eq. (9.188) must equal Eq. (9.189) at $z = h$, and Eq. (9.188) evaluated at $z = 0$ must equal Eq. (9.189) evaluated at $z = h_\star$. The resulting expressions for the constants when used in (9.188) and (9.189) yield the solutions

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$$\chi_i(z,s) = \beta_i e^{-(s+\lambda_i)z/V}\left[\int_0^{z/V}\Phi(Vy,s)e^{(s+\lambda_i)y}\,dy + \frac{e^{-(s+\lambda_i)\vartheta_\star}}{1-e^{-(s+\lambda_i)\vartheta_\star}}\int_0^{\vartheta}\Phi(Vy,s)e^{(s+\lambda_i)y}\,dy\right] \quad \text{for } 0 \le z \le h \tag{9.191}$$

$$\chi_i(z,s) = \frac{\beta_i e^{-(s+\lambda_i)z/V}}{1-e^{-(s+\lambda_i)\vartheta_\star}}\int_0^{\vartheta}\Phi(Vy,s)e^{(s+\lambda_i)y}\,dy \quad \text{for } h \le z \le h_\star \tag{9.192}$$

The symbol $\vartheta_\star$ denotes the transit time around the complete circuit, i.e.,

$$\vartheta_\star = \vartheta + \vartheta_e \tag{9.193}$$

where ϑ_e is the transit time through the external loop.

The solution for the precursor concentrations $S_i(z,t)$ is obtained by the inversion of Eqs. (9.191) and (9.192). These are easily determined by the application of the transform

$$\mathfrak{L}\{F_b(t)\} = f(s)e^{-bs} \tag{9.194}$$

where $F_b(t) = 0$ when $0 < t < b$
$= F(t-b)$ $t > b$

The second term in (9.191) and the expression (9.192) may be cast into the appropriate form by using the expansion

$$\frac{1}{1-e^{-(s+\lambda_i)\vartheta_\star}} = \sum_{n=0}^{\infty} e^{-(s+\lambda_i)\vartheta_\star n}$$

which is valid since the exponential in the denominator is less than unity by virtue of the fact that $s > 0$. In this way it may be shown that the inversions of (9.191) and (9.192) are

$$S_i(z,t) = \nu\Sigma_f\beta_i\left[\int_0^{z/V}\varphi(z-Vy,\,t-y)e^{-\lambda_i y}\,dy + \sum_{n=0}^{\infty}e^{-\lambda_i\vartheta_\star n}\int_{z/V+\vartheta_e}^{z/V+\vartheta_\star}\varphi(V\vartheta_\star+z-Vy,\,t-n\vartheta_\star-y)e^{-\lambda_i y}\,dy\right] \quad \text{for } 0 \le z \le h \tag{9.195}$$

$$S_i(z,t) = \nu\Sigma_f\beta_i\sum_{n=0}^{\infty}e^{-\lambda_i\vartheta_\star n}\int_{z/V-\vartheta}^{z/V}\varphi(z-Vy,\,t-n\vartheta_\star-y)e^{-\lambda_i y}\,dy \quad \text{for } h \le z \le h_\star \tag{9.196}$$

This expression for $S_i(z,t)$ is to be solved simultaneously with the relation for the neutron flux (9.183). The solution is obtained by expanding the flux $\varphi(z,t)$ in a Fourier series (appropriate to the boundary condition in z); thus, for $z \ge 0$,

$$\varphi(z,t) \equiv \sum_{m=1}^{\infty}\phi_m(t)\,Z_m(z) \qquad \text{with} \begin{cases} Z_m(z) \equiv \sin\mu_m z \\ \mu_m \equiv \pi m/h \end{cases} \tag{9.197}$$

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and developing suitable expressions for the Fourier coefficients $\phi_m(t)$. The substitution of (9.197) into (9.195) and (9.183), followed by the multiplication of (9.183) by $Z_j(z)$ and integration over the interval $(0,h)$ yields an equation for $\phi_j(t)$.

$$c_j l_{\text{th}} \phi_j(t) = c_j \phi_j(t) \left\{ \frac{p_L}{B^2} (\mu^2 - \mu_j^2) + [(1-\beta)\delta k - \beta] \right\}$$
$$+ k \sum_i \beta_i \lambda_i \sum_{m=1}^{\infty} \left\{ \int_0^h Z_j(z)\, dz \int_0^{z/V} \phi_m(t-y) Z_m(z - Vy) e^{-\lambda_i y}\, dy \right.$$
$$+ \sum_{n=0}^{\infty} e^{-\lambda_i \vartheta_\star n} \int_0^h Z_j(z)\, dz$$
$$\left. \times \int_{z/V+\vartheta_e}^{z/V+\vartheta_\star} \phi_m(t - n\vartheta_\star - y) Z_m(z + V\vartheta_\star - Vy) e^{-\lambda_i y}\, dy \right\} \quad (9.198)$$

Note that,

$$\int_0^h Z_j(z)\, Z_m(z)\, dz = 0 \qquad j \neq m$$
$$\int_0^h Z_j(z)\, Z_m(z)\, dz \equiv c_j \qquad j = m \quad (9.199)$$

If we invert the order of integration in (9.198) and call

$$L_{jm}(y) \equiv \frac{1}{c_j} \int_{Vy}^h Z_j(z)\, Z_m(z - Vy)\, dz$$
$$M_{jm}(y) \equiv \frac{1}{c_j} \int_0^{V(y-\vartheta_e)} Z_j(z)\, Z_m(z + V\vartheta_\star - Vy)\, dz \quad (9.200)$$
$$N_{jm}(y) \equiv \frac{1}{c_j} \int_{V(y-\vartheta_\star)}^h Z_j(z)\, Z_m(z + V\vartheta_\star - Vy)\, dz$$

then (9.198) reduces to

$$l_{\text{th}} \phi_j(t) = \phi_j(t) \left\{ \frac{p_L}{B^2} (\mu^2 - \mu_j^2) + [(1-\beta)\delta k - \beta] \right\}$$
$$+ k \sum_i \beta_i \lambda_i \sum_{m=1}^{\infty} \left\{ \int_0^{\vartheta} \phi_m(t-y) L_{jm}(y) e^{-\lambda_i y}\, dy \right.$$
$$+ \sum_{n=0}^{\infty} e^{-\lambda_i \vartheta_\star n} \left[\int_{\vartheta_e}^{\vartheta_\star} \phi_m(t - n\vartheta_\star - y)\, M_{jm}(y) e^{-\lambda_i y}\, dy \right.$$
$$\left. \left. + \int_{\vartheta_\star}^{\vartheta_\star+\vartheta} \phi_m(t - n\vartheta_\star - y)\, N_{jm}(y) e^{-\lambda_i y}\, dy \right] \right\} \quad (9.201)$$

An adequate approximation to this equation is obtained by retaining the first term only in the series (9.197); thus neglecting all terms for $j > 0$ and dropping the subscripts on ϕ_1, L_{11}, M_{11}, and N_{11}, (9.201) becomes

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(note that $\mu_1 = \mu$)

$$l_{th}\,\phi(t) = \phi(t)[(1-\beta)\delta k - \beta] + k\sum_i \beta_i\lambda_i \left\{\int_0^{\vartheta} \phi(t-y)\,L(y)e^{-\lambda_i y}\,dy + \sum_{n=0}^{\infty} e^{-\lambda_i\vartheta_\star n}\left[\int_{\vartheta_e}^{\vartheta_\star} \phi(t-n\vartheta_\star-y)\,M(y)e^{-\lambda_i y}\,dy + \int_{\vartheta_\star}^{\vartheta_\star+\vartheta} \phi(t-n\vartheta_\star-y)\,N(y)e^{-\lambda_i y}\,dy\right]\right\} \quad (9.202)$$

The kernels L, M, and N are obtained from Eqs. (9.200); the integrations yield

$$L(\zeta) = (1-\zeta)\cos\pi\zeta + \frac{1}{\pi}\sin\pi\zeta \qquad 0 \le \zeta \le 1 \quad (9.203)$$

$$M(\zeta) = \left(\zeta - \frac{\vartheta_e}{\vartheta}\right)\cos\pi\left(\frac{\vartheta_\star}{\vartheta} - \zeta\right) + \frac{1}{\pi}\sin\pi\left(\zeta - \frac{\vartheta_e}{\vartheta}\right) \qquad \frac{\vartheta_e}{\vartheta} \le \zeta \le \frac{\vartheta_\star}{\vartheta} \quad (9.204)$$

$$N(\zeta) = \left(1 - \zeta + \frac{\vartheta_\star}{\vartheta}\right)\cos\pi\left(\zeta - \frac{\vartheta_\star}{\vartheta}\right) + \frac{1}{\pi}\sin\pi\left(\zeta - \frac{\vartheta_\star}{\vartheta}\right) \qquad \frac{\vartheta_\star}{\vartheta} \le \zeta \le 1 + \frac{\vartheta_\star}{\vartheta} \quad (9.205)$$

where $\zeta \equiv y/\vartheta$. It may be seen that these functions are all closely related; thus $N(\zeta) = L[\zeta - (\vartheta_\star/\vartheta)]$, and $M(\zeta)$ is a "reflection" of $L(\zeta)$ about $\zeta = \vartheta_e/\vartheta$. Figure 9.12 shows the kernel $L(\zeta)$, along with the linear approximation $1 - \zeta$.

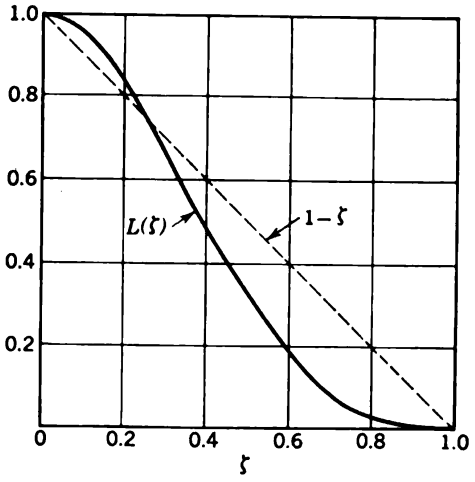

FIG. 9.12 Comparison of kernel $L(\zeta)$ and linear approximation.

The results (9.203) through (9.205) may be used in (9.202) to obtain an expression for the inhour equation appropriate to the circulating-fuel reactor and a sinusoidal axial variation in the neutron flux. However, instead of using these complicated expressions, we apply some rough approximations that will suffice for the present purpose and at the same time simplify the calculation. Thus we introduce the following linear expressions for the kernels:

$$L(\zeta) \equiv 1 - \zeta$$
$$M(\zeta) \equiv \zeta - \frac{\vartheta_e}{\vartheta} \quad (9.206)$$
$$N(\zeta) \equiv 1 - \zeta + \frac{\vartheta_\star}{\vartheta}$$

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These approximations give reasonably good estimates of the set (9.203) through (9.205); note that the values of these functions at the limits of their respective intervals are properly matched and that the average value of each comes within 20 per cent of that for the corresponding exact function. This is the relevant comparison since the exponential weighting factors $e^{-\lambda_i v}$ which appear in (9.202) are slowly varying over the transit period ϑ for most systems of practical interest. The inhour relation is obtained from (9.202) by introducing the expression

$$\phi(t) = \phi_0 e^{\alpha t} \tag{9.207}$$

If we use the set (9.206), then it may be shown, after some algebra, that (9.202) reduces to the form

$$l_{\text{th}}\alpha = (1 - \beta)\delta k - \beta + k \sum_i \frac{\lambda_i \beta_i}{\gamma_i} \left[1 - \frac{(1 - e^{-\gamma_i \vartheta_*})(1 - e^{-\gamma_i \vartheta})}{\vartheta \gamma_i (1 - e^{-\gamma_i \vartheta_\star})} \right] \tag{9.208}$$

where

$$\gamma_i \equiv \lambda_i + \alpha \tag{9.209}$$

This result may also be written in terms of the reactivity ω,

$$\omega = \frac{l_{\text{th}}\alpha}{1 + l_{\text{th}}\alpha} + \frac{1}{1 + l_{\text{th}}\alpha} \sum_i \frac{\beta_i \alpha}{\gamma_i} \left[1 + \frac{\lambda_i (1 - e^{-\lambda_i \vartheta_*})(1 - e^{-\lambda_i \vartheta})}{\alpha \vartheta \gamma_i (1 - e^{-\lambda_i \vartheta_\star})} \right] \tag{9.210}$$

The comparison of this expression with (9.98) reveals the influence of the circulation on the reactivity of the system. This effect is seen more clearly after a rearrangement of the exponential factors in the sum. We first show, however, that this sum may be written in terms of the fraction of the total number of precursors in the circuit which may be found in the core. For this purpose we require the total number of precursors in the core and in the loop. These are obtained from (9.195) and (9.196); for the case of a flat axial distribution in power, $\varphi(z,t) \equiv \phi_0 e^{\alpha t}$ and the precursor relations become

$$\begin{aligned} S_i(z,t) &= \frac{\nu \Sigma_f \beta_i \phi_0 e^{\alpha t}}{\gamma_i (1 - e^{-\gamma_i \vartheta_\star})} \{1 - e^{-\gamma_i z/V} + e^{-\gamma_i \vartheta_\star} [e^{-\gamma_i (z/V - \vartheta)} - 1]\} \\ &\qquad\qquad 0 \le z \le h \\ &= \frac{\nu \Sigma_f \beta_i \phi_0 e^{\alpha t}}{\gamma_i (1 - e^{-\gamma_i \vartheta_\star})} [(e^{\gamma_i \vartheta} - 1) e^{-\gamma_i z/V}] \qquad h \le z \le h_\star \end{aligned} \tag{9.211}$$

The integration of the first of these over the interval $(0,h)$ and the integration of the second over $(h,h_\star)$ give the total number of precursors in core and loop, respectively. The fraction a_i of precursors of type i in the core is computed from

$$a_i = 1 - \frac{S_i(\text{loop})}{S_i(\text{total})} \qquad \text{with } S_i(\text{total}) = \frac{\nu \Sigma_f \beta_i \phi_0 h e^{\alpha t}}{\gamma_i} \tag{9.212}$$

then

$$a_i(\alpha) = 1 - \frac{(1 - e^{-\gamma_i \vartheta_*})(1 - e^{-\gamma_i \vartheta})}{\vartheta \gamma_i (1 - e^{-\gamma_i \vartheta_\star})} \tag{9.213}$$

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Of particular interest will be the fraction of precursors in the core at steady state; in that case we have, as $\alpha \to 0$,

$$a_i(0) = 1 - \frac{(1 - e^{-\lambda_i \vartheta_e})(1 - e^{-\lambda_i \vartheta})}{\vartheta \lambda_i (1 - e^{-\lambda_i \vartheta_\star})} \tag{9.214}$$

and when $\lambda_i \vartheta_\star \ll 1$,

$$a_i(0) \simeq \frac{\vartheta}{\vartheta_\star} \tag{9.215}$$

Thus for transit times appreciably less than the precursor mean life (see Table 9.1), the fraction in the core is given approximately by the fraction

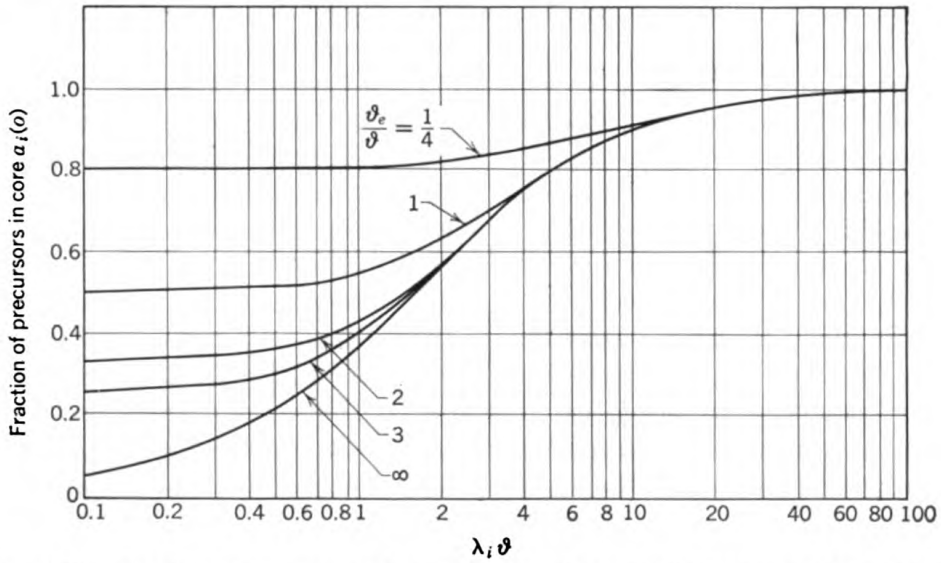

FIG. 9.13 Fraction of precursors in core of circulating-fuel reactor at steady state.

of the total circuit time spent in the core. It is also of interest to note that for all values of α, when $\vartheta \to \infty$ (i.e., no circulation),

$$\lim_{\vartheta \to \infty} a_i(\alpha) = 1 \tag{9.216}$$

as is to be expected. Figure 9.13 shows the fraction of precursors in the core at steady state as a function of $\lambda_i \vartheta$ for various ϑ_e/ϑ.

It is evident that Eq. (9.213) for $a_i(\alpha)$ may be used to reduce the expression (9.210) for the reactivity; thus we obtain

$$\omega = \frac{l_{\text{th}}\alpha}{1 + l_{\text{th}}\alpha} + \frac{1}{1 + l_{\text{th}}\alpha} \sum_i \frac{\beta_i}{\gamma_i} [\gamma_i - \lambda_i a_i(\alpha)] \tag{9.217}$$

This result has several interesting features. Consider first the reactivity ω_{circ} required to maintain the circulating-fuel system at steady state. For this condition $\alpha \to 0$, and (9.217) takes the form

$$\omega_{\text{circ}} = \sum_i \beta_i [1 - a_i(0)] \tag{9.218}$$

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where $a_i(0)$ may be obtained from (9.214) or Fig. 9.13. Note that, when ϑ and $\vartheta_e \to \infty$ (no circulation), $\omega \to 0$ by (9.218). This agrees with the result for the stationary-fuel reactor (9.98). Also, in the case of infinite circulation velocity V, ϑ and $\vartheta_e \to 0$. If we let $\vartheta_e/\vartheta \equiv \zeta = \text{constant}$ and take the limit as $\vartheta \to 0$, (9.218) yields

$$\lim_{\vartheta \to 0} \omega_{\text{circ}} = \beta \left(\frac{\vartheta_e}{\vartheta_\star} \right) \tag{9.219}$$

Thus we find that to maintain a circulating-fuel reactor at steady state we require some positive reactivity varying from 0 for the stationary-fuel system to a maximum of $\beta\vartheta_e/\vartheta_\star$ for the system with infinite circulation velocity. It is obvious from these observations that, if a circulating-fuel reactor is at steady state at some given circulation velocity, a sudden reduction in this velocity will cause the reactor to become supercritical since a smaller amount of positive reactivity is required to maintain the system at steady state at the lower velocity than at the higher.

Another interesting conclusion which may be drawn is that the kinetic behavior of a circulating-fuel reactor (within the limits of the present model) may be described in terms of an equivalent stationary-fuel system in which the fractions of the delayed neutrons $\beta_i^{\text{circ}} < \beta_i$, the actual delay fractions. To demonstrate this we determine the "excess reactivity" for the circulating-fuel system. By this we mean the reactivity over and above that required for criticality at any specified circulation velocity. This is the quantity $\omega - \omega_{\text{circ}}$. From (9.217) and (9.218) we obtain

$$\omega - \omega_{\text{circ}} = \frac{l_{\text{th}}\alpha}{1 + l_{\text{th}}\alpha} + \frac{1}{1 + l_{\text{th}}\alpha} \sum_i \frac{\beta_i}{\gamma_i} [\gamma_i(1 + l_{\text{th}}\alpha)a_i(0) - \lambda_i a_i(\alpha) - l_{\text{th}}\alpha\gamma_i] \tag{9.220}$$

It is of interest to evaluate this expression for the case wherein the reactor is on a slow period, i.e., when $\alpha \ll \lambda_i$. For this computation we use a first-order approximation to $a_i(\alpha)$. The appropriate form is easily shown to be

$$a_i(\alpha) \simeq a_i(0) \left(1 + \frac{\alpha}{\lambda_i} h_i \right) - \frac{\alpha}{\lambda_i} h_i$$

where

$$h_i \equiv \frac{\lambda_i\vartheta}{e^{\lambda_i\vartheta} - 1} + \frac{\lambda_i\vartheta_e}{e^{\lambda_i\vartheta_e} - 1} - \frac{\lambda_i\vartheta_\star}{e^{\lambda_i\vartheta_\star} - 1} - 1$$

If we further impose the condition that $\lambda_i\vartheta_\star \ll 1$ [cf. Eq. (9.215)], then $h_i \simeq -\frac{1}{6}\vartheta\vartheta_\star\lambda_i^2(1 - \vartheta/\vartheta_\star)$; both this quantity and $l_{\text{th}}\lambda_i$ are much less than unity. Thus to zeroth-order approximation, the expression within the bracket of the sum (9.220) is $\alpha a_i(0)$. The result (9.220) then reduces to

$$\omega - \omega_{\text{circ}} \simeq \frac{l_{\text{th}}\alpha}{1 + l_{\text{th}}\alpha} + \frac{1}{1 + l_{\text{th}}\alpha} \sum_i \frac{\beta_i\alpha a_i(0)}{\alpha + \lambda_i} \tag{9.221}\star$$

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This result[1] is to be compared to (9.98). We find, then, that the circulating-fuel system responds like a stationary-fuel reactor with the delay fraction β_i replaced by $a_i(0)\beta_i < \beta_i$. Thus the circulating system is the more sensitive to reactivity changes. Note that the proper correlation is made in terms of the so-called "excess reactivity." Also, it follows from the above result, and (9.216), that as $\vartheta \to \infty$, (9.221) reduces to (9.98).

Finally, we can estimate the effect on the multiplication constant of a change in the circulation velocity. Consider the multiplication k_{circ} required to maintain the system at steady state when the fuel is circulating at some velocity V. From (9.218),

$$k_{\text{circ}} = \frac{1}{1 - \omega_{\text{circ}}} = \frac{1}{1 - \sum_i \beta_i[1 - a_i(0)]} \tag{9.222}$$

and it follows that

$$\frac{\partial k_{\text{circ}}}{\partial V} = -\frac{\vartheta}{V}\frac{\partial k_{\text{circ}}}{\partial \vartheta} = \frac{\vartheta k_{\text{circ}}^2}{V}\sum_i \beta_i \frac{\partial}{\partial \vartheta} a_i(0) \simeq \frac{k_{\text{circ}}^2 \beta}{V}\left(\frac{\vartheta}{\vartheta_*}\right) \tag{9.223}$$

where we have used (9.215) for $a_i(0)$. Thus, as previously recognized, a decrease in circulation velocity causes an increase in the multiplication.

b. Flux-temperature Coupling without Delayed Neutrons. The effect of flux-temperature coupling upon the kinetics of circulating-fuel reactors is studied by means of a model wherein all fission neutrons are taken to be prompt. As in the analysis of the low-power systems, it is assumed that the one-velocity model applies and that the circulation of the fuel is well represented by slug flow.

The general line of attack is similar to that applied in the preceding analysis and follows the approach used by J. A. Fleck.[2] The problem of temperature-dependent kinetics has received the attention also of Ergen,[3] Weinberg,[4] and Welton,[5] who formulated the problem in terms of equations which involve integral operations on the past history of the neutron flux. This approach leads directly to a set of equations in terms of the

[1] An alternative derivation is given by W. K. Ergen, "The Inhour Formula for a Circulating-fuel Nuclear Reactor with Slug Flow," Oak Ridge National Laboratory, ORNL CF 53-12-108, Dec. 22, 1953.

[2] J. A. Fleck, Jr., "The Temperature-dependent Kinetics of Circulating Fuel Reactors," Brookhaven National Laboratory, BNL 357 (T-65), July, 1955.

[3] W. K. Ergen, Kinetics of the Circulating-fuel Nuclear Reactor, *J. Appl. Phys.*, **25**(6), 702–711 (June, 1954).

[4] W. K. Ergen and A. M. Weinberg, Some Aspects of Nonlinear Reactor Dynamics, *Physica*, **20**, 413 (1954).

[5] T. A. Welton, "Kinetics of Stationary Reactor Systems," *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, 1955, vol. 5, P/610, pp. 377–388.

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time variable alone. The method used by Fleck begins with the usual set of partial differential equations (in space and time) for both the neutron flux and for the energy balance. It is shown that by expanding the solution for the flux in a Fourier series the resulting expressions for the time-dependent Fourier coefficients are in fact the relations used by the authors mentioned above. Thus the present method offers a somewhat more general approach and has the added advantage in that it contains implicitly the spatial dependence of the neutron flux and temperature and permits solutions for space-dependent perturbations and initial conditions.

The calculation is carried out for an *open-loop* circuit, that is, a circuit in which the entrance conditions to the reactor core are constant in time. This implies that the external loop removes all the heat generated in the core. The core proper is a right circular cylinder of height h and radius ρ_0. All fluid particles and heat are transported around the circuit at a uniform velocity V, and it is assumed that thermal equilibrium and density changes occur instantaneously with variations in temperature. Moreover, it is assumed that no lateral exchange of heat occurs between adjacent fluid streamlines; i.e., the reactor is taken to be a *line* from the standpoint of heat generation in the fuel. Thus all axial paths through the core are identical.

On this basis the appropriate differential equations for the neutron flux ϕ and the energy content H (per unit volume) of the fuel are seen to be[1]

$$-D\nabla^2\phi(\mathbf{r},t) + \Sigma_a\phi(\mathbf{r},t) = \nu\Sigma_f p_{\text{th}} g_{\text{th}}\phi(\mathbf{r},t) - \frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},t) \tag{9.224}$$

$$\frac{\partial}{\partial t}H(\mathbf{r},t) + V\frac{\partial}{\partial z}H(\mathbf{r},t) = \varepsilon\Sigma_f\phi(\mathbf{r},t) \tag{9.225}$$

where ε is the energy release per fission. These two equations are readily reduced to functional relations in z and t by the application of the assumptions listed above and the relation (9.178) for ϕ. By following the procedure outlined between Eqs. (9.178) and (9.184), Eq. (9.224) may be reduced to the form [cf. Eq. (9.183)]

$$l_{\text{th}}\frac{\partial}{\partial t}\varphi(z,t) = \frac{p_L}{B^2}\left[\frac{\partial^2}{\partial z^2}\varphi(z,t) + \mu^2\varphi(z,t)\right] + (k-1)\varphi(z,t) \tag{9.226}\star$$

where we have used the definitions (9.182). Since we have assumed that the system is to be represented by a line reactor, the heat generation and therefore temperature T are functions only of time t and axial position z (measured from entrance to core). Then, after (9.127),

$$H(\mathbf{r},t) \to H(z,t) = c[T(z,t) - T_0(0)] \equiv c\Theta(z,t) \tag{9.227}$$

[1] Energy transport by conduction is neglected; for reference see Goldstein, *op. cit.*, vol. II, chap. XIV.

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where $T_0(0)$ denotes the temperature at the inlet to the core and c is the heat capacity of the fuel. Equation (9.225) may now be written

$$\frac{\partial}{\partial t}\Theta(z,t) + V\frac{\partial}{\partial z}\Theta(z,t) = \frac{\varepsilon\Sigma_f}{c}\varphi(z,t) \qquad 0 \le z \le h \tag{9.228}$$

This equation is to be solved simultaneously with (9.226) and a suitable expression for the multiplication. For the latter we take

$$\delta k = \delta k_0 - \gamma\Theta(z,t) \tag{9.229}$$

In the present calculation we solve the temperature equation (9.228) in terms of the flux $\varphi(z,t)$ and use this result in (9.229) and thence in (9.226). The temperature equation may be solved by the method of Laplace transforms as in the preceding analysis or by the general methods for treating first-order partial differential equations.[1] It may be shown that the application of the latter method leads to the results

$$\Theta(z,t) = \Theta_0(z - Vt) + \frac{\varepsilon\Sigma_f}{c}\int_0^t \varphi(z - Vs,\, t - s)\, ds \qquad z \ge Vt \quad 0 \le t \le \vartheta \tag{9.230}\star$$

$$\Theta(z,t) = \frac{\varepsilon\Sigma_f}{c}\int_0^{z/V} \varphi(z - Vs,\, t - s)\, ds \qquad z < Vt \quad 0 \le t \le \vartheta$$

and

$$\Theta(z,t) = \frac{\varepsilon\Sigma_f}{c}\int_0^{z/V} \varphi(z - Vs,\, t - s)\, ds \qquad t \ge \vartheta \tag{9.231}\star$$

where ϑ is the transit time through the core, and

$$\Theta_0(z) \equiv T_0(z) - T_0(0) \tag{9.232}$$

with $T_0(z)$ as the initial temperature distribution in the core. Note that the solutions for the interval $0 \le t \le \vartheta$ accommodate the fact that Θ is different from zero only in this interval and identically zero for $z \le 0$.

The results (9.230) and (9.231) are also easily deduced by physical arguments. Note that if $t < \vartheta$, i.e., before all the original fuel particles in the core at $t = 0$ are swept out, the temperature (rise) at z and t is simply the initial value at $z - Vt$ plus the temperature rise due to the fission heating experienced by the slug from $z - Vt$ as it travels along the core to point z. Clearly, when $t \ge \vartheta$, all the original fuel in the core has been swept out. In this case, the initial condition to be used with (9.228) is the entrance condition $\Theta(0,t) = 0$; thus the temperature rise experienced by a slug which has reached z at time t is due entirely to the

[1] See, for example, F. B. Hildebrand, "Advanced Calculus for Engineers," pp. 368–378, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1950; and R. Courant and D. Hilbert, "Methoden der Mathematischen Physik," vol. II, chap. 2, especially pp. 51–57, Springer-Verlag, Berlin, Vienna, 1937.

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heat it picks up from fissions between the point $z - Vs$ (where it was located at time $t - s$) and the observation point z (its present location).

The solutions (9.230) and (9.231) may now be applied to (9.226). For this purpose we first express the flux by means of a Fourier series. If we use the form (9.197), then the solution (9.231) for $\Theta(z,t)$ may be written

$$\Theta(z,t) = \frac{\epsilon\Sigma_f}{c} \sum_{n=1}^{\infty} \int_0^{z/V} \phi_n(t - s) Z_n(z - Vs)\, ds \qquad t \geq \vartheta \tag{9.233}$$

The substitution of this expression into (9.229) and thence into (9.226), followed by multiplication by $Z_j(z)$ and integration over the interval $(0,h)$, yields [cf. Eq. (9.198)]

$$\begin{aligned} l_{\text{th}}\phi_j(t) &= \phi_j(t) \left[\frac{p_L}{B^2} (\mu^2 - \mu_j^2) + \delta k_0 \right] \\ &\quad - \frac{\gamma\epsilon\Sigma_f}{cc_j} \sum_{m,n=1}^{\infty} \phi_m(t) \int_0^h Z_j(z)\, Z_m(z)\, dz \int_0^{z/V} \phi_n(t - s)\, Z_n(z - Vs)\, ds \end{aligned} \tag{9.234}$$

where we have used also the orthogonality properties (9.199). If we invert the order of integration and call

$$K_{jmn}(z) \equiv \frac{1}{c_j} \int_z^h Z_j(y)\, Z_m(y)\, Z_n(y - z)\, dy \tag{9.235}$$

Eq. (9.234) becomes

$$\begin{aligned} l_{\text{th}}\phi_j(t) &= \phi_j(t) \left[\frac{p_L}{B^2} (\mu^2 - \mu_j^2) + \delta k_0 \right] \\ &\quad - \frac{\gamma\epsilon\Sigma_f}{c} \sum_{m,n=1}^{\infty} \phi_m(t) \int_0^{\vartheta} \phi_n(t - s) K_{jmn}(Vs)\, ds \end{aligned} \tag{9.236}$$

which applies for $t \geq \vartheta$. In the same way, it may be shown that when $t < \vartheta$ the corresponding relation is

$$\begin{aligned} l_{\text{th}}\phi_j(t) &= \phi_j(t) \left[\frac{p_L}{B^2} (\mu^2 - \mu_j^2) + \delta k_0 \right] \\ &\quad - \frac{\gamma}{c_j} \sum_{m=1}^{\infty} \phi_m(t) \int_0^h \Theta_0(z - Vt)\, Z_j(z)\, Z_m(z)\, dz \\ &\quad - \frac{\gamma\epsilon\Sigma_f}{c} \sum_{m,n=1}^{\infty} \phi_m(t) \left[\int_{Vt}^h Z_j(z)\, Z_m(z)\, dz \int_0^t \phi_n(t - s)\, Z_n(z - Vs)\, ds \right. \\ &\qquad \left. + \int_0^{Vt} Z_j(z)\, Z_m(z)\, dz \int_0^{z/V} \phi_n(t - s)\, Z_n(z - Vs)\, ds \right] \end{aligned}$$

where we have applied the appropriate forms for Θ as indicated in (9.230).

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If we invert the order of the s and z integrations and note that $\Theta_0(z) \equiv 0$ for $z \leq 0$, the above equation reduces to

$$l_{th}\dot{\phi}_j(t) = \phi_j(t)\left[\frac{p_L}{B^2}(\mu^2 - \mu_j^2) + \delta k_0\right] - \frac{\gamma}{c_j}\sum_{m=1}^{\infty}\phi_m(t)\int_{Vt}^{h}\Theta_0(z - Vt)\,Z_j(z)\,Z_m(z)\,dz - \frac{\gamma\varepsilon\Sigma_f}{c}\sum_{m,n=1}^{\infty}\phi_m(t)\int_0^t\phi_n(t-s)K_{jmn}(Vs)\,ds \quad (9.237)$$

Equations (9.236) and (9.237) are an infinite set of integrodifferential equations of the delay type for the Fourier coefficients $\phi_j(t)$. An approximate solution to this set can be obtained when the temperature deviations are small; then, the first term in the Fourier expansion is adequate and we can discard all terms $j > 0$. If we drop the subscripts on ϕ_1 and K_{111} and note that $\mu_1 = \mu$, the set (9.236) and (9.237) reduces to the pair of equations

$$\begin{aligned} l_{th}\dot{\phi}(t) &= \phi(t)\left[\delta k_0 - \frac{\gamma}{c_1}\int_{Vt}^{h}\Theta_0(z - Vt)\,Z_1^2(z)\,dz - \frac{\gamma\varepsilon\Sigma_f}{c}\int_0^t\phi(t-s)\,K(Vs)\,ds\right] \quad \text{for } 0 \leq t \leq \vartheta \quad (a) \\ l_{th}\dot{\phi}(t) &= \phi(t)\left[\delta k_0 - \frac{\gamma\varepsilon\Sigma_f}{c}\int_0^{\vartheta}\phi(t-s)\,K(Vs)\,ds\right] \quad \text{for } t \geq \vartheta \quad (b) \end{aligned} \quad (9.238)$$

where $$K(w) = \frac{2}{h}\int_w^h Z_1^2(z)\,Z_1(z-w)\,dz = \frac{2}{3\pi}\left(1 + \cos\frac{\pi w}{h}\right)^2 \quad (9.239)$$

This result is appropriate to the sinusoidal axial variation in the neutron flux. In the analysis which follows we use, in place of the exact form indicated for $K(w)$, a linear approximation similar to that used for $L(\zeta)$ in (9.206). Thus we introduce the function $K_*(w)$, where

$$K(w) \simeq K_*(w) \equiv 1 - \frac{w}{h} \quad (9.240)$$

If this relation is used in (9.238), along with the substitution (9.132) for $P(t)$, the result is

$$\begin{aligned} l_{th}\dot{P}(t) &= P(t)\left[\delta k_0 - \frac{2\gamma}{h}\int_{Vt}^{h}\Theta_0(z - Vt)\,Z_1^2(z)\,dz - \frac{\gamma}{c}\int_0^t P(t-s)\,K_*(Vs)\,ds\right] \quad 0 \leq t \leq \vartheta \quad (a) \\ l_{th}\dot{P}(t) &= P(t)\left[\delta k_0 - \frac{\gamma}{c}\int_0^{\vartheta}P(t-s)\,K_*(Vs)\,ds\right] \quad t \geq \vartheta \quad (b) \end{aligned} \quad (9.241)$$

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This pair of equations is written more compactly by introducing an *effective temperature* $\theta(t)$, which we define

$$\theta(t) \equiv -\frac{\delta k_0}{\gamma} + \frac{1}{c}\int_0^{\vartheta} P(t-s)\,K(Vs)\,ds \tag{9.242}$$

where $K(Vs)$ is given by (9.239), or approximated by (9.240). Note that

$$\dot{\theta}(t) = \frac{1}{c}\left[P(t)\,K(0) - P(t-\vartheta)\,K(h) + V\int_0^{\vartheta} P(t-s)\,\dot{K}(Vs)\,ds\right] \tag{9.243}$$

If we approximate K by K_*, this equation reduces to

$$\dot{\theta}(t) = \frac{1}{c}\left[P(t) - \frac{1}{\vartheta}\int_0^{\vartheta} P(t-s)\,ds\right] \tag{9.244}$$

This relation and the definition (9.242) may be used to replace Eq. (9.241*b*). The set (9.241) may then be written in terms of the effective temperature $\theta(t)$; summarizing, we have

$$l_{th}\dot{P}(t) = P(t)\left[\delta k_0 - \frac{2\gamma}{h}\int_{Vt}^{h}\Theta_0(z-Vt)\sin^2\left(\frac{\pi z}{h}\right)dz - \frac{\gamma}{c}\int_0^t P(t-s)\left(1-\frac{s}{\vartheta}\right)ds\right] \qquad 0 \le t \le \vartheta \tag{9.245}\star$$

and

$$\begin{aligned} l_{th}\dot{P}(t) &= -\gamma P(t)\,\theta(t) && (a)\\ \theta(t) &= \frac{1}{c}\left[P(t) - \frac{1}{\vartheta}\int_0^{\vartheta} P(t-s)\,ds\right] \qquad t \ge \vartheta && (b) \end{aligned} \tag{9.246}\star$$

Note that the function K_* has been used throughout. It is of interest to mention that the set (9.246) constitute the relations used by Ergen, Weinberg, and Welton in their studies of the kinetics of circulating-fuel reactors. These studies place emphasis on the stability aspects of the nonlinear set (9.246) and include the extension of the present theory to account for the effects of alternate fuel paths through the core with different transit times as well as the effects of delayed neutrons.

Since Eqs. (9.246) apply only for values of t greater than the core transit time, the set may be conveniently called the "asymptotic" solutions for the temperature-dependent kinetics. In the general study of this problem one should properly use first Eq. (9.245), which describes the reactor power transients during the time interval before all the original fuel in the core (at $t = 0$) is swept out. The solution for $t \ge \vartheta$ is then constructed from (9.246), using the results from (9.245) for $t = \vartheta$ as initial conditions.

In order to obtain a solution for the power transients during the time

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interval $(0,\vartheta)$ we require an expression for the steady-state temperature distribution $\Theta_0(z)$. This may be obtained from the steady-state flux distribution $\phi_0(\rho,z)$ at $t = 0$, where

$$\nabla^2\phi_0(\rho,z) + B^2\phi_0(\rho,z) = 0 \qquad \phi_0(\rho,z) \equiv R(\rho)\,Z(z) \tag{9.247}$$

For the case of a sinusoidal axial flux

$$Z(z) = \phi_0 \sin\frac{\pi z}{h} \tag{9.248}$$

The steady-state temperature distribution may be determined from (9.228) by setting the time derivative equal to zero; then

$$\frac{d}{dz}\Theta_0(z) = \frac{\varepsilon\Sigma_f\phi_0}{cV}\sin\frac{\pi z}{h} \tag{9.249}$$

If we call $\varepsilon\Sigma_f\phi_0 \equiv P_0$, and note that $\Theta_0(0) = 0$ [cf. Eq. (9.232)], (9.249) yields

$$\Theta_0(z) = \frac{P_0\vartheta}{c\pi}\left(1 - \cos\frac{\pi z}{h}\right) \tag{9.250}$$

The substitution of this result into (9.245) gives

$$l_{\text{th}}\dot{P}(t) = P(t)\left[\delta k_0 - \frac{\gamma\vartheta P_0}{c\pi}\left(1 - \frac{t}{\vartheta} - \frac{4}{3\pi}\sin\frac{\pi t}{\vartheta} - \frac{1}{6\pi}\sin\frac{2\pi t}{\vartheta}\right) - \frac{\gamma}{c}\int_0^t P(t-s)\left(1 - \frac{s}{\vartheta}\right)ds\right] \qquad 0 \le t \le \vartheta \tag{9.251}$$

This equation is to be solved simultaneously with the pair (9.246).

The expressions (9.251) and (9.246) form a nonlinear set of integro-differential equations in $P(t)$ and $\theta(t)$. An exact solution to this set is not considered here. We seek instead the solution to the corresponding linearized equations. For this purpose we introduce

$$P(t) \equiv P_0[1 + w(t)] \qquad \text{with } P(0) \equiv P_0 \tag{9.252}$$

and require that $w(t) \ll 1$ so that $\ln P(t) \simeq \ln P_0 + w(t)$. With this approximation (9.251) becomes

$$l_{\text{th}}\dot{w}(t) = \delta k_0 - \frac{\gamma\vartheta P_0}{c\pi} - \frac{\gamma\vartheta P_0}{c}\left[\left(1 - \frac{1}{\pi}\right)\frac{t}{\vartheta} - \frac{t^2}{2\vartheta^2} - \frac{4}{3\pi^2}\sin\frac{\pi t}{\vartheta} - \frac{1}{6\pi^2}\sin\frac{2\pi t}{\vartheta}\right] - \frac{\gamma P_0}{c}\int_0^t w(t-s)\left(1 - \frac{s}{\vartheta}\right)ds \qquad 0 \le t \le \vartheta \tag{9.253}$$

The set (9.246) in turn reduces to

$$\begin{aligned} l_{\text{th}}\dot{w}(t) &= -\gamma\theta(t) && (a)\\ \theta(t) &= \frac{P_0}{c}\left[w(t) - \frac{1}{\vartheta}\int_0^{\vartheta} w(t-s)\,ds\right] \qquad t \ge \vartheta && (b) \end{aligned} \tag{9.254}$$

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The solution to Eq. (9.253) must satisfy the initial conditions

$$w(0) = 0 \qquad \dot{w}(0) = -\zeta^2$$
$$\ddot{w}(0) = \frac{\delta k_0}{l_{\text{th}}} - \frac{\vartheta\zeta^2}{\pi} \tag{9.255}$$

where
$$\zeta^2 \equiv \frac{P_0\gamma}{l_{\text{th}}c} \tag{9.256}$$

In obtaining the solution to (9.253) it is more convenient to work with the corresponding ordinary differential equation. It is easily shown that this relation results from two differentiations of Eq. (9.253); thus,

$$\dddot{w}(t) + \zeta^2\ddot{w}(t) - \frac{\zeta^2}{\vartheta}\dot{w}(t) = \zeta^2 f(t) \qquad \text{for } 0 \le t \le \vartheta \tag{9.257}$$

with
$$f(t) \equiv \frac{1}{\vartheta}\left(1 - \frac{4}{3}\sin\frac{\pi t}{\vartheta} - \frac{2}{3}\sin\frac{2\pi t}{\vartheta}\right) \tag{9.258}$$

The corresponding solution for the asymptotic forms (9.254) is obtained by eliminating $\theta(t)$ in (9.254*a*) by the application of (9.254*b*). The result is a third-order equation involving a "delayed" term.

$$\dddot{w}(t) + \zeta^2\ddot{w}(t) + \frac{\zeta^2}{\vartheta}[\dot{w}(t-\vartheta) - \dot{w}(t)] = 0 \qquad \text{for } t \ge \vartheta \tag{9.259}$$

The initial conditions for this equation are obtained from the solution to Eq. (9.257) evaluated at $t = \vartheta$. For this purpose we require the function $w(\vartheta)$ and the derivatives $\dot{w}(\vartheta)$ and $\ddot{w}(\vartheta)$ as computed from (9.257).

We will not concern ourselves further with the detailed solution to this problem but will instead examine briefly some of the physical features of the system described by the pair of equations (9.257) and (9.259).

Consider first the form of the power oscillations. This feature may be studied by introducing the elementary solution

$$w(t) = e^{\alpha_\star t} \tag{9.260}$$

and investigating the properties of the quantity $\alpha_\star$. The substitution of (9.260) into (9.259) yields a transcendental equation for $\alpha_\star$ in terms of the parameters ζ and ϑ; thus

$$\alpha_\star^3 + \zeta^2\alpha_\star + \frac{\zeta^2}{\vartheta}(e^{-\vartheta\alpha_\star} - 1) = 0 \tag{9.261}$$

The roots of this equation are in general complex; therefore set

$$\alpha_\star \equiv x_\star + iy_\star$$

The substitution of this relation, after separation of real and imaginary

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parts, yields the pair of equations for the unknowns $x_\star$ and $y_\star$

$$\begin{aligned} x_\star^3 - 3x_\star y_\star^2 + \zeta^2 x_\star + \frac{\zeta^2}{\vartheta}(e^{-\vartheta x_\star}\cos \vartheta y_\star - 1) &= 0 \\ -y_\star^3 + 3x_\star^2 y_\star + \zeta^2 y_\star - \frac{\zeta^2}{\vartheta} e^{-\vartheta x_\star}\sin \vartheta y_\star &= 0 \end{aligned} \tag{9.262}$$

It can be shown by the application of Nyquist's criterion[1] that Eq. (9.259) has no antidamped solutions since Eq. (9.261), or the pair (9.262), has no roots in the right half of the $\alpha_\star$ plane.[2] Thus we find that $x_\star < 0$, and the general solution $w(t) = \exp(\alpha_\star t)$ represents a damped oscillation. The introduction of a δk_0 into the circulating-fuel reactor (with $\gamma > 0$) results, then, in a series of power excursions which eventually decay, bringing the system to a new steady-state condition.

The solutions to (9.262) may be obtained by the usual iterative techniques. It is of some interest to mention that special roots of (9.261) occur when $\zeta = 2n\pi/\vartheta$ ($n = 1, 2, \ldots$). In these cases, $\alpha_\star$ is a pure imaginary, and the solution to (9.259) is an undamped steady sinusoidal oscillation.

The temperature of the fuel at the core exit is obtained from (9.233). In accord with our previous approximations we use only the first term in the series; then,

$$\Theta(z,t) \simeq \frac{1}{c}\int_0^{z/V} P(t-s)\, Z_1(z - Vs)\, ds \tag{9.263}$$

The linearized approximation of this expression gives for the exit temperature

$$\Theta(h,t) \equiv \Theta_{ex}(t) = \frac{\vartheta P_0}{c}\left[\frac{2}{\pi} + \frac{1}{\vartheta}\int_0^{\vartheta} w(t-s)\sin\frac{\pi s}{\vartheta}\, ds\right] \tag{9.264}$$

The temperature oscillations in this system are quite mild as compared to the power oscillations. This trend is evident from a comparison of $\Theta_{ex}(t)$ and $w(t)$. Since the reactor temperature is given by an integral over some past history of the power, it will necessarily exhibit a much smoother variation in time. Thus even though the power oscillations, as represented by $w(t)$, may be very sharp, the temperature oscillations will be much more gradual and of relatively small amplitude. This property makes the circulating-fuel reactor an extremely attractive device as a source of power. Not only does it indicate that there is little danger to the reactor through excessive temperature variations, but since the

[1] See, for example, H. S. Tsien, "Engineering Cybernetics," McGraw-Hill Book Company, Inc., New York, 1954.

[2] This calculation has been performed by F. G. Prohammer, "Note on the Linear Kinetics of the Circulating Fuel Reactor," Oak Ridge National Laboratory, Y-F10-99, Apr. 22, 1952.

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power extracted from the reactor is determined by the temperature of the fuel, the fluctuations in actual power output will not be severe.

Finally, it should be mentioned that, when a certain δk_0 is introduced into the reactor, the resulting oscillations in temperature decay rapidly; however, the exit temperature eventually stabilizes at some new level. This value can be determined from (9.264), and it is easily shown that as $t \to \infty$, $\Theta_{ex}(t) \to 2\vartheta P_0/\pi c$. This result may be evaluated in terms of the excess multiplication. If we set $\dot{P}(0) = 0$ in Eq. (9.241*b*), then,

$$\delta k_0 = \frac{\gamma P_0}{c} \int_0^{\vartheta} \left(1 - \frac{s}{\vartheta}\right) ds$$

and we obtain $P_0 = 2c\delta k_0/\gamma\vartheta$. Thus

$$\lim_{t \to \infty} \Theta_{ex}(t) = \frac{4\delta k_0}{\pi\gamma} \tag{9.265}$$

which states that if some positive reactivity is added to the system, then because of the negative temperature coefficient of the fuel, the reactor must operate at some higher temperature level which just compensates for this increase.

9.6 Decay of Fission Products and Burnout Poisons

In the time-dependent problems considered so far, no account has been taken of the appearance of fission products or *burnout poisons* which serve as strong "neutron sinks," that is, remove neutrons from the reactor by radiative capture. Nuclei with large neutron-absorption cross sections are created during the fission of a nucleus, either as direct products of the reaction or as intermediate stages in the decay process of other fragments. Burnout poisons are materials which have large absorption cross sections and which are purposely included in a reactor to aid in controlling the neutron population and thereby to aid in limiting the maximum available reactivity in the system. For some time after a new charge of nuclear fuel is added to the reactor, the reactivity of the system may be far beyond the limits which the control system can suppress. Burnout poisons introduced along with the fuel (possibly mixed in with it) can assist the control system in removing excess neutrons. As the fuel is consumed the available reactivity decreases, but so also does the concentration of burnout poisons. By this time, however, the control system alone is able to maintain a steady-state condition in the reactor. The primary function of these poisons is realized, then, during the initial stages of the reactor fuel cycle.

The purpose of the present section is to examine some of the problems associated with the estimation of the concentration of poisons in the reactor as a function of time for periods of operation which are compa-

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rable with the time required to reduce fuel concentrations by sizable fractions of their initial values. The concentration of these nuclei may be included in a study of the complete neutron economy of the reactor system, and suitable adjustments may then be made to maintain the reactivity within prescribed limits.

a. Decay of Fission Products. The distribution of nuclear fragments which are produced by a fission reaction extends over the entire mass scale (see Fig. 1.3). The majority of the fragments are grouped, however, in two regions—mass numbers 95 and 140. The fragments which are of particular interest in the present problem are those which have reasonably large neutron-capture cross sections as well as relatively large percentage yields. We are concerned, therefore, with fragments having large values of Σ_a. The most important of these are Xe^{135}, Sm^{149}, and I^{135}.

We will consider first the concentration of I^{135} and determine its value as a function of time. Iodine 135 is a first-stage decay product of Te^{135}, which is a direct fission product. The tellurium which appears from fission reactions decays by negative beta emission to form I^{135}, which in turn decays to Xe^{135}, again by negative beta emission. The decay process is indicated below, along with the appropriate half-lives:

$$ {}_{52}\mathrm{Te}^{135} \xrightarrow[<2\ \mathrm{min}]{\beta^-} {}_{53}\mathrm{I}^{135} \xrightarrow[6.7\ \mathrm{hr}]{\beta^-} {}_{54}\mathrm{Xe}^{135} \tag{9.266} $$

The yield of Te^{135} is 0.056 atom per fission. We note from the above relation that the decay rate of the Te^{135} is relatively fast; so a reasonable approximation to this reaction would be to assume that I^{135} appears as a direct fission product of yield 0.056 and to ignore entirely the presence of the tellurium. This approximation has some merit since the half-life of Te^{135} is short, relative to the time periods of interest here.

On the basis of this assumption we can write the following balance equation for $I(t)$, the I^{135} concentration:

$$ \dot{I}(t) \simeq -\lambda_I I(t) - \sigma_a^{(I)} I(t)\phi(t) + \mu_I \Sigma_f(t)\phi(t) \tag{9.267} $$

where (refer to Table 2.1 for data to compute λ)

$\lambda_I \equiv$ decay constant for $I^{135} = 0.1037\ \mathrm{hr}^{-1}$

$\sigma_a^{(I)} \equiv$ neutron absorption cross section for I^{135}

$\mu_I \equiv$ atoms of I^{135} per fission (0.056)

$\Sigma_f\phi \equiv$ fissions per unit volume per unit time

and $I(t)$ is given in nuclei per unit volume. It should be recognized that the cross section indicated in this equation must correspond to the neutron energy implied in the flux ϕ. Inasmuch as the largest fission-fragment cross sections occur in the vicinity of thermal energies, we will confine our attention to the effects of these poisons on the thermal-neutron flux. Thus the cross section $\sigma_a^{(I)}$ and those which follow refer to

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the thermal-energy group defined by the reactor temperature. Note, furthermore, that (9.267) includes only the time dependence of the iodine concentration. Spatial variations due to a nonuniform neutron flux are not treated in the calculations which follow. These effects are properly handled by perturbation methods or numerical analyses.

Equation (9.267) states that the concentration of I^{135} nuclei is decreased at the rate $\lambda_I I(t)$ from decay and at the rate $\sigma_a^{(I)} I(t)\phi(t)$ from thermal-neutron capture. Iodine 135 nuclei are produced at the rate $\mu_I \Sigma_f(t)\phi(t)$. The solution to this equation may be written in the general form

$$I(t) = e^{-\int_0^t L(\theta)\,d\theta}\left[\mu_I \int_0^t \Sigma_f(\theta)\,\phi(\theta) e^{\int_0^\theta L(\theta')\,d\theta'}\,d\theta + I(0)\right] \quad (9.268)\star$$

where

$$L(t) \equiv \lambda_I + \sigma_a^{(I)}\phi(t) \quad (9.269)$$

Thus, given the time behavior of the flux, (9.268) yields an explicit relation for the iodine concentration.

The balance relation for the Xe^{135} may now be given in terms of the I^{135} concentration. If $X(t)$ denotes the Xe^{135} concentration, then the appropriate differential equation is

$$\dot{X}(t) = -[\lambda_x + \sigma_a^{(x)}\phi(t)]\,X(t) + \lambda_I I(t) + \mu_x \Sigma_f(t)\,\phi(t) \quad (9.270)$$

where the quantities λ_x, $\sigma_a^{(x)}$, and μ_x are the decay constant, the thermal-absorption cross section, and the fission yield, respectively, for xenon. The value of the yield is $\mu_x \simeq 0.003$ and $\lambda_x = 0.0754\ \text{hr}^{-1}$. Note that the first two terms on the right side of this equation represent the reduction in the Xe^{135} concentration as a result of radioactive decay and neutron capture and that the last two terms give the rate of production from the decay of I^{135} [the parent nucleus, cf Eq. (9.266)] and as a direct product of fission. The general solution to (9.270) is

$$X(t) = e^{-\int_0^t M(\theta)\,d\theta}\left\{\int_0^t [\lambda_I I(\theta) + \mu_x \Sigma_f(\theta)\,\phi(\theta)] e^{\int_0^\theta M(\theta')\,d\theta'}\,d\theta + X(0)\right\} \quad (9.271)\star$$

with

$$M(t) \equiv \lambda_x + \sigma_a^{(x)}\phi(t) \quad (9.272)$$

The third reactor poison considered in this analysis is Sm^{149}, which appears as the radioactive beta-decay product of Pm^{149} (47-hr half-life). The Pm^{149} is itself the beta-decay product of Nd^{149} (1.7-hr half-life), as well as being a direct fission product. The decay process is given by

$$_{60}Nd^{149} \xrightarrow[1.7\ \text{hr}]{\beta^-} {}_{61}Pm^{149} \xrightarrow[47\ \text{hr}]{\beta^-} {}_{62}Sm^{149} \quad (9.273)$$

Because of the relatively short half-life of Nd^{149}, we will assume that Pm^{149} is a direct fission product. The combined yield of Pm^{149} is, then, the fission yield plus the Nd^{149} production. This number is 0.014 atom

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per fission. The balance relations for the two remaining nuclei, promethium $P_m(t)$ and samarium $S_m(t)$, are given by

$$\dot{P}_m(t) = \mu_p \Sigma_f(t)\ \phi(t) - [\lambda_p + \sigma_a^{(p)}\phi(t)]P_m(t) \tag{9.274}$$

$$\dot{S}_m(t) = \lambda_p P_m(t) - \sigma_a^{(s)} S_m(t)\ \phi(t) + \mu_s \Sigma_f(t)\ \phi(t) \tag{9.275}$$

where the index p refers to P_m, and s to S_m. Samarium 149 has a stable nucleus, and therefore its balance equation contains no decay term. The solution to (9.274) is

$$P_m(t) = e^{-\int_0^t G(\theta)\,d\theta}\left[\mu_p \int_0^t \Sigma_f(\theta)\ \phi(\theta) e^{\int_0^\theta G(\theta')\,d\theta'}\,d\theta + P_m(0)\right] \tag{9.276}$$

with

$$G(t) \equiv \lambda_p + \sigma_a^{(p)}\phi(t) \tag{9.277}$$

and the solution for $S_m(t)$ is

$$S_m(t) = e^{-\sigma_a^{(s)}\int_0^t \phi(\theta)\,d\theta}\left\{\int_0^t [\lambda_p P_m(\theta) + \mu_s \Sigma_f(\theta)\ \phi(\theta)] e^{\sigma_a^{(s)}\int_0^\theta \phi(\theta')\,d\theta'}\,d\theta + S_m(0)\right\} \tag{9.278}\star$$

The results (9.268), (9.271), and (9.278) are the general expressions for the time dependence of the principal fission products. Detailed calculations for these functions can be carried out once the $\phi(t)$ and $\Sigma_f(t)$ are specified. Some simple cases are now considered.

b. Steady-state Poison Concentrations. The concentrations of the poisons I^{135}, Xe^{135}, and Sm^{149} in a reactor which is operating at steady state are easily obtained from the original differential equation. At steady state we assume that the flux is stabilized to some value ϕ_0. As a simplification we also take Σ_f to be a constant; that is, the rate of fuel consumption is small, so that Σ_f at any time instant is not much different from its initial value. We find then, on the basis of these assumptions, that the steady-state I^{135} concentration is

$$I_0 = \frac{\mu_I \Sigma_f \phi_0}{\lambda_I + \sigma_a^{(I)}\phi_0} \tag{9.279}$$

The steady-state concentration of Xe^{135} is obtained from (9.270):

$$X_0 = \frac{\lambda_I I_0 + \mu_x \Sigma_f \phi_0}{\lambda_x + \sigma_a^{(x)}\phi_0} \tag{9.280}$$

where I_0 is computed from (9.279). The steady-state concentration of Sm^{149} is derived from (9.274) and (9.275):

$$S_m^{(0)} = \frac{\Sigma_f}{\sigma_a^{(s)}}\left[\frac{\lambda_p \mu_p + \lambda_p \mu_s + \mu_s \sigma_a^{(p)}\phi_0}{\lambda_p + \sigma_a^{(p)}\phi_0}\right] \tag{9.281}$$

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It is of interest to determine the limiting values of these concentrations for arbitrarily large values of the thermal flux. These are

$$\lim_{\phi_0 \to \infty} I_0 = \frac{\mu_I \Sigma_f}{\sigma_a^{(I)}} \qquad \lim_{\phi_0 \to \infty} X_0 = \frac{\mu_x \Sigma_f}{\sigma_a^{(x)}} \qquad \lim_{\phi_0 \to \infty} S_m^{(0)} = \frac{\mu_s \Sigma_f}{\sigma_a^{(s)}} \quad (9.282)$$

Thus the steady-state concentrations of Xe and I increase with increasing flux whereas that of Sm decreases, reaching in the limit the asymptotes given above.

c. Poison Concentrations following Reactor Shutdown. The time-dependent behavior of the poison concentrations in a reactor following shutdown can be obtained from the general results given above if $t = 0$ is taken as the instant of shutdown and the flux is set equal to zero for all subsequent time. In the case of samarium, this procedure easily yields the result

$$S_m(t) = S_m(0) + P_m(0)(1 - e^{-\lambda_p t}) \quad (9.283)$$

where $P_m(0)$ and $S_m(0)$ now represent the concentrations of promethium and samarium in the reactor at shutdown. This result shows, then, that the samarium concentration increases monotonically to the asymptotic value $S_m(0) + P_m(0)$. The buildup is slow, however, since $\lambda_p = 0.01472\ \text{hr}^{-1}$.

Although the postshutdown concentration of samarium can be a serious consideration in some reactor situations, the control of the xenon concentration is a problem of more general importance in all thermal reactors.[1] This problem is especially serious when the I^{135} concentration at shutdown is relatively large. Under these circumstances the Xe^{135} concentration increases rapidly after shutdown and can reach appreciable magnitudes before decaying away. This buildup is important from the standpoint of control, since a large positive reactivity change must be available if it is to be required that there be a restart of the reactor around the time the maximum occurs. Thus special "xenon-override" control rods would be required in a mobile nuclear power plant if short-time startups were to be an essential operational feature.

At the time of shutdown, there exists a particular combination of iodine and xenon concentrations in the reactor. This combination depends, of course, upon the operating history (i.e., flux program) as indicated by Eqs. (9.268) and (9.271). In any case, these can be determined in detail, by measurement if necessary. The concentrations after shutdown are obtained by setting $\phi(t) \equiv 0$ and letting $t = 0$ be the instant of shutdown. Then,

$$I(t) = I_s e^{-\lambda_I t} \quad (9.284)$$

$$X(t;z_s) = \frac{I_s \lambda_I}{\lambda_I - \lambda_x} \left\{ \left[1 + \frac{z_s}{\lambda_I} (\lambda_I - \lambda_x) \right] e^{-\lambda_x t} - e^{-\lambda_I t} \right\} \quad (9.285)$$

[1] Especially solid-fuel reactors.

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where I_s and X_s denote the concentrations of I^{135} and Xe^{135} at shutdown, and

$$z_s \equiv \frac{X_s}{I_s} \tag{9.286}$$

We have introduced the parameter z_s in the symbol for the xenon concentration since the absolute magnitude of the concentration depends upon this ratio, as well as upon the value of I_s.

The expression (9.285) is generally applicable and gives the xenon concentration after shutdown for a specified value of the ratio z_s. For each value of z_s, a separate calculation for $X(t;z_s)$ is required, and each will yield a different time behavior. This complete system of solutions, however, can be combined into a single generalized relation. The appropriate form is obtained by applying a displacement transformation on the time coordinate. Thus we define a new timelike variable θ which is related to t by

$$t \equiv \theta - \theta_\star \tag{9.287}$$

where $\theta_\star$ is to be determined for convenience. The substitution of this relation into (9.285) yields

$$X(t;z_s) \to \hat{X}(\theta;z_s) = \frac{I_s\lambda_I}{\lambda_I - \lambda_x}\left\{\left[1 + \frac{z_s}{\lambda_I}(\lambda_I - \lambda_x)\right]e^{-\lambda_x(\theta-\theta_\star)} - e^{-\lambda_I(\theta-\theta_\star)}\right\} \tag{9.288}$$

We now choose $\theta_\star$ so that $\hat{X}(0;z_s) = 0$. The result is

$$\theta_\star \equiv \frac{\ln\left[1 + (z_s/\lambda_I)(\lambda_I - \lambda_x)\right]}{\lambda_I - \lambda_x} \tag{9.289}$$

The substitution of this expression into (9.288) yields

$$\hat{X}(\theta;z_s) = \frac{I_s\lambda_I}{\lambda_I - \lambda_x}\left[1 + \frac{z_s}{\lambda_I}(\lambda_I - \lambda_x)\right]^{\lambda_I/(\lambda_I-\lambda_x)}(e^{-\lambda_x\theta} - e^{-\lambda_I\theta}) \tag{9.290}$$

This result may be written more simply if we evaluate $\hat{X}$ in terms of its maximum value $\hat{X}(\theta_m;z_s)$. The maximum is seen to occur at

$$\theta_m = \frac{\ln(\lambda_I/\lambda_x)}{\lambda_I - \lambda_x} \tag{9.291}$$

and it follows that

$$\frac{\hat{X}(\theta;z_s)}{\hat{X}(\theta_m;z_s)} = \frac{\lambda_x e^{\lambda_I\theta_m}}{\lambda_I - \lambda_x}(e^{-\lambda_x\theta} - e^{-\lambda_I\theta}) \equiv x(\theta) \tag{9.292}\star$$

This function gives the so-called *universal xenon curve*[1] shown in Fig. 9.14.

[1] A. G. Ward, "A Universal Curve for the Prediction of Xenon Poison after a Reactor Shutdown," Atomic Energy of Canada Limited, CRRP-685, Jan. 25, 1957.

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Note that it represents the xenon concentration after shutdown for the special case when the concentration at shutdown (X_s) is precisely zero and normalized so that the maximum value of the function (9.292) is unity. Thus we find that the displacement transformation (9.287) along with the specification (9.289) transforms the "truncated" function (9.285)

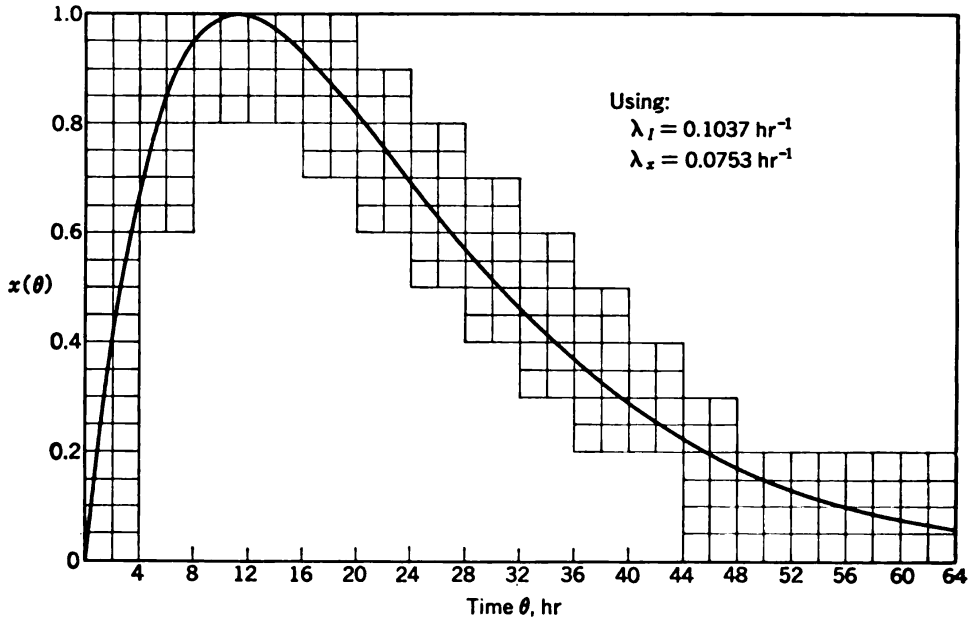

FIG. 9.14 Universal xenon curve.

to a more simple form which passes through the origin. A further observation to be made is that the X and $\hat{X}$ functions are related by

$$\frac{\hat{X}(t;z)}{\hat{X}(t_m;z)} = \frac{X(t;0)}{X(t_m;0)} \tag{9.293}$$

The procedure for determining the xenon concentration in a particular situation from the universal curve is as follows. Given the ratio of xenon to iodine concentration at shutdown z_s, Eq. (9.289) is solved to obtain the "effective" shutdown time $\theta_\star$. These solutions may be read directly from Fig. 9.15. If this value of θ is then marked on the abscissa of Fig. 9.14, the curve to the right of this point gives the relative variation in xenon concentration for all subsequent time. Thus if X_s is the actual concentration at shutdown and $x(\theta_\star)$ the value read from the curve of Fig. 9.14, then $X_s x(\theta)/x(\theta_\star)$ is the concentration at time $\theta - \theta_\star$ *after* shutdown.

In the event that the reactor has been operating at a steady-state flux ϕ_0 for a long time, the iodine and xenon concentrations will have reached the equilibrium values given by Eqs. (9.279) and (9.280). Then,

$$\frac{X_0}{I_0} = \frac{\lambda_I(\mu_I + \mu_x) + \mu_x\sigma_a^{(I)}\phi_0}{\lambda_x\mu_I + \mu_I\sigma_a^{(x)}\phi_0} \simeq \frac{\lambda_I}{\lambda_x + \sigma_a^{(x)}\phi_0} \tag{9.294}$$

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This function is shown in Fig. 9.16. The specific xenon curve for this case is obtained by selecting X_0/I_0 from Fig. 9.16 for the known ϕ_0. This value is then used in Fig. 9.15 to determine the effective shutdown time θ_* and, thence, the xenon curve from Fig. 9.14.

d. Constant-flux Reactor. We consider next the general solutions for the I^{135}, Xe^{135}, and Sm^{149} concentrations given in Sec. 9.6a for the

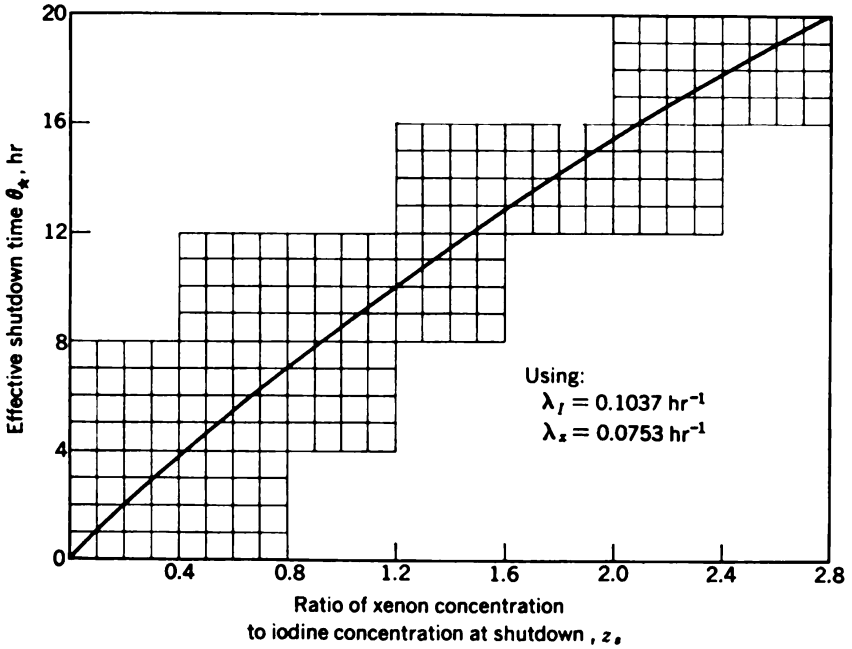

FIG. 9.15 Effective shutdown time as function of xenon-iodine ratio.

case of a reactor which is operating at a fixed flux level ϕ_*. The system is not at steady state, however. For convenience we assume that the reactor is being started up for the first time, so that the initial concentrations $I(0) = X(0) = P_m(0) = S_m(0) \equiv 0$. Equation (9.268) gives the I^{135} concentration in this case to be (again taking Σ_f constant)

$$I(t) = I_*(1 - e^{-L_* t}) \tag{9.295}$$

where $$I_* \equiv \frac{\mu_I \Sigma_f \phi_*}{L_*} \quad \text{and} \quad L_* \equiv \lambda_I + \sigma_a^{(I)} \phi_* \tag{9.296}$$

Note that the I_* corresponds to I_0 of (9.279) if we identify ϕ_* with ϕ_0. For Xe^{135}, we have [from (9.271)],

$$X(t) = X_* \left\{ 1 + \frac{\lambda_I \mu_I}{L_* \mu_x} + \left[\frac{\lambda_I \mu_I}{\mu_x (M_* - L_*)} - 1 \right] e^{-M_* t} - \frac{\lambda_I \mu_I M_* e^{-L_* t}}{L_* \mu_x (M_* - L_*)} \right\} \tag{9.297}$$

with $$X_* \equiv \frac{\mu_x \Sigma_f \phi_*}{M_*} \quad \text{and} \quad M_* \equiv \lambda_x + \sigma_a^{(x)} \phi_* \tag{9.298}$$

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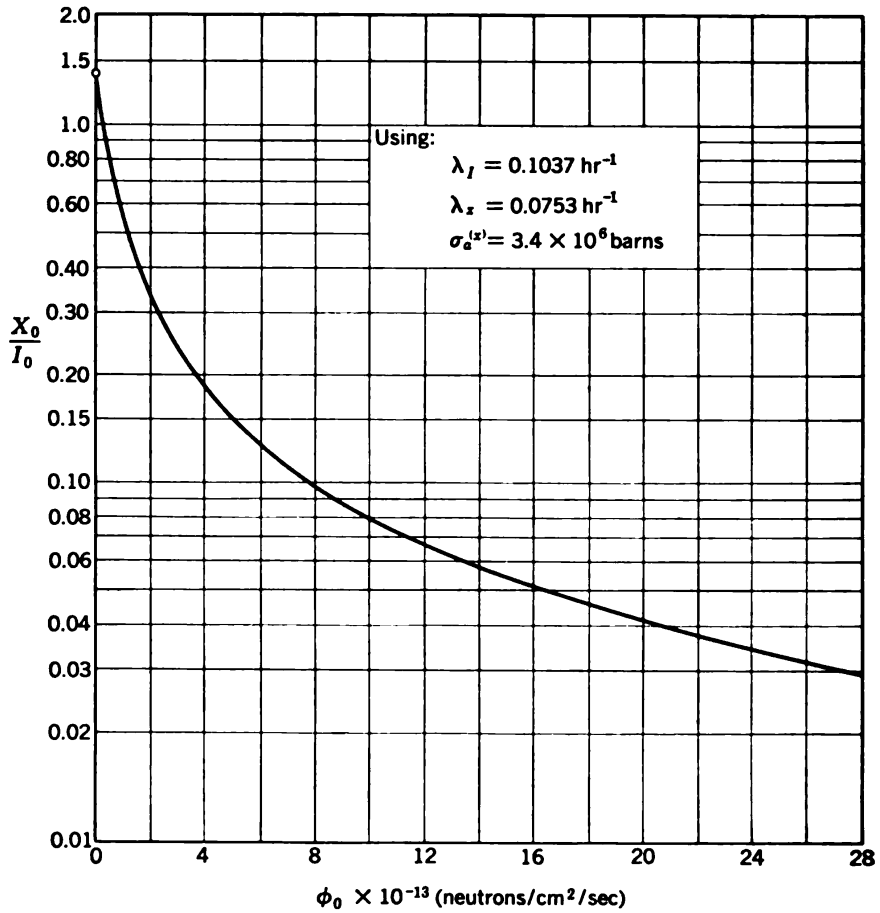

FIG. 9.16 Ratio of equilibrium concentrations of xenon to iodine as function of steady-state flux.

and, finally, for Sm^{149} [from (9.278)],

$$S_m(t) = \lambda_p S_m^* \left[\frac{1 - e^{-\sigma_a^{(s)}\phi_* t}}{\sigma_a^{(s)}\phi_*} + \frac{e^{-\sigma_a^{(s)}\phi_* t} - e^{-G_* t}}{\sigma_a^{(s)}\phi_* - G_*} \right] + \frac{\mu_s \Sigma_f}{\sigma_a^{(s)}} [1 - e^{-\sigma_a^{(s)}\phi_* t}] \tag{9.299}$$

where $$S_m^* \equiv \frac{\mu_p \Sigma_f \phi_*}{G_*} \quad \text{and} \quad G_* \equiv \lambda_p + \sigma_a^{(p)}\phi_* \tag{9.300}$$

These results give the buildup of the fission products for a reactor starting at $t = 0$ and operating thereafter at flux $\phi(t) = \phi_*$. Note that for $t \to \infty$ these solutions reduce to the steady-state values derived in Sec. 9.6b if we take $\phi_* \equiv \phi_0$.

e. Constant-power Reactor. The assumption of a constant flux used in the preceding calculations has merit from a practical viewpoint since reactors designed and operated specifically as research tools are frequently required to maintain uniform flux levels for prolonged periods of time in order to accommodate various irradiation experiments. There are other

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reactor applications, however, in which a constant-flux program would be unsatisfactory. A good example is the use of a reactor as a source of power. It is clear that in this case the principal requirement on the system could very well be that the rate of energy production remain fixed for long periods of time. Thus, the operational condition which would be imposed on the system would not be that the flux remain constant but, instead, that the total number of fissions per unit time remain constant for the specified time period. If this time interval is long in comparison with the operating life of the reactor, then the rate of fuel consumption must be taken into account in computing the total fissions. In this case Σ_f is treated as a time-dependent function. For such systems the appropriate condition to be imposed is

$$P(t) \equiv P_0 = \varepsilon\Sigma_f(t)\,\phi(t) = \varepsilon\sigma_f N_F(0)\,\phi(0) \tag{9.301}$$

where $P(t)$ is the power generated per unit volume by the reactor, P_0 is some specified power level, $N_F(t)$ is the concentration of fuel (nuclei per unit volume), and ε is the energy release per fission.

As noted previously, the calculation of the poison concentrations is dependent on the time behavior of the flux. In the case of the constant-power reactor, the flux must be inversely proportional to the fuel concentration. Thus a complete description of the operating condition of this type of reactor will require, in addition to the differential equations for the poison concentrations, a statement of the mass balance for the fuel nuclei; namely,

$$\dot{N}_F(t) = -\sigma_a^{(F)} N_F(t)\phi(t) \tag{9.302}$$

where $\sigma_a^{(F)}$ denotes the total absorption cross section for the fuel. If the constant-power relation (9.301) is applied to this equation, the solution is seen to be

$$N_F(t) = N_F(0)[1 - \sigma_a^{(F)}\phi(0)t] \tag{9.303}$$

and the flux is given by

$$\phi(t) = \frac{\phi(0)}{1 - \sigma_a^{(F)}\phi(0)t} \tag{9.304}$$

as mentioned above. The substitution of this expression for $\phi(t)$ into the general solutions for the fission-product poisons yields the appropriate time-dependent functions for the constant-power reactor. These results can be expressed in terms of the incomplete gamma function.

It will be convenient for the remainder of the present section to rewrite Eqs. (9.303) and (9.304) in terms of the dimensionless variable z, where

$$z \equiv \frac{t}{t_b} \tag{9.305}$$

The symbol t_b denotes the time it takes for enough fuel to be consumed

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so that the over-all system reactivity is reduced to zero. In a solid-fuel system t_b would be related to the "life" of a fuel element. The quantity $\sigma_a^{(F)}\phi(0)t$ in the expression for the flux may be written in terms of z; thus

$$\sigma_a^{(F)}\phi(0)t = \frac{\sigma_a^{(F)}t_b P_0 z}{\sigma_f N_F(0)\varepsilon} \tag{9.306}$$

It is clear that $P_0 t_b/\varepsilon$ is the total fissions per unit volume which occur in time t_b; but this may also be given by

$$(\text{Fuel nuclei destroyed})\frac{\sigma_f}{\sigma_a^{(F)}} = [N_F(0) - N_F(t_b)]\frac{\sigma_f}{\sigma_a^{(F)}}$$

If we call β_F the burnup fraction of the fuel, where

$$\beta_F = \frac{N_F(0) - N_F(t_b)}{N_F(0)} \tag{9.307}$$

then

$$\sigma_a^{(F)}\phi(0)t = \beta_F z \tag{9.308}$$

The fuel concentration and the flux, when written in these terms, take the form

$$N_F(z) = N_F(0)(1 - \beta_F z) \qquad \phi(z) = \frac{\phi(0)}{1 - \beta_F z} \tag{9.309}\star$$

f. Secondary Fission Fragments. In addition to the primary products I^{135}, Xe^{135}, and Sm^{149}, a fission reaction yields various secondary fragments which, experience has shown, can be treated in terms of a single effective absorption cross section. Some detailed information as to the nature of these fragments and the sequence of their appearance following a fission is available in the literature.[1] It will suffice for the present treatment to consider an elementary approach for estimating the concentration of these secondary fragments. The fundamental assumption will be that all the secondary poison nuclei which appear can be regarded as a *single* "composite" nucleus with an absorption cross section $\sigma^{(FP)}$. If we call $N_{FP}(t)$ the concentration of this composite (or effective) nucleus, then the balance relation is clearly

$$\dot{N}_{FP}(t) = \sigma_f N_F(t)\,\phi(t) \tag{9.310}$$

The cross section $\sigma^{(FP)}$ must be selected from experience. We have omitted a term for removal by neutron capture in Eq. (9.310) on the basis that the cross section $\sigma^{(FP)}$ represents a time-averaged quantity which accounts for both the continuous production and loss of secondary

[1] W. B. Lewis, "Reactivity Change Expected and Observed in the Long Irradiation of Natural Uranium in the NRX Reactor," DR-21, Feb. 4, 1952; L. D. P. King, "Los Alamos Power Reactor Experiment and Its Associated Hazards," LAMS-1611, Dec. 2, 1953; J. Halperin and R. W. Stoughton, "Some Effects of Transmutation Products on U^{233} Breeder Pile Operation," ORNL-1368, Sept. 29, 1952.

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poison nuclei. The actual buildup of such fragments during reactor operation is discussed in the reference reports.

The concentration $N_{FP}(t)$ is easily determined for the two flux programs previously considered. For the case of *constant power*, we use (9.301) for the total fissions and obtain

$$\frac{N_{FP}(z)}{N_F(0)} = \left[\frac{\sigma_f}{\sigma_a^{(F)}}\right] \beta_F z \tag{9.311}$$

where we have used the relation $\beta_F = \sigma_a^{(F)}\phi(0)t_0$, which follows from (9.307). For the case of constant flux ϕ_*, the fuel-concentration equation (9.302) gives

$$N_F(t) = N_F(0)e^{-\sigma_a^{(F)}\phi_* t} \tag{9.312}$$

Substitution into (9.310) yields [using (9.307)]

$$\frac{N_{FP}(z)}{N_F(0)} = \frac{\sigma_f}{\sigma_a^{(F)}}[1 - (1 - \beta_F)^z] \tag{9.313}$$

g. Burnout Poisons. The use of burnout poisons was discussed in the introduction to the present section. In the development which follows we compute the nuclear concentration for some arbitrary poison $N_j(t)$ for the cases of constant flux and constant power. The results for the case of constant power are then used in a sample calculation to demonstrate the use of a burnout poison in controlling the reactivity of a reactor.

The mass balance for the burnout-poison nucleus is given by the equation

$$\dot{N}_j(t) = -\sigma_a^{(j)} N_j(t)\ \phi(t) \tag{9.314}$$

where $\sigma_a^{(j)}$ denotes the neutron-absorption cross section of the poison j. The solutions for $N_j(t)$ for the two cases mentioned are easily shown to be

$$N_j(z) = N_j(0)(1 - \beta_F z)^{s_j} \quad \text{for constant power} \tag{9.315}$$

$$N_j(z) = N_j(0)(1 - \beta_F)^{s_j z} \quad \text{for constant flux} \tag{9.316}$$

where

$$s_j \equiv \frac{\sigma_a^{(j)}}{\sigma_a^{(F)}} \tag{9.317}$$

The result (9.315) is now applied to the criticality calculation of a bare homogeneous reactor operating at constant power. The appropriate relation for the multiplication constant is

$$k = \eta f \epsilon p_{th} g_{th} p_{NL} \tag{6.81}$$

We use this relation to compute the effect on k caused by introducing burnout poisons which have the time dependence given by (9.315). For convenience in computation, we choose $\epsilon p_{th} \equiv 1$, so that (6.81) reduces to

$$k = \eta f g_{th} p_{NL} \tag{9.318}$$

In these calculations we are concerned primarily with time-dependent

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phenomena, and, strictly speaking, the relations (6.81) and (9.318) are meaningful only when $k \equiv 1$, that is, for steady-state operation. Thus if we speak of $k \neq 1$, then the flux is experiencing rapid variations in time, so that the reactor either is shutting down or is supercritical. Nevertheless, it will be useful in this analysis to think in terms of a time-dependent multiplication $k(t)$ which will describe the influence of the various poisons and the fuel on the reactivity during the operating life of the system. Actually, $k(t) \equiv 1$ all through this period, but this is accomplished only by the continual action of a control system; $k(t)$ represents, then, the available multiplication at any given instant and is the amount which the control system must counter in order to maintain the reactor at steady state. This concept of a $k \neq 1$ was previously treated by introducing a ν_c so that $k \equiv \nu/\nu_c$. Thus ν_c is a fictitious number of neutrons which must be produced per fission in order to maintain the system at steady state. In the present application we may think of a corresponding $\nu_c(t)$ which specifies the neutron yield per fission at each instant during the life of the reactor so that the system remains just critical. The appropriate expression for k is then (for g_{th} and η independent of time)

$$k(t) = \eta g_{th} f(t)\, p_{NL}(t) \tag{9.319}$$

If we call

$$a(t) \equiv \frac{\Sigma_a^{(x)}(t)}{\Sigma_a^{(F)}(t)} \tag{9.320}$$

where $\Sigma_a^{(x)}$ represents the absorption cross section of all reactor materials other than fuel, then (for D independent of time)

$$\left.\begin{aligned} f(t) &= \frac{1}{1 + a(t)} \\ p_{NL}(t) &= \frac{1}{1 + B^2 L^2(t)} \end{aligned}\right\} \quad \text{with } L^2(t) = \frac{D}{\Sigma_a^{(F)}(t)[1 + a(t)]} \tag{9.321}$$

It will be convenient to separate $\Sigma_a^{(x)}$ into a component $\Sigma_a^{(c)}$ for materials whose cross sections do not vary with time and component $\Sigma_a^{(b)}(t)$ for those which do; thus

$$\Sigma_a^{(x)}(t) \equiv \Sigma_a^{(c)} + \Sigma_a^{(b)}(t) \tag{9.322}$$

For the present calculation we include in $\Sigma_a^{(b)}(t)$ only the burnout poisons and omit the production of fission fragments.

The fuel concentration for this system is given by (9.309) and the poison concentration by (9.315). If these expressions are used in (9.320), we obtain

$$a(z) = \frac{s_c n_c}{1 - \beta_F z} + \sum_{j=1}^{N} s_j n_j (1 - \beta_F z)^{a_j - 1} \tag{9.323}$$

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where we have assumed that the reactor contains N burnout poisons, $j = 1, 2, \ldots, N$. The symbol s_c is defined according to (9.317), and n_j is the loading fraction of poison j:

$$n_j \equiv \frac{N_j(0)}{N_F(0)} \tag{9.324}$$

The result (9.323) may be used in (9.319):

$$k(z) = \frac{\eta g_{th} p_{NL}(z)}{1 + a(z)} \tag{9.325}$$

The reactivity ω is, then,

$$\omega(z) = \frac{k(z) - 1}{k(z)} = 1 - \frac{1}{\eta g_{th}} \left[1 + a(z) + \frac{B^2 D}{\Sigma_a^{(F)}(0)(1 - \beta_F z)} \right] \tag{9.326}\star$$

where z is the time variable given as a fraction of the operating life of the fuel elements and β_F is the burnup fraction of the fuel. It is possible, by making a suitable choice of the loading fractions of the poisons

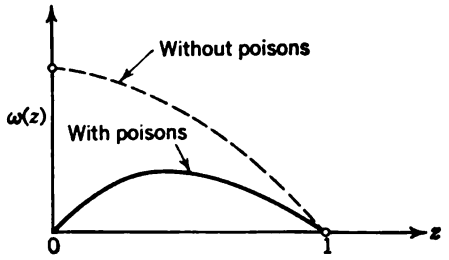

FIG. 9.17 Reactivity change with and without burnout poisons.

n_j and their types, represented by the cross section ratios s_j, to control the reactivity to variations which lie within the capabilities of the control system. If no poison were added to the constant-power reactor, the reactivity would vary as shown in Fig. 9.17. The initial reactivity $\omega(0)$, here, may be far beyond the magnitudes which can be suppressed by the controls. Thus by introducing sufficient poison to make $\omega(0) \equiv 0$, the reactivity can be made to take the form shown by the solid-line curve in the figure. Note that the maximum excursion in $\omega(z)$ has been diminished and might be brought within the range of the controls. In some situations the maximum rise in reactivity may still be beyond the capabilities of the controls; such is sometimes the case when B^{10}, which has a large σ_a, is used. Although enough B^{10} can be added to suppress the initially large amount of reactivity available, it "burns out" too fast, and in a short time $\omega(z)$ will rise to uncontrollable magnitudes. In these cases it is possible to use a combination of B^{10} and some other poison (e.g., Li^6) which does not have so large a cross section. Now the combined initial loadings of these two poisons will suppress the large $\omega(0)$, and yet a sufficient amount of the lower cross section poison will always

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remain to keep the reactivity within controllable bounds during the remaining life of the system. In carrying out these calculations, it is useful to impose the following conditions on $\omega(z)$:

$$\omega(0) = 0 \qquad \omega(1) = 0 \tag{9.327}$$

These conditions will yield two simultaneous equations for the loading fractions n_j of the two poisons.

PROBLEMS

9.1 Consider a bare thermal reactor [described by the constants D, B^2, $\Sigma_a(0)$, $\Sigma_f(0)$, p_{th}, and τ_{th}] which is operating at steady state. At some time instant, say $t = 0$, the physical state of the system is caused to vary in time, so that the macroscopic cross sections become functions of time. Such behavior can be achieved, for example, by changing the pressure on a compressible fluid fuel.

a. Derive the expression for the thermal flux $\phi(\mathbf{r},t)$ for this system on the basis of the following model: (1) $\Sigma_a(t) = \Sigma_a(0)(1 + a \sin wt)$, $\Sigma_f(t) = \Sigma_f(0)(1 + b \sin wt)$, and $D(t) \equiv D$; (2) neglect delayed neutrons; (3) neglect temperature effects on system; (4) include $g_{th} \equiv e^{-B^2\tau_{th}}$ and p_{th}; (5) assume that both g_{th} and p_{th} are time-independent quantities.

b. Determine the function $\delta k(t) \equiv k(t) - 1$ in terms of the steady-state constants of the system. At what values of wt does $\delta k(t)$ attain its maximum value?

c. Sketch the functions $\delta k(t)$ and $\phi(\mathbf{r},t)$ in terms of the variable wt.

d. Find an expression for $\phi(\mathbf{r},t)$ in terms of the mean lifetime $l \equiv \{v[DB^2 + \Sigma_a(0)]\}^{-1}$ and the maximum positive value of $\delta k(t)$.

e. Determine the conditions on the frequency w such that $\phi_{max} \geq c\phi_{min}$, where c is an arbitrary constant.

9.2 Determine the one-velocity flux distribution $\phi(x,t)$ in a nonmultiplying slab of width $2a$ for which the diffusion medium is characterized by the one-velocity constants Σ_a and D. Apply the condition that the flux vanish at the geometric boundaries of the medium and that the initial condition is

$$\phi(x,0^+) = 0$$

At $t = 0$, a source function $S(x,t)$ is introduced:

$$S(x,t) \equiv \phi_0 \cos\left(\frac{\pi x}{2a}\right)(1 + \sin wt)$$

9.3 The flux in a bare homogeneous one-velocity reactor is time dependent. Find this time behavior under the following two (different) physical assumptions: (*a*) all neutrons are prompt; (*b*) a fraction β of the neutrons from fission are delayed time t_0. Assume that the form

$$\phi(t) = Ae^{t/T}$$

applies, and find values of T which allow a solution to each case. Compare these values for T, using the form (where a and b refer to the assumptions given above)

$$T_b = T_a(1 + \alpha)$$

and give an explicit form for α. In each case only *one* time-dependent equation is needed; do *not* consider the precursors of the delayed neutrons. In model b assume that $t_0/T_b \ll 1$ in order to solve the equation for T_b.

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9.4 A bare homogeneous thermal reactor is critical and operating in steady state with flux $\phi_0(\mathbf{r})$. At time $t = 0$, a small positive reactivity is introduced and held constant until $t = t_1$. At time $t = t_1$ the reactivity is reduced to zero.

a. By using one group of delayed neutrons and assuming that t_1 is sufficiently long that the reactor flux is rising with the stable reactor period α_1 at $t = t_1$, find the expression for the time dependence of the neutron flux and the concentration of the precursor for all $t \geq 0$.

b. What is the final asymptotic value of the flux long after the reactivity is reduced to zero?

c. Compute and plot the flux and precursor concentration as a function of time using the following data:

$$\begin{aligned} \beta &= 0.0076 \\ \omega &= 0.001 \\ \lambda &= 0.08 \text{ sec}^{-1} \\ t_1 &= 10 \text{ sec} \\ l_{\text{th}} &= 0.001 \text{ sec} \end{aligned}$$

9.5 A bare homogeneous thermal reactor is operating in steady state up to time $t = 0$, when some additional multiplication δk_0 is introduced into the system. Examine the time behavior of this system on the basis of the following model:

(1) Delayed neutrons (fraction β) are present but have infinite delay time;

(2) The added multiplication may be expressed by the relation

$$k(t) \equiv \delta k_0 - \gamma\Theta(t)$$

where the temperature rise above the steady-state value $T(0) = T_0$ is

$$\Theta(t) \equiv T(t) - T(0)$$

(3) The power removal from the reactor is given by

$$\mathcal{P}(t) = \mathcal{P}_0[1 + \mu\Theta(t)] \qquad \text{for } \mu \geq 0$$

(4) Solutions of the form $\mathcal{P}_0 e^{w(t)}$ describe the power generated in the reactor, and $w(t) \ll 1$.

a. Write down an appropriate set of differential equations and initial conditions which describe the reactor power $P(t)$ and the temperature rise $\Theta(t)$ for all $t \geq 0$. Use the fact that $P(t)$ may be linearized.

b. Solve the equations of part *a* and show that the results can be written in the form

$$\Theta(t) = \frac{\delta k_0}{\gamma + \beta\mu}\left[1 - e^{-\chi t}\left(\frac{\chi}{\psi}\sin \psi t + \cos \psi t\right)\right]$$

$$w(t) = \frac{\delta k_0}{l_{\text{th}}}\left\{\frac{\mu l_{\text{th}}}{\gamma + \beta\mu} + e^{-\chi t}\left[\left(1 - \frac{\mu\chi l_{\text{th}}}{\gamma + \beta\mu}\right)\frac{\sin \psi t}{\psi} - \frac{\mu l_{\text{th}} \cos \psi t}{\gamma + \beta\mu}\right]\right\}$$

where

$$\chi \equiv \frac{1}{2}\left(\frac{\mu\mathcal{P}_0}{c} + \frac{\beta}{l_{\text{th}}}\right) \qquad \text{and} \qquad \psi \equiv \left[\frac{\mathcal{P}_0}{c l_{\text{th}}}(\gamma + \beta\mu) - \chi^2\right]^{\frac{1}{2}} \qquad \text{real}$$

and l_{th} is the neutron lifetime in a finite reactor and c is the specific heat of a homogenized sample of the reactor material.

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CHAPTER 10

HETEROGENEOUS REACTORS

10.1 Introduction and Survey

a. Merits of Heterogeneous Fuel-moderator Configurations. Some remarks were made in Sec. 1.2 about the advantages to be gained in using heterogeneous arrangements and the basis for the choice of this configuration for the earliest reactor designs. Aside from the physical separation of fuel and moderator, the most important feature of such systems is the fact that they are nearly completely "thermal" (i.e., at least 95 per cent of the fissions are caused by thermal neutrons). These unique characteristics lead to significant improvements in some of the factors which affect the multiplication constant. By establishing a thermal system it is possible to exploit the relatively large fission cross sections of the nuclear fuels in the low-energy range, and by separating the fuel and moderator, such quantities as the resonance-escape probability and the fast effect can be measurably increased.

These advantages were well appreciated in the early days of reactor technology, and many experiments and analytical studies were devised to explore them. It is important to recognize that it was only because of the superior nuclear characteristic of heterogeneous configurations that it was possible to sustain a nuclear chain reaction at all. As noted earlier, the only abundant and efficient moderator materials available in the early days were graphite and water; but even graphite, with its significantly smaller absorption cross section, is incapable of supporting a chain reaction with natural uranium in a homogeneous arrangement. This fact is easily demonstrated. To begin with we note that if a finite system is to be made critical then $k_\infty > 1$. Now $k_\infty = \eta\epsilon f p_{\text{th}}$, and for natural uranium $\eta = 1.33$. If we take $\epsilon = 1.05$ (a relatively large value), then clearly we must have $f p_{\text{th}} > 0.73$. Figure 10.1 shows a plot of the thermal utilization and the resonance-escape probability for homogeneous mixtures of natural uranium and graphite. It is seen that the product of these two quantities (the broken-line curve) is at most 0.59, which is well below the minimum value estimated above. Thus the homogeneous system with an optimum ratio of moderator to fuel is still subcritical. A heterogeneous arrangement of these materials, however, can be made into a critical assembly. This is achieved in main part through the

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increase in the resonance-escape probability which is realized when the fuel is lumped instead of being uniformly distributed.

The properties of the graphite systems are representative of the general trends to be expected with other moderators. Except for heavy water, homogeneous mixtures of moderator and natural uranium are subcritical; however, critical assemblies can be obtained by going to heterogeneous arrangements. A convenient criterion for selecting a heterogeneous configuration is the maximization of the multiplication

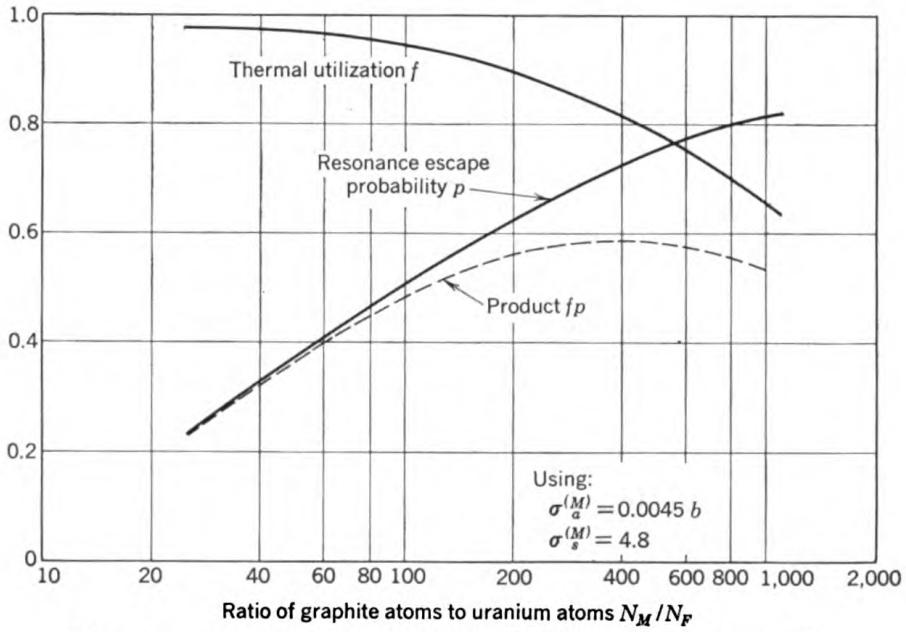

FIG. 10.1 Homogeneous system of natural uranium and graphite.

constant k_∞. Thus the bulk of the effort required in the analysis of heterogeneous-reactor designs is directed toward the determination of the various factors which enter into the criticality relation, particularly as given by the four-factor formula. It is interesting to note in passing that much of the terminology and notation of reactor physics presently in use originated with the early studies of heterogeneous systems. The four-factor formula was developed to describe the nuclear characteristics of natural uranium lattices and is best suited to reactors with low fuel concentration. However, as already demonstrated in the homogeneous-reactor calculations of the preceding chapters, this general relationship is easily adapted to systems with high enrichments of fissionable materials.

When the system of interest is to be charged with natural uranium, each of the factors in the criticality relation must be computed with a fair degree of accuracy since for these systems, even with the most favorable arrangement of fuel and moderator, the resulting multiplication

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constant exceeds unity by only a few per cent. Under these circumstances small errors in the estimation of the quantities f, p_{th}, ϵ, and η are greatly magnified and can lead to large errors[1] in the estimation of quantities dependent on $\delta k \equiv k - 1$, for example, the critical mass.[2] Once these factors have been determined to the desired accuracy, the k_∞ for the system is known and suitable expressions can be obtained for the effective diffusion length and migration area of the lattice and, thence, estimates of the over-all size of the core and reflector configuration.

The present treatment of the subject of heterogeneity begins with an elementary discussion organized in the form of a summary. This portion of the treatment is given in the present section. Attention is directed here to the delineation of the essential features of heterogeneous systems, the over-all effects of "lumping," and to simplified methods for estimating the characteristics of such systems. Whenever possible, experimental results and formulas are presented which can be used directly in computations. The results and discussion given in this section should suffice for a survey-type treatment of the general subject.

In subsequent sections of this chapter we examine in greater detail various facets of the problems mentioned above. For these portions of the treatment we delve more deeply into the theoretical basis for several of the semiempirical relations customarily employed in lattice calculations.

b. Methods of Analysis. The first step in the analysis of a heterogeneous-reactor configuration is the evaluation of the various factors in k_∞. In order to obtain a reliable estimate of these quantities it is essential that one take into account the actual heterogeneous arrangement of the core materials; in point of fact, most any elementary mathematical model which does so will yield superior results to those based on an equivalent homogeneous system. Two analytical models devised for this purpose have gained general acceptance. The first of these is the so-called *unit-cell model* of Wigner and Seitz, and the other is the *heterogeneous method* of Feinberg and Galanin.

The essential feature of the Wigner-Seitz model[3] is that it represents a typical lattice element of a heterogeneous array by an equivalent unit cell. For example, if a heterogeneous core were to be composed of cylindrical fuel rods placed in a square lattice of spacing a, then a reasonable choice of the unit cell would be the circle of radius $a/\sqrt{\pi}$ concentric with a rod. In performing a calculation based on this model the usual practice is to specify that the net neutron current at the boundary of the

[1] It is easily shown that in first approximation the per cent error in δk is inversely proportional to δk and directly proportional to the sum of the errors in the factors f, p_{th}, ϵ, and η.

[2] See, for example, Eq. (6.191) and subsequent discussion.

[3] Originally conceived for the calculation of wave functions of crystal lattices.

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cell be zero. The neutron distribution in the unit cell is then obtained with the aid of diffusion theory or the first-flight transport model of Sec. 5.8a, depending on the "size" of the cell. Having determined both the energy spectrum and the spatial distribution of the neutron population in the cell, the thermal utilization and resonance-escape probabilities may then be computed and, finally, the multiplication constant for the reactor. Most of the calculations carried out in the present treatment of heterogeneous systems are based on this method.

The method of Feinberg and Galanin, on the other hand, makes a direct assault on the problem of computing the multiplication constant of a given configuration by treating it as an integral system. In this so-called "heterogeneous method" the fuel-rod lattice is pictured as a collection of discrete line (or point) neutron sources and sinks. A set of neutron-balance equations may be written for this system, and the solution of the set leads to the criticality condition for the reactor. This general approach to the over-all problem is both elegant and precise; moreover, it offers great flexibility in treating complex heterogeneous arrays involving various lattice geometries, several degrees of periodicity, and irregularities in rod size and spacing. The method is also useful for analyzing control-rod effectiveness, the influence of voids and cavities, as well as some aspects of the problem of fuel storage. It should be mentioned, too, that the mathematical treatment is somewhat more complicated than that involved in the unit-cell method, and that even though the method possesses the many attributes mentioned above, its greatest usefulness is realized when applied to small heterogeneous arrays and irregular configurations. The analysis of large regular geometries by this method offers no significant advantage over the Wigner-Seitz model.

The last section of this chapter is devoted to a general review and discussion of this method. Although several of the more important ideas involved are considered in some detail, the method as developed by Feinberg and Galanin is not presented here in all its generality. The present treatment is intended primarily as an introduction. This section is concluded with an application to reactors with small number of fuel lumps, followed by a numerical example.

c. Characteristic Parameters. The most important parameters involved in the description of a heterogeneous system are the four factors in k_∞, the effective diffusion length, and the age; the latter two are of course required in the computation of the neutron leakage. In the presentation which follows we consider some elementary methods for calculating f, p_{th}, and L based on the Wigner-Seitz model. These parameters are considered first since the transition to heterogeneous arrangements affects these more than the others of the group. Although the fast effect is also measurably improved in going to heterogeneous configura-

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tions, it plays a lesser role than the others.[1] A discussion of ϵ and M, the migration area, is presented in later sections.

1. *Thermal Utilization f.* If the fuel and moderator are distributed uniformly throughout a system, then it is clear that the nuclei of both materials will be exposed to the same neutron flux at all energies. If, on the other hand, the two materials are physically separated, then the neutron density at a given energy will differ in the two media. In the case of the thermal flux, there is, in fact, a depression in the neutron population throughout the fuel region. This behavior leads to several significant differences in the characteristics of heterogeneous and homogeneous systems.

One factor which is extremely sensitive to these differences in flux distribution is the thermal utilization. The important trends can be readily displayed with the aid of the unit-cell model. Let V_{cell} denote the volume of such a cell and V_F and V_M the volumes of the fuel and moderator regions, respectively, in this cell. Then if $\phi_i(\mathbf{r})$ denotes the thermal flux in region i of the cell, we can define the thermal utilization for the heterogeneous system as

$$f_{\text{het}} \equiv \frac{\int_{V_F} \Sigma_a^{(F)} \phi_F(\mathbf{r})\, d\mathbf{r}}{\int_{V_F} \Sigma_a^{(F)} \phi_F(\mathbf{r})\, d\mathbf{r} + \int_{V_M} \Sigma_a^{(M)} \phi_M(\mathbf{r})\, d\mathbf{r}} \tag{10.1}$$

If the fuel and moderator regions are separately homogeneous and uniform, then the $\Sigma_a^{(i)} \equiv N_i \sigma_a^{(i)}$ are constants. (Here N_i denotes the actual nuclear density of the pure material i.) The above expression may then be written

$$f_{\text{het}} = \frac{\Sigma_a^{(F)} V_F \bar{\phi}_F}{\Sigma_a^{(F)} V_F \bar{\phi}_F + \Sigma_a^{(M)} V_M \bar{\phi}_M} \tag{10.2}$$

where the average fluxes $\bar{\phi}_i$ are computed from

$$\bar{\phi}_i \equiv \frac{1}{V_i} \int_{V_i} \phi_i(\mathbf{r})\, d\mathbf{r} \tag{10.3}$$

We introduce next the ratio of the average fluxes

$$\zeta \equiv \frac{\bar{\phi}_M}{\bar{\phi}_F} \tag{10.4}\star$$

which we call the *disadvantage factor*, since the fuel is at a "disadvantage" if the average flux throughout its region is less than that throughout the moderator, i.e., if $\zeta > 1$. With this definition, (10.2) reduces to

$$f_{\text{het}} \equiv \frac{\Sigma_a^{(F)}}{\Sigma_a^{(F)} + \Sigma_a^{(M)} \zeta (V_M/V_F)} \tag{10.5}\star$$

[1] Note that, although the major changes appear in f, p_{th}, and L, k_∞ is still highly sensitive to ϵ, by the arguments presented in Sec. 10.1a.

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Some appreciation of the significance of this result may be had from a comparison of this heterogeneous system to an equivalent homogeneous system. We define such a system as one which has the same volume V_{cell} as the unit cell and moreover contains the same total mass of fuel and moderator as the heterogeneous system, but now distributed uniformly throughout V_{cell}. Then $V_{\text{cell}} \equiv V_F + V_M$, and if we call $N_i^{(\text{hom})}$ the nuclear density of material i distributed uniformly over V_{cell}, then clearly

$$N_i^{(\text{hom})} V_{\text{cell}} \equiv N_i V_i \tag{10.6}$$

With this definition we can write the thermal utilization of the homogeneous system in the form

$$f_{\text{hom}} \equiv \frac{N_F^{(\text{hom})} \sigma_a^{(F)}}{N_F^{(\text{hom})} \sigma_a^{(F)} + N_M^{(\text{hom})} \sigma_a^{(M)}} \tag{10.7}$$

where $\sigma_a^{(i)}$ is the actual microscopic cross section of material i. The application of (10.6) yields

$$f_{\text{hom}} = \frac{\Sigma_a^{(F)}}{\Sigma_a^{(F)} + \Sigma_a^{(M)}(V_M/V_F)} \tag{10.8}$$

By comparison with (10.5) we note that the two expressions for f have the same form, except for the factor ζ which appears in f_{het}. Now as a rule $\zeta \geq 1$, i.e., the flux in the fuel is depressed; thus we find that

$$f_{\text{het}} \leq f_{\text{hom}} \tag{10.9}$$

We conclude therefore that, in going from a homogeneous system to a heterogeneous, keeping the mass of each material constant, the thermal utilization is *reduced*.

Another way of thinking about this behavior is to assign "effective" cross sections to the materials in the heterogeneous system such that

$$[\sigma_a^{(F)}]_{\text{eff}} \equiv \frac{\sigma_a^{(F)}}{\zeta} \qquad \text{and} \qquad [\sigma_a^{(M)}]_{\text{eff}} \equiv \sigma_a^{(M)} \tag{10.10}$$

Then, f_{het} has a form identical to f_{hom}, but now expressed in terms of $[\sigma_a^{(F)}]_{\text{eff}}$, which is generally less than $\sigma_a^{(F)}$. Thus the equivalent homogeneous system utilizes a fuel with a smaller absorption cross section than the actual heterogeneous cell and, consequently, exhibits a smaller value of f.

The disadvantage factor is a function of the fuel and moderator cross sections as well as of the unit-cell dimensions. As the scale of the heterogeneity decreases, the homogeneous case is approached. In the limit, as the scale is reduced to zero, the flux in the fuel region approaches that in the moderator region, ζ approaches unity, and the effective cross section for the fuel approaches the actual cross section. Various methods for the computation of this quantity are discussed in the next section.

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2. *Diffusion Length.* The effect of lumping the materials in a reactor on the thermal nonleakage probability is difficult to compute accurately. Certainly, the most important quantity for consideration is the diffusion length. This calculation could be approached from the viewpoint of the relation between the diffusion length and the mean square distance from source point to point of absorption, as in Eq. (5.228) for the homogeneous system. In a heterogeneous system such a calculation would involve finding the flux, over the medium, arising from sources at various positions, determining the mean square distance to absorption for each source point, then averaging over source positions.[1] This is a complicated procedure and does not easily yield useful results. Further, it should be noted that there would be various directional effects in a heterogeneous array because of nonsymmetries in the shape and arrangement of the fuel lumps.[2] Thus the apparent nuclear properties would vary with direction, and in a bare reactor the separation of variables previously used would not be successful; the form $(1 + L^2B^2)^{-1}$ for p_{NL} would hold only approximately.

However, it seems clear that, in the case in which the unit-cell dimensions are small in comparison with the diffusion length, the form $(1 + L^2B^2)^{-1}$ would be a good approximation to the thermal nonleakage probability, and it would also be acceptable to use the general form $L^2 \equiv D/\Sigma_a$. Then, if we use

$$D \equiv \frac{1}{3\Sigma_{\text{tr}}} = \frac{1}{3[\Sigma_{\text{tr}}^{(M)} + \Sigma_{\text{tr}}^{(F)}]} \tag{10.11}$$

along with (where V_{cell} is the volume of the cell and $\bar{\phi}$ is the average flux throughout V_{cell})

$$\Sigma_{\text{tr}}\bar{\phi}V_{\text{cell}} \equiv \Sigma_{\text{tr}}^{(F)}\bar{\phi}_F V_F + \Sigma_{\text{tr}}^{(M)}\bar{\phi}_M V_M \tag{10.12}$$

and similarly for Σ_a; then,

$$\begin{aligned} L_{\text{het}}^2 \equiv \frac{D}{\Sigma_a} &= \frac{(\bar{\phi}V_{\text{cell}})^2}{3[\Sigma_{\text{tr}}^{(M)}\bar{\phi}_M V_M + \Sigma_{\text{tr}}^{(F)}\bar{\phi}_F V_F][\Sigma_a^{(F)}\bar{\phi}_F V_F + \Sigma_a^{(M)}\bar{\phi}_M V_M]} \\ &= \left(\frac{\bar{\phi}V_{\text{cell}}}{\bar{\phi}_M V_M}\right)^2 \frac{(1 - f_{\text{het}})L_M^2}{1 + \Sigma_{\text{tr}}^{(F)}V_F/\Sigma_{\text{tr}}^{(M)}V_M\zeta} \end{aligned} \tag{10.13}$$

where ζ is given by (10.4) and f_{het} by (10.5). For small fuel lumps $V_{\text{cell}} \simeq V_M$, $\bar{\phi} \simeq \bar{\phi}_M$, and $\Sigma_{\text{tr}}^{(F)}V_F/\Sigma_{\text{tr}}^{(M)}V_M\zeta \ll 1$; therefore [cf. Eq. (6.155)],

$$L_{\text{het}}^2 \simeq \frac{(1 - f_{\text{het}})L_M^2}{1 + \Sigma_{\text{tr}}^{(F)}V_F/\Sigma_{\text{tr}}^{(M)}V_M\zeta} \simeq L_M^2(1 - f_{\text{het}}) \tag{10.14}\star$$

3. *Resonance-escape Probability.* We found previously that in the case of a homogeneous system with low fuel concentration a good esti-

[1] Some of these considerations are incorporated into the Feinberg-Galanin model.

[2] Refer to Sec. 10.5.

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mate of the neutron flux was given by the asymptotic result. For these systems the resonance-escape probability took the form

$$p_{\text{th}} = \exp\left[- \int_{E_{\text{th}}}^{\infty} \frac{\Sigma_a(E)\, dE}{\xi \Sigma_t(E) E} \right] \tag{6.174}$$

where ξ denotes some suitable energy-averaged value for the entire fast range as defined, for example, in Eq. (4.121). For the present, more general treatment of resonance absorption, we require a broader definition which can be applied to a greater variety of physical situations. Following Wigner, we define the resonance-escape probability $p(E)$ as the ratio of neutrons which reach a given energy E in the resonance range to the number which enter the resonance range. If the source of high-energy neutrons supplies to the system one neutron per unit volume per unit time, then the appropriate statement for $p(E)$ is

$$p(E) \equiv 1 - \int_{E}^{\infty} \frac{F(E')\, \Sigma_a(E')\, dE'}{\Sigma_t(E')} \tag{10.15}$$

where

$$F(E) \equiv \Sigma_t(E)\, \phi(E) \tag{10.16}$$

denotes the total collision density in the system. The corresponding expression for p_{th} is, of course,

$$p_{\text{th}} = 1 - \int_{E_{\text{th}}}^{\infty} \frac{F(E)\, \Sigma_a(E)\, dE}{\Sigma_t(E)} \tag{10.17}$$

For many situations of practical interest, the resonance-escape probability p_{th} is close to unity so that $1 - p_{\text{th}} \ll 1$, and (10.17) is well approximated by [cf. Eq. (6.174)]

$$p_{\text{th}} = \exp\left[- \int_{E_{\text{th}}}^{\infty} \frac{F(E)\, \Sigma_a(E)\, dE}{\Sigma_t(E)} \right] \tag{10.18}\star$$

In the present survey of the subject we consider only the so-called first-order approximations in developing the theoretical basis of the discussion. These consist of the *narrow resonance* (NR) and the *narrow resonance–infinite absorber* (NRIA) approximations (after the terminology of Chernick[1]) which were developed[2] to treat particular types of resonance structure. As will be discussed later in Sec. 10.3, the former model is useful in describing absorptions by very narrow resonances, such as appear at the higher energy levels in U^{238} and Th^{232}, and the latter, for the few low-lying, rather broad resonances. It is of interest to mention

[1] J. Chernick and R. Vernon, *Nuclear Sci. Eng.*, **4**, 649–672 (1958).

[2] J. B. Sampson and J. Chernick, "Reasonance Escape Probability in Thermal Reactors," in D. J. Hughes and J. E. Sanders (eds.), Progress in Nuclear Energy, Series I, Physics and Mathematics, vol. 2, pp. 223–270, Pergamon Press, New York, 1958.

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that this distinction is required primarily because of the crudeness of the models used; more precise theoretical studies indicate that in higher-order approximations both yield results which are in good agreement.

The criterion for using the NR approximation is that the fuel resonances be widely spaced and so narrow that a single collision of a neutron with a fuel nucleus is sufficient to hurdle the resonance. In these cases the asymptotic expression is a good approximation for the flux over the entire resonance range, and we can take

$$\Sigma_t(E)\ \phi(E) \simeq \frac{1}{E\xi} \tag{10.19}$$

which follows from (4.134) for a normalized source function. With this relation, Eq. (10.18) reduces to the familiar form (6.174). In studying the resonance-escape probability, it is convenient to write the integral in (6.174) in the form

$$I = \frac{1}{\xi}\int_{E_{th}}^{\infty} \frac{[N_F\sigma_a^{(F)}(E) + N_M\sigma_a^{(M)}(E)]\,dE/E}{N_F[\sigma_a^{(F)}(E) + \sigma_{sr}^{(F)}(E) + \sigma_{si}^{(F)}(E) + \sigma_p^{(F)}] + N_M[\sigma_a^{(M)}(E) + \sigma_s^{(M)}(E)]} \tag{10.20}$$

where we have introduced the resonance-scattering cross section $\sigma_{sr}^{(F)}$ and the potential-scattering cross section $\sigma_p^{(F)}$ of the fuel and the concept of an interference-scattering cross section $\sigma_{si}^{(F)}$ to account for the interference effects between the two.[1] We define the *effective resonance integral* $\mathcal{I}$ by the relation

$$\mathcal{I} \equiv \xi\sigma_p I \tag{10.21}\star$$

where σ_p is the scattering cross section per fuel (or absorber) atom

$$\sigma_p \equiv \sigma_p^{(F)} + \frac{N_M}{N_F}\sigma_s^{(M)} \tag{6.177}$$

and $\sigma_s^{(M)}$ is the scattering cross section of the moderator (here taken to be energy independent); then,

$$\mathcal{I} = \int_{E_{th}}^{\infty} \frac{\sigma_p[\sigma_a^{(F)}(E) + N_M\sigma_a^{(M)}(E)/N_F]\,dE/E}{[\sigma_a^{(F)}(E) + \sigma_{sr}^{(F)}(E) + \sigma_{si}^{(F)}(E)] + \sigma_p + N_M\sigma_a^{(M)}(E)/N_F} \tag{10.22}$$

A good approximation is obtained by utilizing the fact that frequently $\sigma_a^{(F)} \gg \sigma_a^{(M)}N_M/N_F$. Thus by neglecting all terms involving $\sigma_a^{(M)}$, Eq. (10.22) reduces to

$$\mathcal{I} \simeq \int_{E_{th}}^{\infty} \frac{\sigma_a^{(F)}(E)\,dE/E}{1 + \frac{1}{\sigma_p}[\sigma_a^{(F)}(E) + \sigma_{sr}^{(F)}(E) + \sigma_{si}^{(F)}(E)]} \tag{10.23}\star$$

[1] J. M. Blatt and V. F. Weisskoff, "Theoretical Nuclear Physics," pp. 398–411, John Wiley & Sons, Inc., New York, 1952.

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which is the usual form in which the effective resonance integral is defined. It is important to point out that this result applies only to a system with narrow resonances; as noted above, the low-energy resonances of some absorbers are not well represented by this model.

When the fuel resonances are broad, it is better to use the NRIA approximation. The corresponding expression to (10.23) for a homogeneous system is

$$\mathcal{I} \simeq \int_{E_{\text{th}}}^{\infty} \frac{\sigma_a^{(F)}(E)\, dE/E}{1 + \sigma_a^{(F)}(E) N_F / \sigma_s^{(M)} N_M} \tag{10.24}\star$$

The approximations involved in deriving this formula from (10.17) are discussed in Sec. 10.3d.

Both expressions (10.23) and (10.24) may be written in the form

$$\mathcal{I} = \int_{E_{\text{th}}}^{\infty} s_a(E) \frac{dE}{E} \tag{10.25}$$

The functions s_a and $\mathcal{I}$ have several properties worth pointing out. These may be illustrated with the aid of the expression for s_a developed for the NR approximation. (Similar arguments may be applied for the case of the NRIA approximation.) Using then the form given in (10.23),

$$s_a \equiv \frac{\sigma_a^{(F)}}{1 + \frac{1}{\sigma_p}[\sigma_a^{(F)} + \sigma_{sr}^{(F)} + \sigma_{si}^{(F)}]} \tag{10.26}$$

The first observation we make is that for a very dilute system ($N_F \to 0$) the cross section $\sigma_p \to \infty$, and

$$\frac{s_a}{\sigma_a^{(F)}} \to 1 \qquad \frac{N_M}{N_F} \to \infty \tag{10.27}$$

Second, we note that as the fuel concentration increases, $\sigma_p \to \sigma_p^{(F)}$, and the ratio $s_a/\sigma_a^{(F)}$ approaches the minimum:

$$\frac{s_a}{\sigma_a^{(F)}} \to \frac{\sigma_p^{(F)}}{\sigma_t^{(F)}} \qquad \frac{N_M}{N_F} \to 0 \tag{10.28}$$

Thus in the NR approximation the *effective absorption* cross section s_a lies within the limits $\sigma_p^{(F)}/\sigma_t^{(F)} < s_a/\sigma_a^{(F)} < 1$, and it follows that the effective resonance integral $\mathcal{I}$ lies in the range $\mathcal{I}_F < \mathcal{I} < \mathcal{I}_\infty$ where $(\ )_F$ refers to the fuel material alone, and $(\ )_\infty$ refers to the infinitely dilute fuel-moderator mixture. We find, therefore, that as the fuel concentration increases, the effective resonance integral decreases from its greatest value $\mathcal{I}_\infty$. The fact that $\mathcal{I} < \mathcal{I}_\infty$ may be explained on the basis of the markedly different behavior of the total collision density and the flux in the vicinity of a sharp resonance. According to Eq. (10.19), the collision density is a reasonably smooth function of energy, provided that the sys-

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tem is not too highly absorbing. The flux, on the other hand, is seen to dip sharply at each resonance. This behavior is seen from the expression for the absorptions per unit energy interval:

$$\Sigma_a(E)\,\phi(E) = \frac{\Sigma_a(E)}{\Sigma_t(E)}\,F(E) = \frac{\Sigma_a(E)}{[\Sigma_a(E) + \Sigma_s(E)]\xi E} \simeq \frac{\sigma_a^{(F)}}{[\sigma_a^{(F)} + \sigma_p]\xi E}$$

When the number of fuel nuclei per scattering nucleus is large, $\sigma_a^{(F)}/\sigma_p$ is large, and ϕ varies roughly as $1/\Sigma_a E$; however, when the fuel concentration is low, $\sigma_a^{(F)}/\sigma_p$ is relatively small and ϕ varies roughly as $1/E$. Thus the higher fuel concentrations result in more pronounced dips in the spectrum of the resonance flux and thereby fewer absorptions per fuel nucleus per unit energy interval, hence, the smaller integral.

Another interesting observation involves the physical interpretation of the effective resonance integral. According to (10.21)

$$\mathcal{I} = \frac{I}{N_F/\xi\Sigma_p} \tag{10.29}$$

Now I/N_F is the total resonance absorptions per fuel atom, and $\xi\Sigma_p$ is the moderating power of the mixture [see Eq. (4.139)]; thus we may identify $\mathcal{I}$ as the *resonance absorption per fuel atom per unit moderating power*. It follows that, by the trends noted above, the resonance absorption per fuel atom per unit moderating power is smaller in a system with high fuel concentration than in one of lower. However, the moderating power in a dilute system is greater than that in a system of high fuel concentration, and since $\xi\Sigma_p$ increases more rapidly than s_a as N_F increases, the total resonance integral I of the dilute system is less than that of the concentrated system; therefore, p(dilute) $>$ p(concentrated).

The application of theoretical models of the sort given in (10.23) and (10.24) is developed in Sec. 10.3f, using the Breit-Wigner line shape to describe each of the resonances implied in the various absorption and scattering cross sections of the fuel nucleus. Although this approach appears to be fruitful, the theoretical basis is still in a stage of development. Thus until detailed information is available on each of the important resonances in a nucleus of interest, and more accurate calculational methods are developed, one must resort principally to experiment for reliable estimates of the effective resonance integrals. These integrals have been measured for many of the materials important in reactor technology, and some are available as general functions of the fuel concentration. A general listing of the available data on infinitely dilute mixtures is given by Macklin and Pomerance[1] and by Dresner.[2]

[1] R. L. Macklin and H. S. Pomerance, Resonance Capture Integrals, *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, P/833, vol. 5, pp. 96–101, August, 1955.

[2] L. Dresner, The Evaluation of Resonance Integrals, *Nuclear Energy*, **2**, 118–127 (1955).

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The effective-resonance integrals for the two fertile fuels U^{238} and Th^{232} have been studied in greatest detail, and empirical formulas have been obtained for computing these integrals as functions of fuel concentration. The best available experimental data are plotted in Fig. 10.2. Dresner[1] has computed from resonance parameters, by methods to be discussed

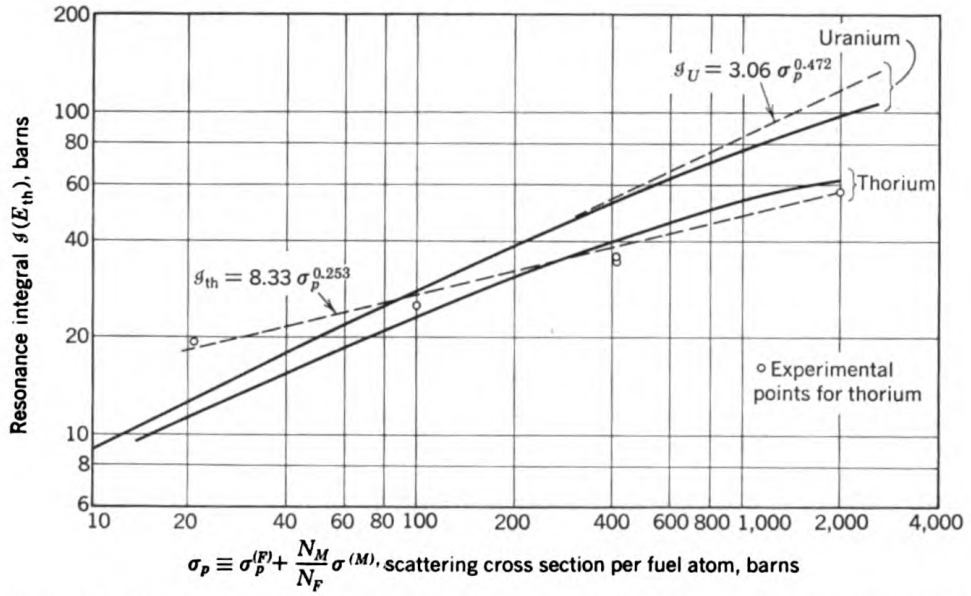

FIG. 10.2 Resonance integrals for uranium 238 and thorium 232 as a function of scattering cross section per fuel atom.

later, the appropriate functional behavior and gives, as the best power-law mathematical fit to his computed data, the formula

$$g(E_{th}) = 3.06\sigma_p^{0.472} \qquad \text{for uranium} \tag{10.30}$$

This function is given by the solid-line curve labeled "uranium" in the figure. These data were computed for a mixture temperature of 300°K, and should be adequate for most reactor calculations.

Similar calculations and measurements are given for thorium 232. The empirical relation obtained by Hughes and Eggler[2] is

$$g(E_{th}) = 8.33\sigma_p^{0.253} \qquad \text{for thorium} \tag{10.31}$$

Figure 10.2 shows the available experimental points and the curve (solid line) computed by Dresner. (It is believed that the experimental point at $\sigma_p = 25$ barns is in error.)

The resonance integrals for *infinitely dilute* mixtures of the various

[1] L. Dresner, *Nuclear Sci. Eng.*, **1**, 68–79 (1956).

[2] D. J. Hughes and C. Eggler, "Resonance Absorption of Thorium," Argonne Laboratory, CP-3093, July 13, 1945.

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nuclear fuels have also been determined. These data are listed in Table 10.1.

The problem of determining the resonance-escape probability in heterogeneous configurations of fuel and moderator has received continued attention since the very early days of reactor technology. Because of the difficulty of the problem and the paucity of cross-section data, much of the initial effort was directed along experimental lines; however,

TABLE 10.1 EFFECTIVE RESONANCE INTEGRALS FOR INFINITELY DILUTE MIXTURES OF VARIOUS NUCLEAR FUELS
(in barns)

Th^{232}	U^{233}	U^{235}	U^{238}	Pu^{239}	Reference
93‡	...	....	269‡	...	Chernick-Vernon[a]
69.8§	...	....	282	...	Macklin-Pomerance[b]
86.2‡	...	....	291	...	
...	442	400	....	520	Weinberg-Noderer[c]
...		467‡	....	...	Trubey-Lessig[d]

‡ Calculated value.

§ Measured value.

[a] J. Chernick and R. Vernon, *Nuclear Sci. Eng.*, **4**, 649–672 (1958).

[b] R. L. Macklin and H. S. Pomerance, Resonance Capture Integrals, *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, P/833, vol. 5, pp. 96–101, August, 1955.

[c] A. M. Weinberg and L. C. Noderer, "Theory of Neutron Chain Reactions," vol. I, chap. IV, Oak Ridge National Laboratory, CF 51-5-98, May 15, 1951.

[d] D. K. Trubey, personal communication; see also D. K. Trubey and R. H. Lessig, Neutron Physics Division Annual Progress Report, Oak Ridge National Laboratory, ORNL-2609, Oct. 29, 1958.

theoretical studies were attempted from the very beginning, of particular importance being the contributions of E. P. Wigner.[1] Until more recently it was necessary, for the reasons mentioned, to rely principally upon measured values of the effective resonance integral. Most recently, more and better cross section data have become available, and it has become profitable again to apply theoretical methods to this problem. As a result, some new analytical models have evolved which have proved quite helpful in further understanding and describing the resonance behavior of lumped fuel. It is especially encouraging that the recent analyses appear to be in reasonably good agreement with measurement. Some of the more elementary models used for this purpose are presented in Sec. 10.3. The present discussion, however, is limited to a brief summary of the essential features of resonance absorption in heterogeneous arrays.

[1] E. P. Wigner, "Resonance Absorption of Neutrons by Spheres," Argonne Metallurgical Laboratory, C-4, 1942; also, E. P. Wigner, E. Creutz, H. Jupnik, and T. Snyder, *J. Appl. Phys.*, **26**, 257–279 (1955).

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Much of the resonance behavior of fuel lumps can be explained in terms of the effective absorption cross section s_a of the fuel in the lump by direct analogy with the concepts developed in connection with the homogeneous systems. In drawing this analogy we compare two systems containing the same fuel and moderator materials, but in one the fuel is lumped and in the other it is homogeneously mixed with the moderator. If we specify that the total masses of the fuel and of the moderator in one system be the same as those in the other, then it follows that the volume-weighted moderating power $\xi\Sigma_p$ will be the same in the two systems [cf. Eq. (10.6)]. However, the fuel in the lumped system behaves as if it were in a mixture of high fuel concentration, and so by the arguments presented in the discussion following Eq. (10.25), the effective absorption cross section s_a in the heterogeneous system appears to be less than that for the fuel in the homogeneous. Consequently the effective resonance integral in the heterogeneous system is also smaller than that of the homogeneous, and since $\Sigma_p^{\text{hom}} = \Sigma_p^{\text{het}}$, we find that the resonance-escape probability is noticeably improved. As a rule, then, for "equivalent" systems,

$$p_{\text{het}} > p_{\text{hom}} \tag{10.32}$$

A good illustration of this point is given in the case of natural uranium. If this material were uniformly mixed with moderator, then in the case of an infinitely dilute concentration the resonance integral is approximately 280 barns. On the other hand, if the uranium and moderator were to be physically separated, but with the value of Σ_p the same for both systems, then the corresponding value of the resonance integral is about 9 barns, a factor of about 26 reduction. Advantages of this order of magnitude are generally achieved in going to heterogeneous configurations.

The earliest formulations of the resonance-absorption problem in heterogeneous arrangements were given by Wigner[1] and by Dancoff and Ginsburg,[2] who showed on the basis of theoretical arguments that the effective resonance integral could be conveniently written as the sum of a so-called *volume-absorption* term and a *surface-absorption* term. The validity of this approach and its particular usefulness in treating thick lumps were confirmed experimentally by Creutz[3] et al. These important advances led in the United States to the adoption of the so-called *standard formula* which expresses the effective absorption cross section as a linear function of the ratio of surface area to volume of the fuel lump. More

[1] *Ibid.*

[2] S. M. Dancoff and M. Ginsburg, "Resonance Absorption in Lumps and Mixtures," Argonne Metallurgical Laboratory, CP-1589, 1944.

[3] E. Creutz, H. Jupnik, and T. Snyder, "Effect of Geometry on Resonance Absorption of Neutrons by Uranium," Argonne Metallurgical Laboratory, C-116, 1951; also *J. Appl. Phys.*, **26**, 257–279 (1955).

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recently it has been shown[1] that for many situations of practical interest a given heterogeneous system can be described in terms of an equivalent homogeneous system; thus many of the results developed for the homogeneous systems may be carried over directly to applications for heterogeneous arrangements.

In the present summary we limit attention to a discussion of the standard formula, in part because of its historic interest and in part because of the wide usage it has received throughout the technology. The derivation of the equivalence equations and their relationship to the formulation of the two-term standard formula are given in Secs. 10.3e and i.

The United States, and generally Western, usage of the standard formula was based for some time on the following definition of the effective absorption cross section:

$$s_a(u) \equiv a(u) + \frac{A_F}{M_F} b(u) \tag{10.33}$$

where A_F denotes the surface area of the fuel lump and M_F its mass. The appropriate expression for the effective resonance integral is then developed by means of the following arguments:

Let $Q(u)$ be the total number of neutrons slowing down per unit time past lethargy u in some specified region of a heterogeneous lattice, say, for example, in a unit cell. At steady state the change in the slowing-down density is precisely the rate at which neutrons are removed from the cell by absorptions. If these removals occur primarily in the fuel component, then we must require that

$$\frac{dQ}{du} = -V_F N_F s_a(u) \bar{\phi}_M(u) \tag{10.34}$$

where, by definition, $s_a(u)$ is that cross section which when multiplied by the flux in the moderator gives the correct number of absorptions in the fuel. In this calculation $\bar{\phi}_M(u)$ denotes the space-averaged flux in the moderator at lethargy u. The slowing-down density is related to the flux, however, by the relation

$$Q(u) = \xi_M V_M N_M \sigma_s^{(M)} \bar{\phi}_M(u) \tag{10.35}$$

If we use this equation in (10.34) and carry out the integration, it is seen that

$$Q(u) = Q(0)\, p(u)$$

[1] See, for example, L. Dresner, "Resonance Absorption of Neutrons in Nuclear Reactors," thesis, Princeton University, chap. 6, 1958; also, Chernick and Vernon, *op. cit.*

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where

$$p(u) = \exp\left[- \frac{V_F N_F}{\xi_M V_M \Sigma_s^{(M)}} \int_0^u s_a(u')\, du' \right] = \exp\left[- \frac{V_F N_F \mathcal{I}(u)}{\xi_M V_M \Sigma_s^{(M)}} \right] \tag{10.36}\star$$

is the resonance-escape probability. Thus it follows that the effective resonance integral $\mathcal{I}$ is simply the integral of s_a as in the homogeneous systems [cf. Eq. (10.25)]. If we use (10.33), then,

$$\mathcal{I} \equiv \int_0^{u_{th}} a(u)\, du + \frac{A_F}{M_F} \int_0^{u_{th}} b(u)\, du \tag{10.37}$$

These integrals are conveniently expressed in the form

$$\int_0^{u_{th}} a(u)\, du \equiv \frac{A_r}{\zeta_r} \tag{10.38}$$

$$\int_0^{u_{th}} b(u)\, du \equiv \left(\frac{\rho'_F}{\zeta_r N_F} \right) B_r \tag{10.39}$$

where ζ_r denotes the ratio of the "average resonance flux" in the moderator to the average resonance flux in the fuel lump; thus we call ζ_r the *resonance disadvantage factor* in analogy to the ζ given by Eq. (10.4) for the thermal flux. The N_F denotes the nuclear density of the fuel material, and for the present, we have used the symbol ρ'_F for the mass density. In terms of the above definitions, $\mathcal{I}$ may be written

$$\mathcal{I} = \frac{A_r}{\zeta_r} \left[1 + \left(\frac{\rho'_r B_r}{N_F A_r} \right) \frac{A_F}{M_F} \right] \tag{10.40}\star$$

where the first term is attributed to "volume" and the second to "surface" absorptions. The functions A_r and B_r have been measured for various materials and shapes of fuel lumps. Of particular interest are the values for natural uranium; the measured results[1] are

$$A_r = 8.6 \text{ barns} \qquad \text{and} \qquad \left(\frac{\rho'_F}{N_F} \right) B_r = 27.5 \text{ g}$$

now, if we take $\zeta_r = 1$, then

$$\mathcal{I} = 8.6 + 27.5 \frac{A_F}{M_F} \qquad \text{uranium metal} \tag{10.41}$$

where A_F is given in square centimeters and M_F in grams. We observe from this relation that, as the lump size increases, A_F/M_F decreases and the effective resonance integral decreases. In the limit for a very large

[1] Sampson and Chernick, *op. cit.*

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lump $(A_F/M_F) \to 0$, and

$$\mathcal{J} \sim \int a \frac{dE}{E} = 8.6 \text{ barns}$$

which is approximately $\frac{1}{26}$ of the value for the infinitely dilute homogeneous mixture of natural uranium and moderator. Expressions of the type (10.41) have been developed also for the various compounds of uranium:[1,2]

$$\mathcal{J} = 12.8 + 24.9 \frac{A_F}{M_F} \qquad U_3O_8 \tag{10.42}$$

$$\mathcal{J} = 20.4 + 3.16 \frac{A_F}{M_F} \qquad 5U_3O_8 + 2C \text{ (by weight)} \tag{10.43}$$

$$\mathcal{J} = 11.0 + 24.5 \frac{A_F}{M_F} \qquad UO_2 \tag{10.44}$$

In the case of thorium, two useful relations are[3]

$$\mathcal{J} = 8.4 + 24.9 \frac{A_F}{M_F} \qquad \text{thorium metal} \qquad 0.1 \leq \frac{A_F}{M_F} \leq 1.3 \tag{10.45}$$

$$\mathcal{J} = 10.4 + 27.4 \frac{A_F}{M_F} \qquad ThO_2 \qquad 0.2 \leq \frac{A_F}{M_F} \leq 1.3 \tag{10.46}$$

An independent formulation of the problem of resonance capture in heterogeneous systems by investigators in the U.S.S.R. led to the development of an alternative expression for the effective absorption cross section which varied as $(A_F/M_F)^{\frac{1}{2}}$. This work was published by Egiazarov, Dikarev, and Madeev[4] and by Burgov,[5] and was summarized later by Gurevich and Pomeranchouk[6] in a paper presented at the Geneva Conference in 1955. The form for $\mathcal{J}$ reported in these works is [cf. Eq. (10.40)]

$$\mathcal{J} = A_r' \left[1 + \mu \left(\frac{A_F}{M_F}\right)^{\frac{1}{2}}\right] \tag{10.47}\star$$

Although this formula differs in form from the corresponding American or Western relation, the constants in standard use are such that the

[1] *Ibid.*

[2] E. Hellstrand, *J. Appl. Phys.*, **28**(12), 1493–1502 (1957).

[3] I. E. Dayton and W. G. Pettus, *Nuclear Sci. Eng.*, **3**, 286–295 (1958).

[4] V. B. Egiazarov, V. S. Dikarev, and V. G. Madeev, *Akad. Nauk. S.S.S.R. Conf. on Peaceful Uses of Atomic Energy*, Session of Physical and Mathematical Sciences, p. 59, English translation by Consultants Bureau, New York.

[5] N. A. Burgov, *Akad. Nauk. S.S.S.R. Conf. on Peaceful Uses of Atomic Energy*, Session of Physical and Mathematical Sciences, p. 69, English translation by Consultants Bureau, New York.

[6] I. I. Gurevich and I. Y. Pomeranchouk, Paper P/649, vol. 5, pp. 466–471, Geneva Conference.

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effective resonance integrals for uranium metal are in adequate agreement[1] for fuel rods ranging from 0.5 to 6 cm in diameter. When cast in the form (10.47), the effective resonance integrals for the various materials[2] considered above become:

$$g = 2.81 + 24.7\left(\frac{A_F}{M_F}\right)^{\frac{1}{2}} \qquad \text{uranium metal} \qquad (10.48)$$

$$g = 4.15 + 26.6\left(\frac{A_F}{M_F}\right)^{\frac{1}{2}} \qquad UO_2 \qquad (10.49)$$

and[3]

$$g = 4.1 + 21.1\left(\frac{A_F}{M_F}\right)^{\frac{1}{2}} \qquad \text{thorium metal} \qquad (10.50)$$

$$g = 0.2 + 25.2\left(\frac{A_F}{M_F}\right)^{\frac{1}{2}} \qquad ThO_2 \qquad (10.51)$$

d. Effects of Lumping. The distribution of the fissionable material in heterogeneous arrays leads to three important advantages and one disadvantage from the standpoint of the reactor physics. The advantages to be gained by utilizing a heterogeneous configuration, however, generally outweigh the principal disadvantage, namely, the reduction in thermal utilization. This effect has already been discussed with the aid of an elementary analytical model, and it was observed that by separating the fuel and moderator materials the fuel is placed at a disadvantage in competing for the available supply of thermal neutrons. Without entering into a detailed calculation, it is easy to establish this argument on physical grounds. The basis of the argument rests with the fact that the slowing-down process, which is the primary source of thermal neutrons, occurs in a region which is physically separated from the fuel. Thus the moderator region surrounding each fuel lump is a source of thermals, whereas the fuel lumps are essentially sinks. This large source in the moderator provides a steady stream of thermal neutrons to the fuel lumps; the "driving force" or potential is achieved by maintaining a higher neutron density in the moderator regions. Thus the moderator is exposed to average flux levels higher than those in the fuel.

The most important of the three advantages to be realized from heterogeneity is the increase in the resonance-escape probability due to the *self-shielding* effect in the fuel lumps. Because of height and sharpness of the resonance structure of the fuel nucleus, the spectrum of the reso-

[1] E. P. Wigner, "Review of Resonance Capture by Lumps," Proceedings of the Brookhaven Conference on Resonance Absorption of Neutrons in Nuclear Reactors, BNL-433, pp. 68–73, 1956.

[2] Hellstrand, *op. cit.*

[3] W. G. Pettus, "Geometric and Temperature Effects in Thorium Resonance Capture," Babcock & Wilcox Co., BAW-TM-203, Lynchburg, Va., Apr. 27, 1959.

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nance flux exhibits local depressions around each resonance. Because of the lumping, the majority of the fuel nuclei in the lump are exposed to a depressed flux much like that in a homogeneous mixture of high fuel concentration. As argued previously, this corresponds to a relatively small value of σ_p and of the effective absorption cross section of the fuel s_a [cf. Eq. (10.28)]; consequently the effective resonance integral is reduced and the resonance-escape probability increased.

There is another, somewhat secondary, effect which serves further to enhance the resonance-escape probability. Because of the physical separation of the two materials, the neutrons involved in the slowing-down process have less opportunity to encounter a fuel nucleus while traversing the resonance range; the number of resonance absorptions is thereby reduced. Clearly, this effect is due primarily to geometric considerations.

Finally, the heterogeneous arrangement results in an increase in the fast effect. The explanation is quite straightforward. Since all fissions take place in regions of high fuel density (frequently pure metal), the resulting high-energy fission neutrons have a greater likelihood of encountering a fuel nucleus while en route to the outer boundaries of the fuel lump. Thus there is an increased probability of causing a fast fission before leaving the lump. Moreover, any fast fissions which might occur can produce additional neutrons which can cause further fast fission, thereby producing a cascade effect. Although some advantage is realized from this phenomenon, the gain is somewhat limited because of the occurrence of inelastic-scattering collisions. In the fast range, the inelastic process is an important channel for compound nucleus decay and competes strongly with fission. Nevertheless, the net result is usually a slight gain in the available number of fast neutrons.

e. Multiplication Constant. Some of the difficulties involved in the computation of the multiplication constant of a heterogeneous reactor have been mentioned in connection with the calculation of the thermal utilization and the effective diffusion length of lattices. The principal difficulty stems, of course, from the fact that there are local variations in the neutron flux at all energies in each unit cell of the array. In the central regions of the reactor core this does not lead to any particular hardship since the flux distributions will be relatively flat and variations from cell to cell will be gradual. This is not the case however in cells near the boundaries of the lattice. Here, variations across the cell can be substantial and will influence markedly such factors as the thermal utilization and resonance-escape probabilities by distorting the neutron distribution in the cells. These difficulties are by no means unique to the method based on the unit-cell concept; similar complications arise in the application of the Feinberg-Galanin method. In the latter application the difficulty appears in selecting suitable kernels to describe the

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source-sink behavior of lumps in the vicinity of irregularities and discontinuities of the surrounding medium.

The unit-cell method offers a relatively simple computational procedure for determining the various factors in k_∞. As already demonstrated, the calculation of these factors in each case reduces ultimately to the determination of the neutron spatial distributions for the entire energy range. For the thermal utilization we require the thermal-flux distribution and in particular the thermal-flux depression (i.e., the thermal disadvantage factor). The resonance-escape probability, on the other hand, requires a knowledge of the spatial distribution of resonance neutrons, and finally, of course, the calculation of the fast effect involves, essentially, the determination of the spatial distribution of successive (cascading) generations of fission neutrons.

Once these factors have been determined, the remaining problem is to compute L and τ since these quantities establish the neutron-leakage rates for the finite system. A first approximation to L^2 has already been given, and a few remarks have been made about corrections for directional effects. Some methods have been developed for including anisotropic effects in the computation of the migration area, and these are reported in a subsequent section.

10.2 Disadvantage Factor and Thermal Utilization

a. Analytical Model for Unit Cell. A practical method for the calculation of the disadvantage factor and the thermal utilization is based on the Wigner-Seitz unit-cell model. In order that this model be applicable to a particular situation, it is necessary that certain simplifying assumptions and conditions be satisfied. These may be illustrated by means of an example. Suppose that the reactor-core configuration of interest consists of an alternating series of fuel plates (thickness $2a$) and slabs of moderator [thickness $2(b - a)$] as shown in Fig. 10.3. To begin with we require that:

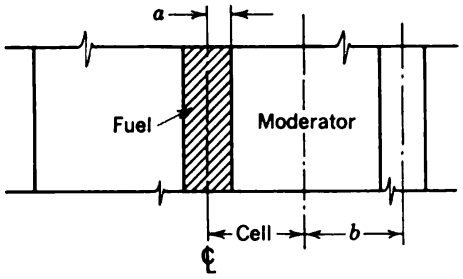

FIG. 10.3 Unit cell in slab geometry.

1. All characteristic reactor dimensions be large compared with the cell half-width b. Then the calculation of the thermal utilization may proceed on the basis of an infinite medium of this structure, since the regions near the edges of the reactor core (for which the disadvantage factor will differ somewhat from the fairly uniform value over the central region) will not play an important role in the criticality calculation. The disadvantage factor ζ we compute from a knowledge of the thermal-flux

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distribution throughout the cell, and this we obtain with the aid of the diffusion theory. We further require, therefore, that:

2. Diffusion theory be valid in both fuel and moderator regions of the unit cell. It must be pointed out that this assumption may be poor on two counts: (*a*) The fuel region is likely to be highly absorbing for thermal neutrons; thus the angular distribution of neutrons in the fuel region and near the fuel-moderator interface in the moderator will very probably deviate considerably from the distribution appropriate to diffusion theory [see Eq. (7.93)]. (*b*) The dimensions of both regions may be of the order of a transport mean free path or smaller, that is, much smaller than a diffusion length in either medium. The nonisotropy of the flux in the vicinity of an absorber-moderator interface was demonstrated in Sec. 7.5a. Figure 7.16*g* shows the typical strong forward bias toward the absorber region which is to be expected in the angular distribution at such boundaries. Finally, it should be mentioned that there is also a third difficulty which is encountered in the use of the diffusion theory, namely, the neutron-energy spectrum around thermal varies appreciably through the moderator and fuel slug. Thus there is some uncertainty in selecting the "one velocity" for the model.

In spite of all these shortcomings we will assume that diffusion theory is applicable. So the results obtained below will not be useful for treating thin, highly absorbing fuel plates (or pins) but could be well applied to large natural uranium lumps in a good moderator such as graphite. A suitable model, based on the "first-flight-transport" concept developed in Sec. 5.8a, is applied in a later analysis to handle these cases. It should be mentioned, however, that disadvantage factors calculated by diffusion theory can be low by about 20 per cent, giving an error in the multiplication constant of about 2 per cent in practical cases for natural uranium with graphite or heavy water.[1] The success of the diffusion-theory treatment during the Manhattan Project was achieved by adjusting the value of η. When more accurate estimates of the thermal utilization are required than possible with the methods discussed here, it is necessary to resort to the direct use of the spherical harmonics or the so-called S_n methods.[2]

Another essential question which must be considered in selecting an analytical model is the specification of the thermal-neutron sources in the cell. In the actual situation, neutrons are born with high energy in the fuel plates; during the slowing-down process they diffuse away from the

[1] Personal communication, L. W. Nordheim.

[2] See, for example, E. Bjørnerud and L. W. Nordheim, "Adaption of the S_n Method to Cell Calculations," General Atomics, GA-126, San Diego, Calif., May 22, 1957; M. Johnson, "Code for a S_4 Cylindrical Cell," General Atomics, GAMD-205, Aug. 28, 1957; and I. Bornstein, "S_n Calculations of a Proposed Chalk River–Ontario Hydro Reactor Lattice," General Atomics, GAMD-242; these documents are company internal publications.

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original plate and may finally become thermal at a distance many slab thicknesses removed from the source point at which they appeared as fast neutrons. In fact (neglecting the difference between the slowing-down properties of the fuel and the moderator), the distribution of neutrons becoming thermal from a source plate at $x = 0$ is, on the basis of age theory, just the Gaussian, $\exp(-x^2/4\tau_{th})$. If the half-width of the Gaussian, or the slowing-down length (both of which are proportional to $\sqrt{\tau_{th}}$), is large in comparison with the moderator slab thickness $2(b - a)$, the number of neutrons becoming thermal will be roughly uniform over a moderator slab (see Fig. 10.4). Actually, the usual

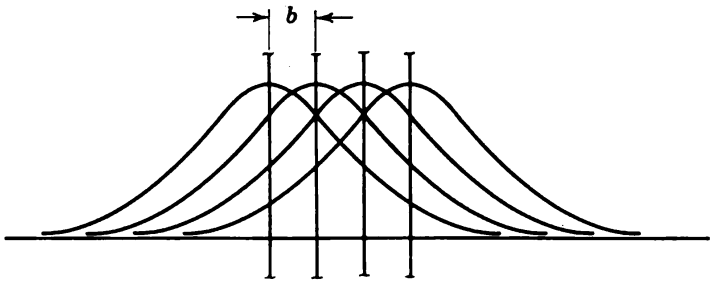

FIG. 10.4 Slowing-down density at thermal from plane sources on the basis of the Fermi age theory.

physical situation is even more favorable for the assumption of uniform thermalization over the moderator region. It will be recalled that the actual distribution of thermal neutrons is better approximated by means of age theory when first-flight corrections are taken into account. This leads to a greater spreading out of the neutrons in most moderators because of the low scattering cross section at high energy. Thus for the analyses which follow we assume:

3. There is a uniform source of thermal neutrons over the entire moderator region, but no neutrons become thermalized in the fuel region. The latter statement is reasonable because most fuel nuclei have large mass.

In the infinite medium of alternating fuel plates and moderator slabs, we may concentrate on a typical region which includes half a fuel plate adjacent to half a moderator slab, as shown in Fig. 10.3. This region is sufficient for the present purpose since we are interested only in the ratio of the average fluxes in the fuel and moderator regions and since the flux throughout the remainder of the medium is obtainable by symmetry from that in the region considered. This region is called a unit cell; in a general geometry the unit cell is established by requiring that (*a*) the fuel-moderator volume ratio in the cell be equal to the average ratio for the entire core and (*b*) the net transport of neutrons across the surface of the cell be zero.

In concluding the present discussion it is of interest to mention that

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some of the assumptions and approximations selected for the unit-cell model are very nicely developed in the general method devised by Feinberg and Galanin. It is especially noteworthy that the method clearly displays how and when these simplifications may best be introduced into the model and the limitations which they impose on the system.

b. Disadvantage Factor by Diffusion Theory. We demonstrate the calculation of the disadvantage factor on the basis of the diffusion-theory model outlined above for the special case of the slab configuration shown in Fig. 10.3. This derivation is followed by a summary of the general results for the elementary lump geometries.

The mathematical problem may be stated in terms of the general relations (5.349) through (5.354). For the present application these may be written

In the fuel $(0 \leq x \leq a)$:

$$-D_F\phi_F''(x) + \Sigma_a^{(F)}\phi_F(x) = 0$$

In the moderator $(a \leq x \leq b)$: (10.52)

$$-D_M\phi_M''(x) + \Sigma_a^{(M)}\phi_M(x) = S_0$$

where S_0 denotes the uniform distribution of thermal neutrons postulated in 3 above. The appropriate boundary conditions are

$$\begin{array}{ll} (1)\ \phi_F(a) = \phi_M(a) & (2)\ D_F\phi_F'(a) = D_M\phi_M'(a) \\ (3)\ \phi_F'(0) = 0 & (4)\ \phi_M'(b) = 0 \end{array} \tag{10.53}$$

The solutions for the thermal-flux distributions $\phi_F(x)$ and $\phi_M(x)$ are easily shown to be

$$\phi_F(x) = \frac{S_0 \cosh \varkappa_F x}{\sigma D_F \varkappa_F \Sigma_a^{(M)} \sinh \varkappa_F a} \tag{10.54}$$

$$\phi_M(x) = \frac{S_0}{\Sigma_a^{(M)}}\left[1 - \frac{\cosh \varkappa_M(b-x)}{\sigma D_M \varkappa_M \sinh \varkappa_M(b-a)}\right] \tag{10.55}$$

where $$\sigma \equiv \frac{\coth \varkappa_F a}{D_F\varkappa_F} + \frac{\coth \varkappa_M(b-a)}{D_M\varkappa_M} \quad \text{and} \quad \varkappa^2 \equiv \frac{\Sigma_a}{D} \tag{10.56}$$

The disadvantage factor for this system is

$$\zeta \equiv \frac{a\int_a^b \phi_M(x)\,dx}{(b-a)\int_0^a \phi_F(x)\,dx} = a\sigma\Sigma_a^{(F)}\left[1 - \frac{1}{\sigma(b-a)\Sigma_a^{(M)}}\right] \tag{10.57}$$

This result may be checked by taking the limiting value of ζ as the fuel and moderator regions become infinitesimally thin. From (10.56) we have

$$\lim_{a,b\to 0} \sigma = \frac{1}{a\Sigma_a^{(F)}} + \frac{1}{\Sigma_a^{(M)}(b-a)}$$

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Substitution into (10.57) yields, in the limit,

$$\lim_{a,b\to 0} \zeta = \left[1 + \frac{a\Sigma_a^{(F)}}{(b-a)\Sigma_a^{(M)}}\right]\left[1 - \frac{1}{\dfrac{(b-a)\Sigma_a^{(M)}}{a\Sigma_a^{(F)}} + 1}\right] = 1$$

where we have kept the ratio a/b constant. This result is, of course, to be expected, since to make the dimensions of both media smaller while holding the volume ratio constant is to approach the properties of the homogeneous system. The disadvantage factor, as calculated by (10.57), has the form shown in Fig. 10.5 when plotted against size.

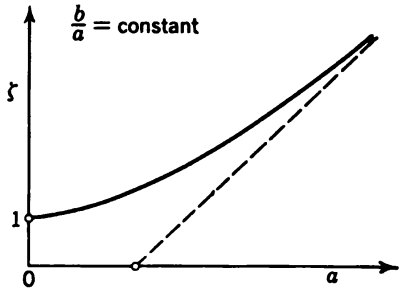

FIG. 10.5 Disadvantage factor as function of cell size.

For convenience in presenting the corresponding results in other geometries, we rewrite Eq. (10.57) in the form

$$\zeta = \frac{a\Sigma_a^{(F)}}{(b-a)\Sigma_a^{(M)}}\left[\frac{(b-a)\Sigma_a^{(M)}}{a\Sigma_a^{(F)}}\kappa_F a \coth \kappa_F a + \kappa_M(b-a)\coth \kappa_M(b-a) - 1\right] \quad (10.58)$$

In the slab geometry, the cell may be taken as a block of *unit cross section* extending from $x = 0$ to $x = b$; so that

V_F = volume of fuel in unit cell = a

V_M = volume of moderator in unit cell = $b - a$

Equation (10.58) may then be written

$$\zeta \equiv \frac{V_F\Sigma_a^{(F)}}{V_M\Sigma_a^{(M)}}\left[\frac{V_M\Sigma_a^{(M)}}{V_F\Sigma_a^{(F)}}F + E - 1\right] \quad (10.59)\star$$

The functions F and E are defined, in slab geometry, by

$$F \equiv a\kappa_F \coth \kappa_F a \qquad E \equiv V_M\kappa_M \coth \kappa_M(b-a) \quad (10.60)$$

It is of interest to note that the quantity F is in fact the ratio of the flux at the surface of the fuel plate to the average value of the flux in the plate. This relationship is basic and applies to other geometries as well.

The form (10.59) is also useful in expressing ζ for two other principal geometries, with suitable expressions for F and E. The two cases are: (1) cylindrical fuel rods in square lattice and (2) spherical lumps of fuel in cubic lattice. The appropriate expressions for the unit-cell dimensions are computed by requiring that the cell volume be equal to the volume of an "element" of the lattice; thus, if a is the lattice spacing, R the

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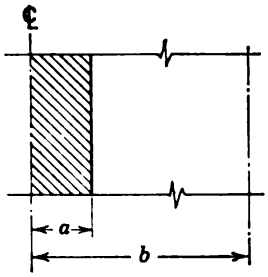
lump radius, and R_{cell} the radius of the unit cell, then the cell characteristics are given by:

Lumps	V_F	V_M	V_{cell}	R_{cell}
Cylindrical.....	πR^2	$a^2 - \pi R^2$	a^2	$a/\sqrt{\pi}$
Spherical.....	$\frac{4}{3}\pi R^3$	$a^3 - \frac{4}{3}\pi R^3$	a^3	$a\left(\frac{3}{4\pi}\right)^{\frac{1}{3}}$

The calculation of the thermal-flux distributions for these geometries, and thence the disadvantage factor, proceeds in the usual way, starting with the basic relations (5.349) through (5.354) and the requirement that the net current vanish at the boundary of the cell. The results for the three basic geometries are summarized in Table 10.2.

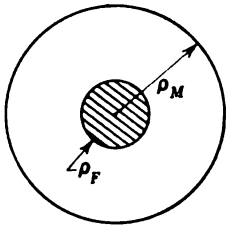
TABLE 10.2 THE FUNCTIONS E AND F FOR VARIOUS CELL GEOMETRIES

Slab



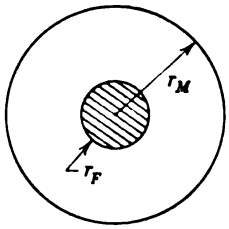
$$F = a\varkappa_F \coth \varkappa_F a$$
$$E = (b - a)\varkappa_M \coth \varkappa_M (b - a)$$

Cylindrical



$$F = \frac{\rho_F \varkappa_F I_0(\varkappa_F \rho_F)}{2I_1(\varkappa_F \rho_F)}$$
$$E = \frac{\varkappa_M(\rho_M^2 - \rho_F^2)}{2\rho_F}\left[\frac{I_0(\varkappa_M\rho_F)\,K_1(\varkappa_M\rho_M) + K_0(\varkappa_M\rho_F)\,I_1(\varkappa_M\rho_M)}{I_1(\varkappa_M\rho_M)\,K_1(\varkappa_M\rho_F) - K_1(\varkappa_M\rho_M)\,I_1(\varkappa_M\rho_F)}\right]$$

Spherical



$$F = \frac{r_F^2 \varkappa_F^2 \tanh \varkappa_F r_F}{3(\varkappa_F r_F - \tanh \varkappa_F r_F)}$$
$$E = \frac{\varkappa_M^2(r_M^3 - r_F^3)}{3r_F}\left[\frac{1 - \varkappa_M r_M \coth \varkappa_M(r_M - r_F)}{1 - \varkappa_M^2 r_M r_F - \varkappa_M(r_M - r_F)\coth \varkappa_M(r_M - r_F)}\right]$$

Fuel
Moderator

The function $E - 1$ for the case of the cylindrical geometry shown in Table 10.2 is plotted in Fig. 10.6. For this purpose we have used the

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definition

$$E(y,z) \equiv \frac{z^2 - y^2}{2y}\left[\frac{I_0(y)\,K_1(z) + K_0(y)\,I_1(z)}{I_1(z)\,K_1(y) - K_1(z)\,I_1(y)}\right]$$

with $y \equiv \varkappa_M \rho_F$ and $z \equiv \varkappa_M \rho_M$

c. Disadvantage Factor for Thin Fuel Plates and Pins. The diffusion-theory results obtained above are not expected to be accurate when

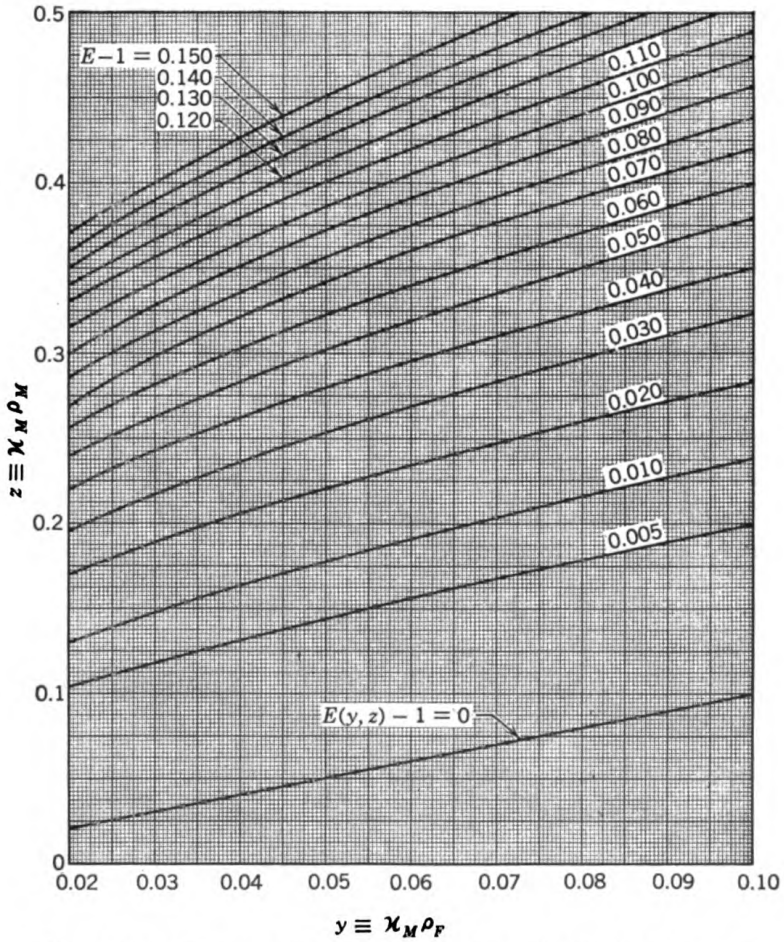

FIG. 10.6 Graph of function $E - 1$ for cylindrical fuel lump.

the fuel is present in the form of thin, highly absorbing plates or in the form of fine pins. For these situations it is better to apply the results of the first-flight transport model developed in Sec. 5.8a. The application of this model is demonstrated for the case of the thin plates, followed by a summary of the corresponding results for the pins.

Consider, then, the slab geometry of Fig. 10.3 and suppose that the fuel plate is very thin, so that we may assume that the plate either absorbs a neutron incident upon it or it does not; scattering collisions are considered to have negligible probability. This formulation is in accord with the definition preceding Eq. (5.275). Let us assume further

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that the diffusion theory is valid in the moderator and that neutrons are being thermalized uniformly throughout the moderator region (with no thermal sources in the fuel). The boundary condition at the edge of the unit cell is as before. The fuel-moderator interface condition on the other hand is now conveniently given in terms of the transmission coefficient α of the plate [see Eq. (5.275)], since the plate is considered too thin to allow the use of diffusion theory. The quantity α may be computed in terms of the plate dimensions, its absorption cross section, and the angular distribution of incident neutrons.

The calculation of the thermal-flux distribution in the moderator is carried out in the usual way by applying (10.52) and (10.53), condition (4). For the interface condition we use the partial current relation

$$j_+(a) = \alpha j_-(a) \tag{10.61}$$

The solution for this system is easily found to be

$$\phi_M(x) = \frac{S_0}{\Sigma_a^{(M)}} \left\{ 1 - \frac{\cosh \varkappa_M(b - x)}{\cosh \varkappa_M(b - a) \left[1 + 2D_M \varkappa_M \left(\frac{1 + \alpha}{1 - \alpha} \right) \tanh \varkappa_M(b - a) \right]} \right\} \tag{10.62}$$

The disadvantage factor is obtained from (10.57). In this application it is convenient to compute $\bar{\phi}_F$ from the relation

$$\Sigma_a^{(F)} \bar{\phi}_F a = D_M \phi'_M(a) \tag{10.63}$$

since the net current *into* the plate must be the rate at which neutrons are lost by absorption. It follows that

$$\zeta = \frac{a\Sigma_a^{(F)}}{(b - a)\Sigma_a^{(M)}} \left\{ \frac{(b - a)\Sigma_a^{(M)}}{a\Sigma_a^{(F)}} \left[\left(\frac{1 + \alpha}{1 - \alpha} \right) 2a\Sigma_a^{(F)} \right] + \varkappa_M(b - a) \coth \varkappa_M(b - a) - 1 \right\} \tag{10.64}$$

Note that the only properties of the fuel plate involved in this result are its absorption cross section and the quantity α. Thus, so long as diffusion theory is valid in the moderator (particularly in the vicinity of the plate), the only new information we require in using this approach is an estimate of α, and this can be obtained by any one of a number of methods, e.g., transport theory or experiment. It should be noted also that Eq. (10.64) implies the result (10.58), since it is easily shown that if the diffusion-theory expression is used for α, namely,

$$\frac{D_M \phi'_M}{\phi_M} = \frac{D_F \phi'_F}{\phi_F} = \frac{1}{2} \left(\frac{1 - \alpha}{1 + \alpha} \right) \tag{10.65}$$

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then the function F is given by [cf. Eq. (10.60)]

$$F = 2a\Sigma_a^{(F)}\left(\frac{1+\alpha}{1-\alpha}\right) \tag{10.66}$$

The analysis for the system with thin fuel pins proceeds in the same way. Without further comment we list the results below:

$$\phi_M(\rho) = \frac{S_0}{\Sigma_a^{(M)}}\left[\frac{L(\rho)}{2D_M\left(\frac{1+\alpha}{1-\alpha}\right)L'(\rho_F) - L(\rho_F)} + 1\right] \tag{10.67}$$

where
$$\begin{aligned} L(\rho) &\equiv I_0(\varkappa_M\rho)\,K_1(\varkappa_M\rho_M) + I_1(\varkappa_M\rho_M)\,K_0(\varkappa_M\rho) \\ L'(\rho) &\equiv \varkappa_M[I_1(\varkappa_M\rho)\,K_1(\varkappa_M\rho_M) - I_1(\varkappa_M\rho_M)\,K_1(\varkappa_M\rho)] \end{aligned} \tag{10.68}$$

The expression for ζ has the form (10.59), with the E given in Table 10.2, and

$$F = \rho_F\Sigma_a^{(F)}\left(\frac{1+\alpha}{1-\alpha}\right) \tag{10.69}$$

The calculation of the transmission coefficients α in the case of *very thin*

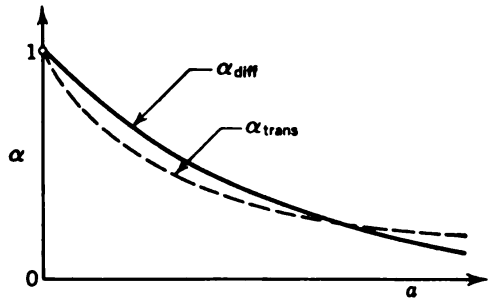

FIG. 10.7 Transmission coefficient for thin fuel plates.

members was carried out in Sec. 5.8a. The results are given in Eqs. (5.331) and (5.332):

$$\begin{aligned} &\textit{Thin fuel plates:} && \alpha \simeq 1 - 4a\Sigma_a^{(F)} \\ &\textit{Thin fuel pins:} && \alpha \simeq 1 - 2\rho_F\Sigma_a^{(F)} \end{aligned} \tag{10.70}$$

It is of some interest to compare the results for α using the diffusion-theory and first-flight transport models for a thin fuel plate. These are found to be, with $2a\Sigma_a^{(F)} \ll 1$,

First-flight transport:

$$\alpha \simeq 1 - 4a\Sigma_a^{(F)} + [2a\Sigma_a^{(F)}]^2E_1[2a\Sigma_a^{(F)}] \tag{10.71}$$

Diffusion theory:

$$\alpha \simeq 1 - 4a\Sigma_a^{(F)} + 2[2a\Sigma_a^{(F)}]^2$$

These functions are shown in Fig. 10.7. The corresponding results for ζ are sketched in Fig. 10.8.

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d. Thermal Utilization. The calculation of the thermal utilization for the heterogeneous reactor is carried out in first approximation by means of Eq. (10.5). As previously mentioned (Sec. 10.1c), the thermal utilization is decreased in going to a heterogeneous arrangement. Moreover,

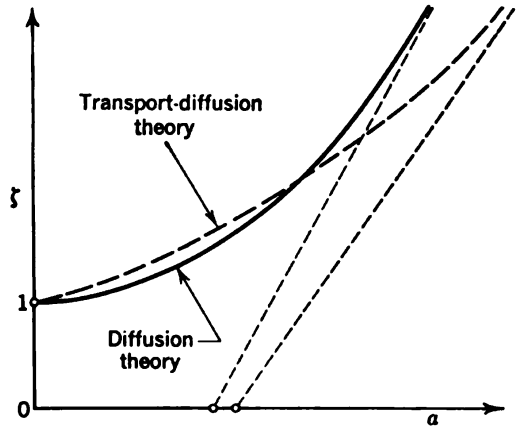

FIG. 10.8 Disadvantage factor for thin fuel plates.

for a given volume ratio V_M/V_F, f is further decreased with increase in cell size. This behavior is due to the increase in ζ, as may be seen from Fig. 10.8. The comparison of f for a homogeneous and a heterogeneous system with the same volume ratio gives

$$\frac{f_{\text{het}}}{f_{\text{hom}}} = \frac{\Sigma_a^{(F)} + \Sigma_a^{(M)}(V_M/V_F)}{\Sigma_a^{(F)} + \Sigma_a^{(M)}(V_M/V_F)\zeta} \tag{10.72}$$

It is understood that the cross sections indicated here are for the pure materials. The general shape of the ratio of f values is shown in Fig. 10.9 as a function of a characteristic cell dimension.

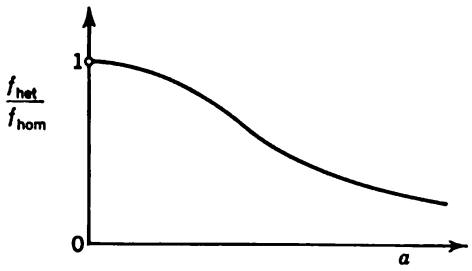

FIG. 10.9 Thermal utilization as function of cell size.

e. Engineering Considerations in Unit-cell Analysis. The unit-cell configurations discussed so far have consisted simply of a region of moderator and a region of fuel. In practice, of course, such simplicity is seldom achieved. Actual heterogeneous lattices will contain a variety of foreign materials and voids or passages whose presence is dictated by engineering requirements. For example, if the reactor of interest is to produce power, flow passages must be provided to carry coolant for extracting the heat. The presence of the coolant must of course be considered in determining the flux distributions in the unit cell. Further, in many instances the fuel and/or the moderator materials may be sensitive to corrosive attack or oxidation by the cooling fluids and therefore

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must be clad or plated with special materials. Thus additional neutron poisons are introduced into the system which may have a significant effect on such parameters as the thermal utilization.

Since an enormous number of materials and passage geometry combinations are encountered in engineering practice, it is not feasible to attempt a general treatment of the subject. Thus in the present analysis we study a particular configuration which includes two important engineering features frequently required in practical designs. The general approach to this specific problem will display several physical concepts that are applicable to other systems as well.

Consider then the cylindrical geometry shown in Fig. 10.10. In this system the fuel rod (radius ρ_F) is clad with a dissimilar metal of thickness t. Concentric with the rod is an annular passage (outside radius ρ_g) for the flow of coolant gas. Beyond this is the moderator region, and the outer boundary of the unit cell is taken as the circle of radius ρ_M. For the analysis of this system we assume that (1) diffusion theory applies in both the fuel and moderator regions, (2) the cladding material has a transmission coefficient α, and (3) there is a uniform source of thermals S_0 in the moderator region. On this basis the pair of equations (5.349) and (5.350) are applicable. These equations are to be solved with the boundary conditions

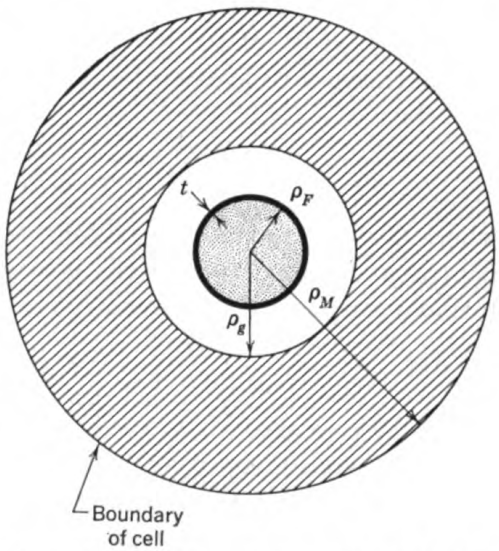

FIG. 10.10 Unit cell for cylindrical fuel rod with cladding and coolant gas passage.

$$\begin{aligned}
&(1)\ \phi'_F(0) = 0 \qquad (2)\ \phi'_M(\rho_M) = 0 \\
&(3)\ \alpha\rho_F j_+^{(F)}(\rho_F) + \beta\rho_g j_-^{(M)}(\rho_g) = \rho_g j_+^{(M)}(\rho_g) \qquad (10.73) \\
&(4)\ \rho_F j_-^{(F)}(\rho_F) = \alpha(1-\beta)\rho_g j_-^{(M)}(\rho_g)
\end{aligned}$$

where

$$\beta \equiv \text{fraction of neutrons leaving the moderator at the surface } \rho_g \text{ heading inward which miss the fuel lump} \qquad (10.74)$$

It is seen then that relations (3) and (4) account for both the absorption losses in the cladding material and for the fact that with a gap there is always a nonzero probability that some neutrons will pass from moderator surface to moderator surface without encountering the fuel rod.

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This probability is the quantity β; some expressions for computing β have been derived by Newmarch.[1]

The general solutions for the fluxes may be written in the form

$$\phi_F(\rho) = AI_0(\varkappa_F\rho) \qquad \phi_M(\rho) = CL(\rho) + \frac{S_0}{\Sigma_a^{(M)}} \tag{10.75}$$

which follow from the application of conditions (1) and (2). The constants A and C are to be determined from conditions (3) and (4), and the function $L(\rho)$ is defined by (10.68).

The thermal utilization for this cell may be written in the form

$$f = \frac{\Sigma_a^{(F)}V_F\bar{\phi}_F}{\Sigma_a^{(F)}V_F\bar{\phi}_F + \Sigma_a^{(M)}V_M\bar{\phi}_M + \Sigma_a^{(C)}V_C\bar{\phi}_C} \tag{10.76}$$

where C refers to the clad region. Note that it is assumed here that no neutrons are absorbed by the gas in the coolant passage. For computational purposes it is convenient to write this expression in terms of $1/f$.

$$\frac{1}{f} - 1 = \left[\frac{V_M\Sigma_a^{(M)}}{V_F\Sigma_a^{(F)}}\right]\zeta + \frac{V_C\Sigma_a^{(C)}\bar{\phi}_C}{V_F\Sigma_a^{(F)}\bar{\phi}_F} \tag{10.77}$$

Thus we require the usual disadvantage factor ζ as well as the ratio of the average flux in the cladding to the average flux in the fuel lump. For the moderator and fuel regions we obtain the following expressions for the average flux:

$$\bar{\phi}_F = \frac{2\pi\rho_F I_1(\varkappa_F\rho_F)A}{V_F\varkappa_F} \qquad \bar{\phi}_M = \frac{S_0}{\Sigma_a^{(M)}} - \frac{2\pi\rho_0 L'(\rho_0)C}{V_M\varkappa_M^2} \tag{10.78}$$

The constants A and C are evaluated next. From the boundary conditions (3) and (4) we obtain two equations involving these quantities. Equation (10.73), condition (4), yields

$$A = \frac{\alpha\rho_0(1-\beta)(1+\gamma_M)}{\rho_F(1+\gamma_F)I_0(\varkappa_F\rho_F)}\left[CL(\rho_0) + \frac{S_0}{\Sigma_a^{(M)}}\right] \tag{10.79}$$

where

$$\gamma_F \equiv \frac{2D_F\phi_F'(\rho_F)}{\phi_F(\rho_F)} \qquad \gamma_M \equiv \frac{2D_M\phi_M'(\rho_0)}{\phi_M(\rho_0)} \tag{10.80}$$

If Eq. (10.79) is used in Eq. (10.73), condition (3), the result may be written in the form

$$\gamma_M = \frac{(1-\beta)(G+\gamma_F)}{1+\beta G + (G+\beta)\gamma_F} \tag{10.81}$$

with

$$G \equiv \frac{1-\alpha^2}{1+\alpha^2} \tag{10.82}$$

[1] D. A. Newmarch, *J. Nuclear Energy*, **2**, 52–58 (1955).

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Equation (10.81) is an explicit statement on C, by virtue of Eqs. (10.75) and (10.80). If these relations are applied, we obtain

$$C = \frac{(1 - \beta)(G + \gamma_F)S_0/\Sigma_a^{(M)}}{2D_M[1 + \beta G + (G + \beta)\gamma_F]L'(\rho_0) - (1 - \beta)(G + \gamma_F)L(\rho_0)} \tag{10.83}$$

Substitution into (10.79) yields

$$A = \frac{2\alpha\rho_0(1 + G)(1 - \beta)L'(\rho_0)D_M S_0}{\rho_F\Sigma_a^{(M)}I_0(\kappa_F\rho_F)\{2D_M[1 + \beta G + (G + \beta)\gamma_F]L'(\rho_0) - (1 - \beta)(G + \gamma_F)L(\rho_0)\}} \tag{10.84}$$

If these results are used in the relations for $\bar{\phi}$, it may be shown, after some algebra, that the disadvantage factor may be written

$$\zeta = \frac{\alpha}{1 - G}\left[\frac{V_F\Sigma_a^{(F)}}{V_M\Sigma_a^{(M)}}\right]\left[\frac{V_M\Sigma_a^{(M)}}{V_F\Sigma_a^{(F)}}F_* + E_* - 1\right] \tag{10.85}$$

where

$$F_* = \frac{\rho_F}{\rho_0}\left[\frac{1 + \beta G}{1 - \beta} - \frac{\pi G\rho_0}{V_M\Sigma_a^{(M)}}\right]F \tag{10.86}$$

$$E_* = \frac{\rho_F}{\rho_0}\left[1 + \frac{GF}{\rho_F\Sigma_a^{(F)}}\right]E + \frac{(G + \beta)V_M\Sigma_a^{(M)}}{\pi(1 - \beta)\rho_0} \tag{10.87}$$

The functions E and F are defined in the usual way (refer to Table 10.2)

$$F \equiv \frac{\rho_F\kappa_F I_0(\kappa_F\rho_F)}{2I_1(\kappa_F\rho_F)} \qquad E \equiv -\frac{V_M\kappa_M^2 L(\rho_0)}{2\pi\rho_F L'(\rho_0)} \tag{10.88}$$

These results may be checked by considering the limiting cases wherein $\alpha \to 1$ and $\rho_0 \to \rho_F$. In the first instance $G \to 0$, and

$$F_* = \frac{\rho_F}{\rho_0}\left(\frac{1}{1 - \beta}\right)F$$

$$E_* = \frac{\rho_F}{\rho_0}E + \frac{\beta V_M\Sigma_a^{(M)}}{\pi\rho_0(1 - \beta)} \tag{10.89}$$

If we also let $\rho_0 \to \rho_F$, then $\beta \to 0$, and $F_* \to F$ and $E_* \to E$.

The miss probability β has been calculated by Newmarch[1] on the basis of the diffusion-theory[2] expression for the angular distribution of neutrons at ρ_0 [refer to Eq. (5.29)]. In the case of round fuel rods, he finds that

$$\beta = \frac{1 - r + \gamma_M f(r)}{1 + \gamma_M}$$

where

$$r \equiv \frac{\rho_F}{\rho_0} \qquad f(r) \equiv 1 - \frac{2}{\pi}\sin^{-1} r - \frac{2r}{\pi}(1 - r^2)^{\frac{1}{2}} \tag{10.90}$$

[1] *Ibid.*

[2] J. H. Tait has performed similar computations using transport theory. The Calculation of the Fine Structure of the Thermal Neutron Flux in a Pile, by the Spherical Harmonics Method, *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, 1955, P/433, pp. 417–422.

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Newmarch performed his analysis for the rod without cladding, i.e., $\alpha = 1$ and $G = 0$. The above expression for β can be taken over directly for the more general case with cladding ($\alpha < 1$), however, by simply applying the appropriate expression for γ_M, namely, (10.81). Then, β takes the form

$$\beta = \frac{(1 - r)(1 + \gamma_F G) + (G + \gamma_F) f(r)}{(1 + \gamma_F G) + (G + \gamma_F) f(r) + r(G + \gamma_F)} \tag{10.91}$$

It is seen that this expression reduces to Newmarch's result when $\alpha \to 1$. When the coolant-gas passage is very narrow so that $\rho_F \simeq \rho_0$, then the substitution of the form $r \equiv 1 - \delta$ with $\delta \ll 1$ yields the approximation

$$\beta \simeq \frac{(1 + \gamma_F G)\delta}{(1 + G)(1 + \gamma_F)} \tag{10.92}$$

Similar results have also been obtained by Newmarch for the case of hollow-tube fuel rods.

Of some practical interest, too, is the special case when the cladding on the fuel rod is very thin. Then $\alpha \simeq 1$, and we define

$$\alpha \equiv 1 - a_* \qquad \text{where } a_* \ll 1$$

For this case $G \simeq a_*$, and we obtain the approximations

$$\begin{aligned} F_* &\simeq \frac{\rho_F}{\rho_0}\left[\frac{1 + \beta a_*}{1 - \beta} - \frac{\pi \rho_0 a_*}{V_M \Sigma_a^{(M)}}\right] F \\ E_* &\simeq \frac{\rho_F}{\rho_0}\left[1 + \frac{F a_*}{\rho_F \Sigma_a^{(F)}}\right] E + \frac{(\beta + a_*) V_M \Sigma_a^{(M)}}{\pi \rho_0 (1 - \beta)} \end{aligned} \tag{10.93}$$

and (10.85) becomes

$$\begin{aligned} \zeta \simeq \frac{V_F \Sigma_a^{(F)}}{V_M \Sigma_a^{(M)}} \Bigg\{ &\left[\frac{V_M \Sigma_a^{(M)} F}{V_F \Sigma_a^{(F)}(1 - \beta)} + E\right]\frac{\rho_F}{\rho_0} - 1 + \frac{\beta V_M \Sigma_a^{(M)}}{\pi \rho_0 (1 - \beta)} \\ &+ a_* \left[\frac{F}{\rho_0 \Sigma_a^{(F)}}\left(E - \frac{\rho_0}{\rho_F} + \frac{\beta V_M \Sigma_a^{(M)}}{\pi \rho_F (1 - \beta)}\right) + \frac{V_M \Sigma_a^{(M)}}{\pi \rho_0 (1 - \beta)}\right]\Bigg\} \end{aligned} \tag{10.94}$$

The computation of the thermal utilization proceeds directly from (10.77). By substituting (10.94), we obtain

$$\begin{aligned} \frac{1}{f} - 1 = &\left[\frac{V_M \Sigma_a^{(M)} F}{V_F \Sigma_a^{(F)}(1 - \beta)} + E\right]\frac{\rho_F}{\rho_0} - 1 + \frac{\beta V_M \Sigma_a^{(M)}}{\pi \rho_0 (1 - \beta)} \\ &+ a_* \left\{\frac{F}{\rho_0 \Sigma_a^{(F)}}\left[E - \frac{\rho_0}{\rho_F} + \frac{\beta V_M \Sigma_a^{(M)}}{\pi \rho_F (1 - \beta)}\right] + \frac{V_M \Sigma_a^{(M)}}{\pi \rho_0 (1 - \beta)}\right\} + \frac{V_C \Sigma_a^{(C)} \bar{\phi}_C}{V_F \Sigma_a^{(F)} \bar{\phi}_F} \end{aligned} \tag{10.95}$$

Similar results obtained from a slightly different viewpoint have been reported by other investigators. It may be shown, after some algebra,

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that the above result agrees term by term with the earlier treatments.[1,2] The only apparent difference between this work and the previous calculations is in the handling of the boundary condition at the gap and the description of the flux distribution through the cladding material. The latter is perhaps the more subtle consideration of the two. The distinction between the two approaches on this point is easily demonstrated. In the earlier work it is assumed that the flux distribution at the outer surface of the cladding is given by the linear function

$$\phi_C(\rho_F + t) \simeq \phi_C(\rho_F) + t\phi'_C(\rho_F) = \phi_F(\rho_F) + \frac{tD_F\phi'_F(\rho_F)}{D_C}$$

The last equality follows by virtue of continuity requirements. This quantity may also be obtained from the condition on the partial current through the cladding,

$$\alpha\rho_F j_+^{(C)}(\rho_F) = \rho_C j_+^{(C)}(\rho_F + t)$$

If we use $\alpha \equiv 1 - a_*$ in this relation, and the approximations $\rho_F \simeq \rho_C$, $\phi'_C(\rho_F) \simeq \phi'_C(\rho_F + t)$ and neglect a_* as small compared to unity, then it may be shown that

$$\phi_C(\rho_F + t) = \phi_F(\rho_F) + 2a_* D_F\phi'_F(\rho_F)$$

Comparison with the expression given above for $\phi_C(\rho_F + t)$ reveals that a_* must be defined by

$$a_* \equiv \frac{t}{2D_C} \tag{10.96}$$

For a very thin layer of cladding it would be appropriate to use

$$a_* = 2\Sigma_a^{(C)}t$$

If the relation (10.96) is used in (10.95), the agreement is complete, as previously mentioned.

There are several approximations which may be generally applied in the analysis of composite unit cell structures of the type considered here: (1) Narrow gaps and channels containing air or other gases are most easily treated by requiring that the flux and the net current be continuous through the gap; thus,

$$\begin{aligned} \phi_F(\rho_F) &= \phi_M(\rho_0) \\ D_F\nabla\phi_F(\rho_F) &= D_M\nabla\phi_M(\rho_0) \end{aligned} \tag{10.97}$$

[1] See, for example, E. A. Guggenheim and M. H. L. Pryce, Uranium Graphite Lattices, *Nucleonics*, **11**(2), 50–60 (February, 1953).

[2] D. J. Littler and J. F. Raffle, "An Introduction to Reactor Physics," pp. 90–95, Pergamon Press, Inc., New York, 1955.

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These are simplifications of the partial current conditions given in (10.73). If the cell is multiregional, then (2) diffusion theory may be used in each, provided, of course, that the regions have reasonable widths, and (3) with continuity conditions to be satisfied at the interfaces. If a region is very thin, then (4) it is convenient to use the transmission coefficient to describe its neutron properties. For composite regions of different moderating materials it is necessary (5) to take the thermal-neutron source in each as proportional to its value of $\xi\Sigma_s$; however, (6) if the region is thin (and especially if it is "metallic"), then the thermal source in the region may be ignored.

Without delving into any further detail it is possible also to draw a few general conclusions about the effect on the thermal utilization of various engineering materials in a cell: (1) In cylindrical geometry the presence of gas gaps between the fuel region and the moderator tends to decrease ζ and thereby increase f. (2) Metallic sheaths on fuel lumps tend to increase ζ since the metal[1] will generally have a large scattering cross section and thus serve as a resistance to the flow of neutrons into the lump, thereby giving rise to a further decrease in the flux available to the fuel. (3) Channels containing hydrogenous materials cause a decrease in f because the neutron-absorption losses tend to exceed the moderating (thermal-neutron producing) properties of the material. In the case of water the situation is further aggravated because of the small diffusion coefficient which creates a large gradient in the neutron density.

10.3 Resonance-escape Probability

a. Neutron-energy Spectrum in Unit Cell. It was mentioned previously that the use of a heterogeneous configuration can lead to substantial increases in the resonance-escape probability. This trend is explained in terms of the neutron-energy spectrum in the unit cell. Figure 10.11 shows the spatial distribution of the neutron flux in a typical slab geometry as a function of neutron energy. It is seen that at neutron energies far removed from the values corresponding to the resonance-absorption peaks in the fuel (and also away from the thermal range) the fuel region is exposed to flux levels comparable to that in the moderator. However, when the neutron energy is in the range where the absorption cross section of the fuel is relatively high, the flux distribution displays a depressed region in and around the fuel lump. The appearance of this depression gives rise to a significant reduction in the removal rate of neutrons from the slowing-down process. This is easily demonstrated if we compute the change in the total number of neutrons slowing down throughout

[1] An important exception is aluminum.

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the cell over the lethargy interval Δu; thus,

$$Q(u + \Delta u) - Q(u) = \int_{\text{cell}} q(x,u + \Delta u)\, dx - \int_{\text{cell}} q(x,u)\, dx$$
$$= - \int_{\text{cell}} \Sigma_a(u)\phi(x,u)\, \Delta u\, dx \qquad (10.98)$$

Clearly, if the fuel cross section is greater than that of the moderator,

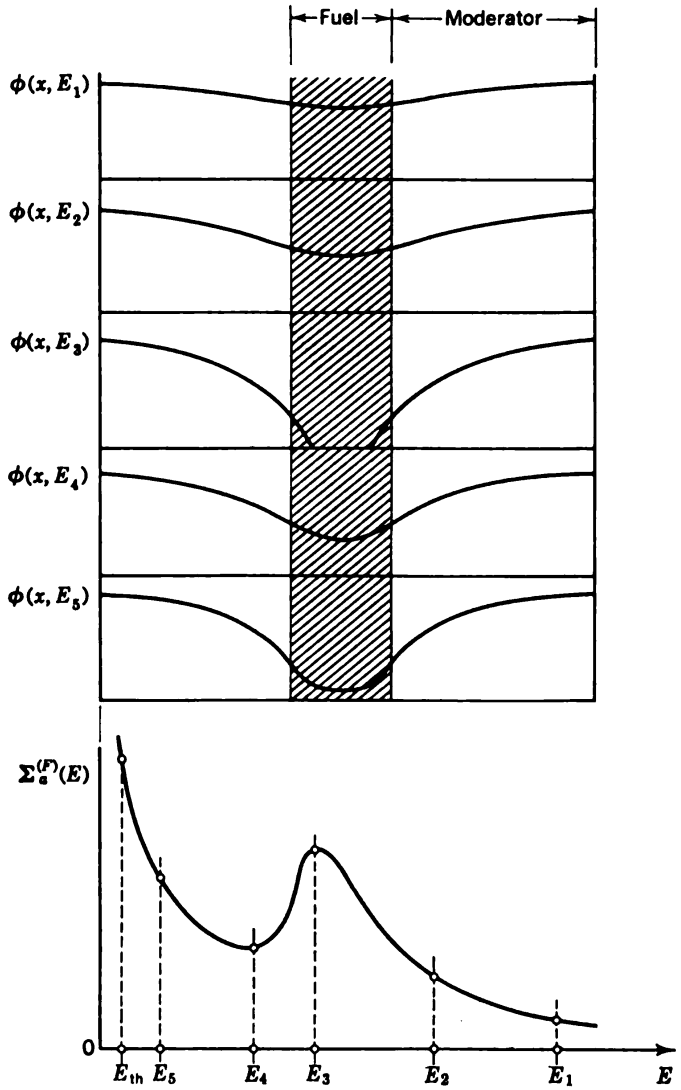

FIG. 10.11 Flux depression as function of neutron energy.

the change in Q will be smaller in the case of the heterogeneous geometry than in the homogeneous, since in the latter the fuel nuclei are in as high a neutron flux as the moderator nuclei, for all neutron energies. As a consequence, the resonance-escape probability will be higher in the heterogeneous system.

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The increase in the resonance-escape probability has been attributed to two effects, namely, the self-shielding of the fuel lump and the physical separation of the two materials (refer to Secs. 10.1c and d). The purpose of the present section is to study these effects in detail. The treatment begins with an elementary analysis of the latter effect, which arises from purely geometric considerations. This study is followed by an outline of an analytical procedure (based on nuclear-resonance parameters) for computing the integrals which appear in the expression for the resonance-escape probability [see, for example, (10.25) and (10.37)]. Application is made first to homogeneous systems, and the results are later extended to heterogeneous systems.

b. Geometric Effect. The influence of the geometric effect on the resonance-escape probability is demonstrated for the case of slab geometry. Consider therefore an infinite heterogeneous array of the type shown in Fig. 10.3. Previously, we considered the diffusion of thermal neutrons in this system with an assumed uniform source of thermal neutrons in the moderator; we consider now the slowing down of neutrons in this geometry. This problem is similar to that of the reflected reactor and, as before, introduces the difficulty of applying an energy-dependent boundary condition.[1] In order to simplify the analysis we eliminate this difficulty by proposing the wholly artificial model based on fuel and moderator materials having energy-independent cross sections. We further assume that fast-fission neutrons (at $u = 0$) begin the slowing-down process in the moderator, and not in the fuel slab. On this basis we apply to the system in Fig. 10.3 the following set of equations for the fast neutrons in the moderator:

$$-D_M \frac{\partial^2}{\partial x^2} \phi_M(x,u) + \Sigma_a^{(M)} \phi_M(x,u) = -\frac{\partial}{\partial u} q_M(x,u) \qquad (10.99)$$
$$q_M(x,u) = \xi_M \Sigma_i^{(M)} \phi_M(x,u)$$

along with the boundary conditions

$$(1)\ \frac{\partial}{\partial x} \phi_M(x,u)\bigg|_{x=b} = 0 \qquad (2)\ j_+^{(M)}(a,u) = \alpha j_-^{(M)}(a,u) \qquad (10.100)$$

Here, $j_+^{(M)}(a,u)$ denotes the partial neutron currents in the moderator at the fuel-moderator interface, and α is the transmission coefficient. For the partial currents we use the diffusion approximation and assume the separability of the space and lethargy dependence of the slowing-down density:

$$q_M(x,u) \equiv F(x)\, q_M(u) \qquad (10.101)$$

[1] See Sec. 6.3a for discussion of this point in connection with the extrapolated boundary condition.

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Substitution into (10.99) yields the requirement that

$$F''(x) + \lambda^2 F(x) = 0 \tag{10.102}$$

and the solutions

$$\begin{aligned} F(x) &= \cos \lambda(b - x) \\ q_M(u) &= q_0 \exp\left\{ - \left[\frac{D_M \lambda^2 + \Sigma_a^{(M)}}{\xi_M \Sigma_t^{(M)}} \right] u \right\} \end{aligned} \tag{10.103}$$

where we have used boundary condition (1) in (10.100). Note that the distribution of source neutrons for the slowing-down process is given by

$$q_M(x,0) = F(x)\, q_M(0) = F(x) q_0 \tag{10.104}$$

The quantity λ resulting from the separation of variables in (10.99) is determined by the application of boundary condition (2) in (10.100). For α independent of lethargy, the appropriate relation is found to be

$$1 = 2\left(\frac{1 + \alpha}{1 - \alpha}\right) D_M \lambda \tan \lambda(b - a) \tag{10.105}$$

In the case of thick fuel plates, α may be computed from diffusion theory, and is given by (5.308); for thin plates the result (5.331) may be used.

The calculation of the resonance-escape probability proceeds from the expression for $q_M(x,u)$. If we compute the total number of neutrons reaching thermal throughout the entire cell, and divide this by the total number of neutrons entering the slowing-down process, we obtain

$$p_{\text{het}}(u_{\text{th}}) \equiv \frac{\int_a^b q_M(x,u_{\text{th}})\, dx}{\int_a^b q_M(x,0)\, dx} = \exp\left\{ - \left[\frac{D_M \lambda^2 + \Sigma_a^{(M)}}{\xi_M \Sigma_t^{(M)}} \right] u_{\text{th}} \right\} \tag{10.106}$$

This result is to be compared with the corresponding relation for the equivalent homogeneous system. We noted previously in the discussion following Eq. (10.57) that the results for the homogeneous system could be derived from those of the heterogeneous by taking the limiting case of very thin fuel plates (holding the ratio a/b constant). Then, using (5.331) for α, we find from (10.105)

$$\lambda^2 \to \frac{\Sigma_a^{(F)}}{D_M(b/a - 1)} \tag{10.107}$$

The substitution of this relation into (10.106) yields

$$p_{\text{hom}}(u_{\text{th}}) = \exp\left\{ - \left[\frac{a\Sigma_a^{(F)}/b + (1 - a/b)\Sigma_a^{(M)}}{\xi_M \Sigma_t^{(M)}(1 - a/b)} \right] u_{\text{th}} \right\} \tag{10.108}$$

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A comparison of this result with (10.106) shows the effect of the scale of the heterogeneity on the resonance-escape probability (see Fig. 10.12). The principal conclusions to be drawn are: (1) as the cell size becomes larger the resonance-escape probability increases monotonically from its initial value corresponding to that in the equivalent homogeneous system; and (2) for large cells the resonance-escape probability approaches a limiting value.

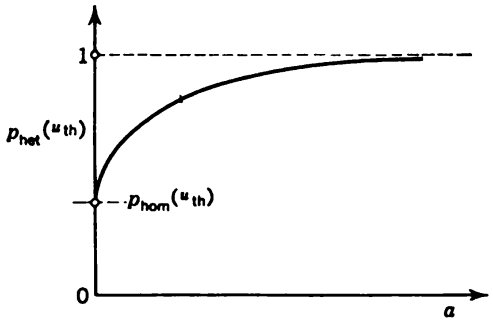

FIG. 10.12 Resonance-escape probability as function of cell size, with b/a held constant.

It is important to point out that the elementary model used in the present calculation was selected only to display the influence of the geometric effect on the resonance-escape probability, and it is not intended that it be used as a procedure for computing this quantity.

c. Calculation of the Collision Density. The general statement of the resonance-escape probability given by Eq. (10.18) was defined in terms of the total collision density throughout the resonance range. An important approximation to this expression was the formula (6.174) which applies in the narrow resonance approximation wherein the collision density takes the asymptotic form $1/E$. In the present treatment we require somewhat better approximations for the collision density in order to develop more generally useful expressions for the effective resonance integral. In fact, the calculation of the energy spectrum and spatial dependence of the collision density in systems containing substantial amounts of fuel (or strong absorber) is the essential problem in resonance absorption.

In the case of homogeneous systems, the various approximations to the collision density may be developed from the general expression (4.119) derived in the slowing-down theory. If we limit attention to a system containing an absorber, denoted by subscript F, and a moderator, denoted by M, then the appropriate form is

$$F(u) = \frac{1}{1-\alpha_F}\int_{u-\ln\frac{1}{\alpha_F}}^{u} F(u')\,\frac{\Sigma_s^{(F)}(u')}{\Sigma_t(u')}\,e^{u'-u}\,du' + \frac{1}{1-\alpha_M}\int_{u-\ln\frac{1}{\alpha_M}}^{u} F(u')\,\frac{\Sigma_s^{(M)}(u')}{\Sigma_t(u')}\,e^{u'-u}\,du' \quad (10.109)$$

where F is defined by (10.16). If Eq. (10.109) is solved for $F(u)$, then the resulting expression may be used in the general formula (10.18) to compute the resonance-escape probability. This calculation may be carried out with whatever precision is consistent with the requirement

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on p. In this development we derive various first-order approximations to F. Nordheim[1] has prepared a direct numerical procedure appropriate to fast computing machines which may be used to achieve greater precision.

A first approximation to Eq. (10.109) may be obtained by introducing the condition, of considerable practical importance, that the resonances in the fuel (or absorber) material are narrow on the energy scale relative to the average energy loss suffered by a neutron in a scattering collision with a moderator atom and, furthermore, are widely separated. In this case we can use the asymptotic expression for the function F which appears in the collision integral for the moderator material and note that it is also consistent to take $\Sigma_s^{(M)} \simeq \Sigma_t$ over the interval of integration. From the analysis of Sec. 4.4b we have $q(u) = \xi\Sigma_t(u)\,\phi(u)$, and it follows that for a normalized source

$$F(u) = \frac{1}{\xi} \tag{10.110}$$

which is a good approximation for a system with light absorption. The use of the asymptotic form in the second integral is permissible since, as previously noted, the collision density is a relatively slowly varying function, even through a strong but sharp resonance (see Fig. 4.21). With these simplifications, Eq. (10.109) reduces to

$$F(u) = \frac{1}{1 - \alpha_F} \int_{u - \ln \frac{1}{\alpha_F}}^{u} F(u') \frac{\Sigma_s^{(F)}(u')}{\Sigma_t(u')}\, e^{u'-u}\, du' + \frac{1}{\xi} \tag{10.111}$$

In the event that all resonances of the fuel are narrow relative to the average energy loss in a collision with a *fuel* atom, the asymptotic form (10.110) may be used throughout in evaluating the effective resonance integral.

Similar approximations may be developed for the collision density in heterogeneous systems. In the development which follows we represent the heterogeneous system by a unit cell of volume V consisting of a segment V_F which is occupied by the fuel material and a segment V_M by the moderator. As a zeroth-order approximation we impose the *flat-flux assumption*, namely, that the neutron flux throughout the resonance range, including the intervals occupied by the resonances, is spatially uniform in both volumes V_F and V_M. This simplification is not valid when: (1) the fuel elements (represented by V_F) are closely packed compared to the mean free path $\lambda_t^{(M)}$ in the moderator material, or (2) when the spacing of the elements is of the order of the slowing-down distance $(\tau_M^{\frac{1}{2}})$ in the moderator, or (3) when the characteristic dimension of the fuel elements is much larger than $\lambda_t^{(M)}$.

[1] L. W. Nordheim, "The Theory of Resonance Absorptions," General Atomics, GAMD-638, San Diego, Calif., Jan. 6, 1959.

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When the requirements for the flat-flux assumption are well satisfied, the total number of neutrons slowing down in the cell Q_0 may be approximated by the asymptotic form; thus we take

$$\phi_M^{\infty}(u)\overline{\xi\Sigma_s V} \equiv Q_0 \equiv 1 \tag{10.112}$$

where one neutron is introduced per unit time in the cell volume V. We define

$$\overline{\xi\Sigma_s V} \equiv \xi_F\Sigma_p^{(F)}V_F + \xi_M\Sigma_s^{(M)}V_M \tag{10.113}$$

with $\Sigma_p^{(F)} = N_F\sigma_p^{(F)}$, and ϕ_M^{∞} denotes the spatially uniform resonance flux in the moderator region. In order to compute the total collision density throughout the fuel volume, we introduce $\bar{\phi}_F(u)$, the spatially uniform (or averaged) flux at lethargy u in V_F from all sources, and $\bar{\phi}_F^s(u)$, the spatially uniform flux at lethargy u in V_F arising wholly from neutrons of lethargy u entering by transport through the surface A_F of the fuel element from the moderator region. We require also the quantity $P_{esc}(u)$, the probability that a neutron appearing in V_F (from some source) at lethargy u escapes without collision from V_F. For the present analysis we take P_{esc} to be independent of u and defined by Eq. (7.234); in this application the appropriate form is

$$P_{esc} = \frac{1}{\bar{s}\Sigma_t^{(F)}}\int_s [1 - e^{-s\Sigma_t^{(F)}}]h(s)\,ds \tag{10.114}$$

with $\bar{s}$ given by (7.232). It should be recalled that $h(s)$ is the normalized distribution of chord lengths in V_F, each chord length weighted by the sum of the cosines of the angles it makes with the normals to the surface of V_F at each end. A cosine weighting is applied for the neutrons at the surface of V_F since this gives a reasonable first approximation to the angular distribution of the neutrons when the resonance absorption of the fuel lump is small (as implied here) and the net transport of neutrons is low.[1]

With these definitions we write the total collision density in V_F as consisting of two parts: one due to fast neutrons in V_F which slow down into the unit lethargy interval about u, and another due to the net transport of neutrons of lethargy u through the surface A_F from the moderator; thus,

$$\bar{\phi}_F(u)\,\Sigma_t^{(F)}(u) = \frac{1 - P_{esc}}{1 - \alpha_F}\int_{u-\ln\frac{1}{\alpha_F}}^{u} \bar{\phi}_F(u')\,\Sigma_s^{(F)}(u')e^{u'-u}\,du' + \Sigma_t^{(F)}(u)\,\bar{\phi}_F^s(u) \tag{10.115}$$

Following (7.227) we take

$$\bar{\phi}_F^s(u) \equiv P_{esc}\,\phi_M^{\infty}(u) \tag{10.116}$$

[1] S. M. Dancoff and M. Ginsburg, "Surface Resonance Absorption in a Close-packed Lattice," Argonne Metallurgical Laboratory, CP-2157, 1944.

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and using

$$F_F(u) \equiv \Sigma_t^{(F)}(u)\ \bar{\phi}_F(u) \tag{10.117}$$

and the expression (10.112), Eq. (10.115) reduces to the form

$$F_F(u) = \frac{1 - P_{esc}}{1 - \alpha_F} \int_{u - \ln \frac{1}{\alpha_F}}^{u} F_F(u') \frac{\Sigma_s^{(F)}(u')}{\Sigma_t^{(F)}(u')} e^{u'-u}\, du' + \frac{P_{esc}\Sigma_t^{(F)}(u)}{\xi\Sigma_s V} \tag{10.118}$$

The solution to this equation which applies in the asymptotic range is obtained by setting $F_F(u) =$ constant. If we take also $\sigma_t \simeq \sigma_s \simeq \sigma_p$ to be consistent, then it is easily shown that

$$F_F(u) \simeq \frac{\Sigma_p^{(F)}}{\xi\Sigma_s V} \qquad \text{and} \qquad \bar{\phi}_F(u) \simeq \frac{1}{\xi\Sigma_s V} \tag{10.119}$$

as is to be expected.

The results developed above may be used to obtain the first-order estimates for the collision densities mentioned previously in Sec. 10.1c. When the resonances are narrow relative to the average loss in energy due to collisions with fuel nuclei and in addition widely separated, we may use the *narrow resonance* (NR) *approximation:*

$$F(u) = \frac{1}{\xi} \qquad \text{NR for homogeneous} \tag{10.110}$$

for the homogeneous system. The corresponding expression for the heterogeneous system is obtained by using the result (10.119) in the integral of Eq. (10.118) and taking $\sigma_t \simeq \sigma_s$ over the integration interval. The solution is seen to be

$$F_F(u) = \frac{N_F}{\xi\Sigma_s V} [\sigma_t^{(F)}(u)\ P_{esc} + (1 - P_{esc})\sigma_p^{(F)}] \qquad \text{NR for heterogeneous} \tag{10.120}$$

Another interesting approximation is obtained when the fuel resonances are narrow compared to the average energy loss in collisions with the moderator, but wide compared to that in collisions with the fuel. These constraints lead to the *narrow resonance–infinite absorber* (NRIA) *approximation.* In the case of the homogeneous system the appropriate form for the collision density is obtained from Eq. (10.111) by withdrawing the factor $F\Sigma_s^{(F)}/\Sigma_t$ from inside the integral sign and solving for F. The solution is

$$F(u) = \frac{\Sigma_t(u)}{\xi[\Sigma_t(u) - \Sigma_s^{(F)}(u)]} \qquad \text{NRIA for homogeneous} \tag{10.121}$$

For the heterogeneous system we use Eq. (10.118) and find, by making a

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similar approximation to the fuel integral,

$$F_F(u) = \frac{\Sigma_t^{(F)}(u)\, P_{esc}}{\overline{\xi\Sigma_s V}\left[1 - (1 - P_{esc})\dfrac{\sigma_s^{(F)}(u)}{\sigma_t^{(F)}(u)}\right]} \qquad \text{NRIA for heterogeneous} \tag{10.122}$$

d. Approximate Formulas for Resonance Integrals. The calculation of the resonance-escape probability (10.18) requires the evaluation of the resonance integral. In terms of the lethargy variable this is

$$I \equiv \int_0^{u_{th}} \frac{F(u)\,\Sigma_a(u)\,du}{\Sigma_t(u)} \tag{10.123}$$

The effective resonance integral may be computed by means of the relation (10.21).

In this development we apply the various approximations for $F(u)$ derived in the preceding analysis to the calculation of $\mathcal{I}$ for both homogeneous and heterogeneous systems, using Eqs. (10.123) and (10.21). We begin with the *homogeneous system.* In the narrow resonance approximation, the appropriate form for F is (10.110). Substitution into (10.123) yields the result previously given in (10.20); in the present notation, we have

$$I = \frac{1}{\xi\sigma_p}\int_0^{u_{th}} \frac{\sigma_a^{(F)}(u)\,du}{1 + \dfrac{1}{\sigma_p}[\sigma_a^{(F)}(u) + \sigma_{sr}^{(F)}(u) + \sigma_{si}^{(F)}(u)]} \tag{10.124}$$

and [cf. Eq. (10.23)]

$$\mathcal{I} = \int_0^{u_{th}} \frac{\sigma_a^{(F)}(u)\,du}{1 + \dfrac{1}{\sigma_p}[\sigma_a^{(F)}(u) + \sigma_{sr}^{(F)}(u) + \sigma_{si}^{(F)}(u)]} \qquad \text{NR for homogeneous} \tag{10.125}\star$$

with σ_p given by (6.177). Again we note that these formulas apply when the resonances are narrow and far apart. The criterion of narrowness is discussed in a subsequent section.

Similar forms may be obtained in the NRIA approximation. The form for F in this case is given by (10.121), and application to (10.123) yields

$$I = \int_0^{u_{th}} \frac{\Sigma_a(u)\,du}{\xi[\Sigma_t(u) - \Sigma_s^{(F)}(u)]}$$

which may be written

$$I = \frac{1}{\xi}\int_0^{u_{th}} \frac{\Sigma_a(u)\,du}{\Sigma_a^{(F)}(u) + \Sigma_a^{(M)}(u) + \Sigma_s^{(M)}} \simeq \frac{N_F}{\xi\Sigma_s^{(M)}}\int_0^{u_{th}} \frac{\sigma_a^{(F)}(u)\,du}{1 + \sigma_a^{(F)}(u)N_F/\sigma_s^{(M)}N_M} \tag{10.126}$$

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if we neglect everywhere $\Sigma_a^{(M)}$ in comparison to $\Sigma_a^{(F)}$. With this result, the effective resonance integral becomes [cf. Eqs. (10.24) and (10.21)]

$$\mathcal{I} \equiv \xi\sigma_m I = \int_0^{u_{th}} \frac{\sigma_a^{(F)}(u)\, du}{1 + \sigma_a^{(F)}(u)/\sigma_m} \qquad \text{NRIA for homogeneous} \quad (10.127)\star$$

where we have introduced

$$\sigma_m \equiv \sigma_a^{(M)} \frac{N_M}{N_F} \tag{10.128}$$

Corresponding formulas may be derived for the resonance integral in *heterogeneous systems* by applying the definition

$$I \equiv V_F \int_0^{u_{th}} \frac{F_F(u)\, \sigma_a^{(F)}(u)\, du}{\sigma_t^{(F)}(u)} \tag{10.129}\star$$

which is given for a normalized source of fast neutrons throughout the unit cell, as specified in (10.112). This definition implies, of course, that essentially all resonance absorption occurs in the fuel element here represented by the volume V_F. The expressions for $F_F(u)$ in the NR and NRIA approximations are given by Eqs. (10.120) and (10.122), respectively. Both forms involve the escape probability which may be defined

$$P_{esc} \equiv 1 - P_c \tag{10.130}$$

where P_c is the probability that a neutron born in V_F makes a collision before escaping. The quantity P_c has been tabulated by Case–de Hoffmann–Placzek for a variety of geometric shapes.[1] It is sometimes more convenient to write P_{esc} in the form

$$P_{esc} \equiv \frac{\mathcal{P}}{\bar{s}\Sigma_t^{(F)}} \qquad \text{and} \qquad \mathcal{P} \equiv \int_s [1 - e^{-s\Sigma_t^{(F)}}]h(s)\, ds \tag{10.131}$$

where we have used (10.114) for P_{esc}; note that the *penetrability* $\mathcal{P}$ is related to the transmission coefficient α of Chap. 5 by [cf. Eq. (5.275)]

$$\mathcal{P} \equiv 1 - \alpha \tag{10.132}$$

In general, as mentioned previously, P_{esc} may be computed from the integral relation (10.114); however, this formula is somewhat complicated, and the resonance integrals will be difficult to calculate. A considerable simplification is possible if we use in place of the exact form Wigner's[2] *rational approximation* which we will designate P_{esc}^{w}; the definition is

$$P_{esc}^{w} \equiv \frac{1}{1 + \bar{s}\Sigma_t^{(F)}} \tag{10.133}\star$$

[1] K. M. Case, F. de Hoffmann, and G. Placzek, "Introduction to the Theory of Neutron Diffusion," Los Alamos Scientific Laboratory, 1953, Tables 2 through 4, (available through Government Printing Office, Washington, D.C.).

[2] E. P. Wigner, *J. Appl. Phys.*, **26**, 255–279 (1955).

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This formula is a good approximation to the actual function when $\bar{s}\Sigma_t^{(F)}$ is either much greater or much less than unity.[1] Some values of P_{esc} are given in Table 10.3 for various elementary shapes, along with the corresponding values of P_{esc}^{w}. These results are also plotted in Fig. 10.13. It is of interest to note that the dependence on geometry is not strong; P_{esc} for spheres and cylinders are very close, with slabs differing only slightly. The rational approximation is seen to be quite good.

TABLE 10.3 THE ESCAPE PROBABILITY FOR VARIOUS GEOMETRIES AND THE RATIONAL APPROXIMATION

$\bar{s}\Sigma_t^{(F)}$	Sphere	Cylinder	Slab	Rational
0.04	0.978	0.974	0.952	0.962
0.1	0.946	0.939	0.902	0.909
0.2	0.896	0.885	0.837	0.823
0.3	0.850	0.819	0.785	0.769
0.5	0.767	0.753	0.701	0.667
1	0.607	0.596	0.557	0.500
2	0.411	0.407	0.390	0.333
3	0.302	0.302	0.295	0.250
5	0.193	0.193	0.193	0.167
10	0.099	0.099	0.100	0.091

In the present formulation of the resonance integral for heterogeneous systems, it will suffice to use the rational approximation for P_{esc} in the expression (10.129) for I. In the NR approximation this formula reduces to

$$I = \frac{N_F V_F}{\overline{\xi\Sigma_s V}} \int_0^{u_{th}} [P_{esc}\sigma_t^{(F)} + (1 - P_{esc})\sigma_p^{(F)}] \frac{\sigma_a^{(F)}\, du}{\sigma_t^{(F)}} \qquad (10.134)$$

and

$$g \equiv \left(\frac{\overline{\xi\Sigma_s V}}{N_F V_F}\right) I = \int_0^{u_{th}} [P_{esc}\sigma_t^{(F)} + (1 - P_{esc})\sigma_p^{(F)}] \frac{\sigma_a^{(F)}\, du}{\sigma_t^{(F)}} \qquad (10.135)$$

If we use now (10.133) for P_{esc}, we obtain

$$g = \int_0^{u_{th}} \frac{[1 + \bar{s}\Sigma_p^{(F)}]\sigma_a^{(F)}\, du}{1 + \bar{s}\Sigma_t^{(F)}} \qquad \text{NR for heterogeneous} \qquad (10.136)\star$$

The appropriate expression in the NRIA approximation is seen to be

$$I = \frac{N_F V_F}{\overline{\xi\Sigma_s V}} \int_0^{u_{th}} \frac{P_{esc}\sigma_a^{(F)}\sigma_t^{(F)}\, du}{\sigma_a^{(F)} + P_{esc}\sigma_s^{(F)}} \qquad (10.137)$$

It follows that, in the rational approximation, the effective resonance

[1] The errors involved in the use of this relation are discussed by Dresner, Thesis, *op. cit.*

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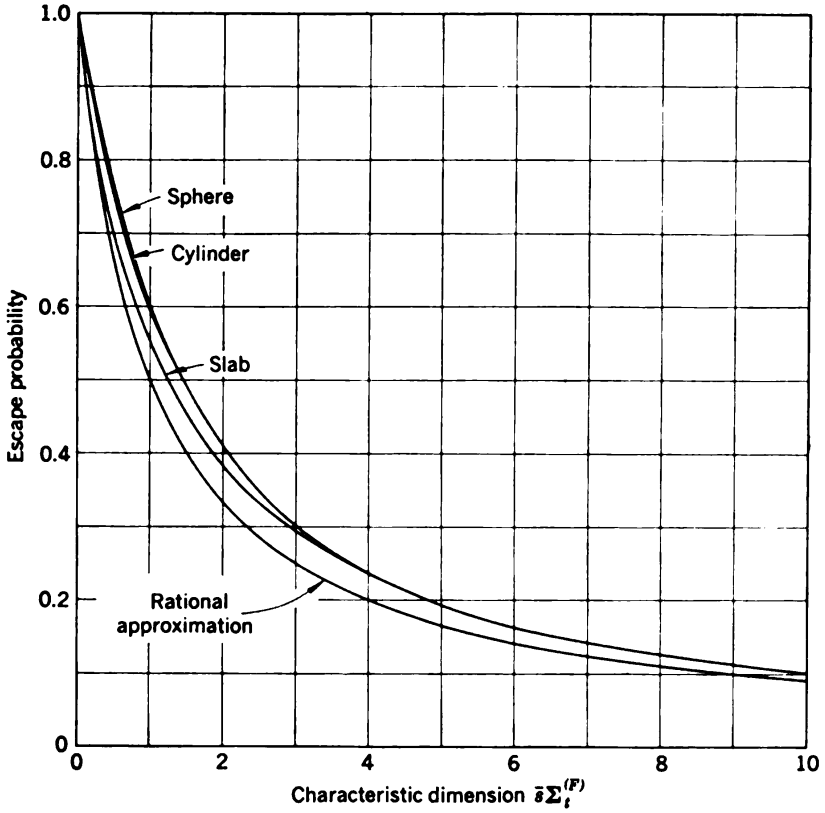

FIG. 10.13 The escape probability and the rational approximation as functions of the characteristic dimension $\bar{s}\Sigma_t^{(F)}$.

integral is [using the definition given in Eq. (10.135)]

$$\mathcal{J} = \int_0^{u_{th}} \frac{\sigma_a^{(F)}\, du}{1 + \bar{s}\Sigma_a^{(F)}} \qquad \text{NRIA for heterogeneous} \qquad (10.138)\star$$

e. Equivalence of Homogeneous and Heterogeneous Systems. The approximate formulas derived above for the effective resonance integral in homogeneous and heterogeneous systems in both the NR and NRIA approximations can be shown to be mathematically identical. Consider first the corresponding expressions in the NR approximation:

$$\mathcal{J} = \int_0^{u_{th}} \frac{\sigma_a^{(F)}\, du}{1 + \dfrac{1}{\sigma_p}\left[\sigma_a^{(F)} + \sigma_{ar}^{(F)} + \sigma_{ai}^{(F)}\right]} \qquad \text{homogeneous} \qquad (10.125)$$

$$\mathcal{J} = \int_0^{u_{th}} \frac{[1 + \bar{s}\Sigma_p^{(F)}]\sigma_a^{(F)}\, du}{1 + \bar{s}\Sigma_t^{(F)}} \qquad \text{heterogeneous} \qquad (10.136)$$

If we define a new quantity σ_p^* such that

$$\sigma_p^* \equiv \sigma_p^{(F)} + \frac{1}{\bar{s}N_F} \qquad (10.139)$$

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then (10.136) may be written

$$\mathcal{g} = \int_0^{u_{th}} \frac{\sigma_a^{(F)}\, du}{1 + (1/\sigma_p^*)[\sigma_a^{(F)} + \sigma_{sr}^{(F)} + \sigma_{si}^{(F)}]} \qquad (10.140)\star$$

which is identical in form to (10.125). Thus we find that a heterogeneous system is equivalent to a homogeneous system in the NR approximation if σ_p for the homogeneous system is the same as the σ_p^* for the heterogeneous system. It is of interest to note also that different heterogeneous systems with the same σ_p^* have the same effective resonance integral. This equivalence relationship has been recognized by P. Bakshi[1] and by L. Dresner.[2,3]

Similar relationships can be established also in the case of the NRIA approximation. We compare, therefore, the formulas

$$\mathcal{g} = \int_0^{u_{th}} \frac{\sigma_a^{(F)}\, du}{1 + \sigma_a^{(F)}/\sigma_m} \qquad \text{homogeneous} \qquad (10.127)$$

$$\mathcal{g} = \int_0^{u_{th}} \frac{\sigma_a^{(F)}\, du}{1 + \bar{s}\Sigma_a^{(F)}} \qquad \text{heterogeneous} \qquad (10.138)$$

If we introduce the symbol σ_m^* for the second term in (10.139), so that,

$$\sigma_m^* \equiv \frac{1}{\bar{s}N_F} \qquad (10.141)$$

then

$$\sigma_p^* = \sigma_p^{(F)} + \sigma_m^* \qquad (10.142)$$

and (10.138) may be written

$$\mathcal{g} = \int_0^{u_{th}} \frac{\sigma_a^{(F)}\, du}{1 + \sigma_a^{(F)}/\sigma_m^*} \qquad (10.143)\star$$

which is mathematically identical to (10.127).

It should be borne in mind that the rational approximation (10.133) was used in establishing the two equivalence relations (10.140) and (10.143). These relations are valid only so long as the width of a resonance can be considered small as compared to the average energy loss from collisions with a moderator atom at energies around that of the resonance. Generally the results obtained above will yield estimates for the heterogeneous integral which are in the order of 10 per cent too small.[4] More precise analyses than those possible with the present approximations to the collision density indicate that the equivalence relations are more generally valid[5] and hold independently of the treat-

[1] Chernick and Vernon, *op. cit.*
[2] L. Dresner, Thesis, *op. cit.*, Chap. 6.
[3] L. Dresner, *Nuclear Sci. Eng.*, **1**, 501 (1956).
[4] Nordheim, *op. cit.*
[5] L. Dresner, unpublished.

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ment of energy degradation in neutron collisions with fuel atoms.[1] Thus the principal modifications needed in the application of these relations are those arising from the geometric properties of the heterogeneous system. These become important when any of the conditions imposed by the flat-flux assumption are not satisfied.

The equivalence relations may be summed up in terms of two very useful practical rules: (1) All fuel lumps containing the same fuel (absorber) material, but with different amounts of other moderator materials, and having the same value of σ_p^*, have the same effective resonance integral regardless of shape or size. (2) The effective resonance integral for a lump is the same as that for a homogeneous mixture of the moderator and lump materials in which the potential-scattering cross section per fuel atom is equal to σ_p^* [as given by (10.139)].

An interesting interpretation of rule 1 is that data on the resonance integral for a series of fuel lumps of one material can be used to compute directly the value for a related series of lumps utilizing the same "absorber" but in an entirely different geometry. Thus, for example, data on uranium metal rods can be used to predict the resonance integrals for UO_2 spheres. It is of considerable importance to note that these rules do not depend on any knowledge whatever of the resonance parameters. They can be used both for predictive purposes and as a check on the theory.

f. Evaluation of Resonance Integrals Using Breit-Wigner Line Shape. The objective of the present calculation is to determine the contribution $\mathcal{I}_i$ to the effective resonance integral by a particular resonance at energy E_i, using the various models previously developed for estimating $\mathcal{I}$. In this treatment we assume that each resonance is well represented by the *single-level Breit-Wigner* formula. The appropriate relations for each of the energy-dependent microscopic cross sections appearing in $\mathcal{I}$ are:[2]

$$\sigma_a(\mathcal{E}) \equiv \frac{\sigma_0\Gamma_\gamma}{\Gamma}\left[1 + \left(\frac{\mathcal{E} - E_i}{\Gamma/2}\right)^2\right]^{-1} \tag{10.144}$$

$$\sigma_{sr}(\mathcal{E}) \equiv \frac{\sigma_0\Gamma_n}{\Gamma}\left[1 + \left(\frac{\mathcal{E} - E_i}{\Gamma/2}\right)^2\right]^{-1} \tag{10.145}$$

$$\sigma_{si}^{(F)}(\mathcal{E}) \equiv \left[\frac{\sigma_0\sigma_p^{(F)}g_J\Gamma_n}{\Gamma}\right]^{\frac{1}{2}}\left(\frac{\mathcal{E} - E_i}{\Gamma/4}\right)\left[1 + \left(\frac{\mathcal{E} - E_i}{\Gamma/2}\right)^2\right]^{-1} \tag{10.146}$$

These relations apply when the nuclei are at rest in the laboratory frame of reference (L). The symbol E_i denotes the position of the resonance, and $\mathcal{E}$ is the kinetic energy associated with the *relative velocity* between neutron and nucleus. It is shown in a subsequent analysis

[1] Chernick and Vernon, *op. cit.*

[2] Blatt and Weisskopf, *op. cit.*, pp. 391–394.

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that, in the case of stationary nuclei, $\mathcal{E}$ is given by

$$\mathcal{E} = \tfrac{1}{2}\mu_* v^2 \tag{10.147}$$

where v is the neutron speed in the laboratory system of coordinates, and μ_*, the reduced mass, is defined by

$$\mu_* \equiv \frac{mA}{1 + A} \tag{10.148}$$

with m and A denoting the mass of the neutron and nucleus, respectively. The symbols Γ_γ and Γ_n are the radiative capture and neutron widths, and

$$\sigma_0 \equiv \frac{4\pi\lambda\!\!\!^{-2}\Gamma_n}{\Gamma} \qquad \Gamma \equiv \Gamma_n + \Gamma_\gamma \tag{10.149}$$

Note that σ_0 is the peak value of the total (absorption plus scattering) resonance. All the relations indicated here apply to the center-of-mass system of coordinates, and $\lambda\!\!\!^{-}$, the reduced neutron wavelength, is given by[1]

$$\lambda\!\!\!^{-} \equiv \frac{\lambda}{2\pi} = \frac{h}{2\pi(2\mu_*\mathcal{E})^{\frac{1}{2}}} = \frac{h}{2\pi\mu_* v} \tag{10.150}$$

The symbol h denotes Planck's constant, and the last equality in this equation follows by virtue of (10.147). The quantity g_J denotes the statistical weight and is defined

$$g_J \equiv \frac{2J' + 1}{2(2I' + 1)} \tag{10.151}$$

where J' is the total angular momentum of neutron and nucleus and I' is the spin of nucleus.[2]

The contribution to $\mathcal{I}$ from an isolated resonance at E_i is obtained by substituting the relations (10.144) through (10.146) into the appropriate form for $\mathcal{I}$ and integrating over the resonance (i.e., the entire energy scale). For these computations it is convenient to introduce the variable

$$x \equiv \frac{2}{\Gamma}(\mathcal{E} - E_i) \tag{10.152}$$

and the parameter

$$\mathfrak{d} \equiv \frac{\sigma_p^{(F)} g_J \Gamma_n}{\sigma_p \Gamma} \tag{10.153}$$

Consider now the application of these functions to the evaluation of

[1] It is shown in Eq. (10.176) that in the case of stationary nuclei the relative energy $\mathcal{E}$ is related to the energy in the laboratory system (L) by $\mathcal{E} = \left(\frac{A}{1+A}\right)E$. Therefore, the relations (10.144) through (10.146) have the same form in (L) if the widths Γ are also measured in (L).

[2] Note that $J' = I' \pm \frac{1}{2}$ for neutrons having zero orbital angular momentum (s waves); note also that for U^{238} and Th^{232}, $g_J = 1$.

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g_i on the basis of the NR approximation. If we use the form (10.23), which is Eq. (10.125) in terms of the energy variable $\mathcal{E}$, then the expression for g_i using the Breit-Wigner formula is

$$g_i = \int_{\text{res}} \frac{\sigma_p \sigma_0 \Gamma_\gamma [\Gamma(1 + x^2)]^{-1} (\Gamma/2\mathcal{E})\, dx}{\sigma_p + \sigma_0(1 + x^2)^{-1} + 2x(\sigma_0 \sigma_p \mathfrak{d})^{\frac{1}{2}}(1 + x^2)^{-1}} \tag{10.154}$$

In the case of *narrow resonances*, the total width Γ is appreciably less than $\mathcal{E}$, and it is a good approximation to ignore the energy dependence of σ_0, Γ, and $\mathcal{E}$ over the range of integration and simply evaluate them at E_i. Moreover, the limits of integration on x can be extended to $\pm\infty$ without introducing serious error. Then, with some minor rearrangement, (10.154) may be integrated to yield

$$g_i = \frac{\sigma_p \Gamma_\gamma}{2E_i} \int_{-\infty}^{\infty} \frac{dx}{\beta x^2 + 2(\beta \mathfrak{d})^{\frac{1}{2}} x + (1 + \beta)} = \frac{g_i^\infty}{\left[1 + \frac{1}{\beta}(1 - \mathfrak{d})\right]^{\frac{1}{2}}}$$

NR for homogeneous (10.155)★

where
$$\beta \equiv \frac{\sigma_p}{\sigma_0} \tag{10.156}$$

and
$$g_i^\infty \equiv \frac{\pi \sigma_0 \Gamma_\gamma}{2E_i} \tag{10.157}$$★

Two limiting cases of this result are of special practical importance. When the mixture is very dilute, $\sigma_p \to \infty$ and

$$g_i \to g_i^\infty \tag{10.158}$$

This result is the so-called *infinite-dilution* effective resonance integral. The sum of g_i^∞ over all resonances has been measured for many materials[1] (see, for example, Table 10.1). At the other extreme, when $\beta \ll 1 - \mathfrak{d}$, as often happens in the low-energy resonances of uranium and thorium or other absorbers in fairly concentrated mixtures, it is seen that g_i is proportional to $\sigma_p^{\frac{1}{2}}$. This rule is roughly verified by experiment.[2]

We found previously that in the NR approximation the effective resonance integral for heterogeneous systems could be written in the form (10.140) which is mathematically identical to (10.125) or (10.23); thus the result (10.155) may be adapted to heterogeneous systems. It follows that in this approximation g_i is given by

$$g_i = \frac{g_i^\infty}{\left[1 + \frac{1}{\beta_*}(1 - \mathfrak{d}_*)\right]^{\frac{1}{2}}}$$

NR for heterogeneous (10.159)★

[1] Macklin and Pomerance, *op. cit.*

[2] See, for example, A. C. G. Mitchell, L. J. Brown, J. R. Pruett, and E. D. Nering, "Resonance Absorption of Uranium in Mixtures," Argonne Metallurgical Laboratory, CP-1676, Mar. 31, 1944.

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where

$$\beta_* \equiv \frac{\sigma_p^*}{\sigma_0} \qquad \text{or} \qquad \frac{\beta_*}{\beta} = 1 + \frac{\sigma_m^*}{\sigma_p^{(F)}} \tag{10.160}$$

and σ_p^* is defined by (10.139), and $\mathfrak{d}_*$ by

$$\mathfrak{d}_* \equiv \frac{\sigma_p^{(F)} g_J \Gamma_n}{\sigma_p^* \Gamma} \tag{10.161}$$

When $\beta_* \ll 1$, a condition nearly always fulfilled for the important resonances in most materials,

$$g_i \simeq g_i^\infty \left(\frac{\beta_*}{1 - \mathfrak{d}_*}\right)^{\frac{1}{2}} = g_i^\infty \beta^{\frac{1}{2}} \left[\frac{\beta_*/\beta}{1 - \mathfrak{d}\beta/\beta_*}\right]^{\frac{1}{2}} \tag{10.162}$$

The factor in the brackets is plotted in Fig. 10.14 as a function of β_*/β

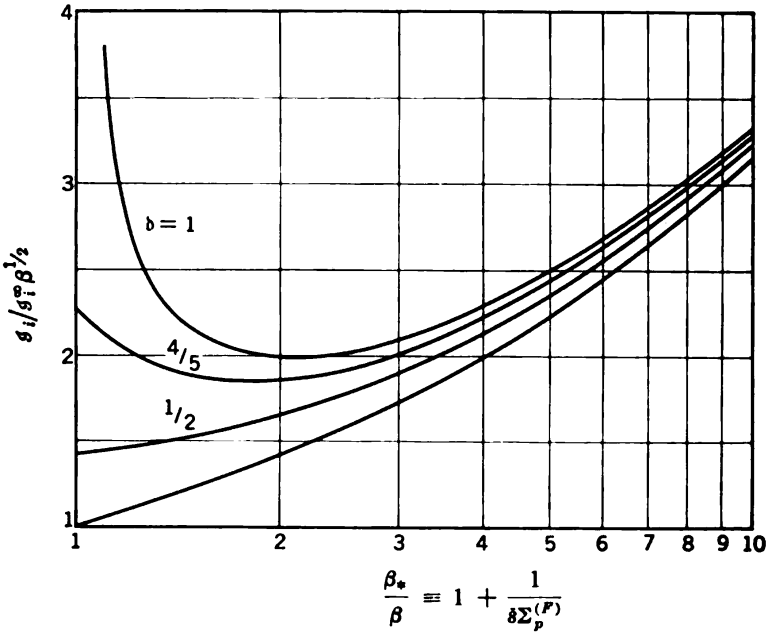

FIG. 10.14 Geometric dependence of resonance integral.

for various $\mathfrak{d}$. It is seen that all the curves are asymptotic to $(\beta_*/\beta)^{\frac{1}{2}}$. Thus with increasing β_*/β, the dependence of the effective resonance integral upon the interference between potential and resonance scattering disappears. This square-root dependence for the heterogeneous systems is analogous to the $\sigma_p^{\frac{1}{2}}$ dependence mentioned above for the homogeneous systems.

The evaluation of g_i using the NRIA approximation follows the same procedure. If we use the form (10.127) for the homogeneous system, then

$$g_i = \int_{\text{res}} \frac{\sigma_a^{(F)}\, du}{1 + \sigma_a^{(F)}/\sigma_m} = \int_{\text{res}} \frac{\sigma_a^{(F)}\, d\mathcal{E}/\mathcal{E}}{1 + \sigma_a^{(F)}/\sigma_m} \tag{10.163}$$

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The substitution of the Breit-Wigner formula (10.144) for $\sigma_a^{(F)}$ and transformation of variable from $\mathcal{E}$ to x according to (10.152) yields

$$g_i = \int_{\text{res}} \frac{\sigma_0 \Gamma_\gamma \, dx/2\mathcal{E}}{(1 + x^2)\,[1 + (\sigma_0 \Gamma_\gamma / \sigma_m \Gamma)(1 + x^2)^{-1}]} \tag{10.164}$$

We use again the approximations applied in going from Eq. (10.154) to Eq. (10.155). Then Eq. (10.164) reduces to

$$g_i = \frac{g_i^\infty}{[1 + (\sigma_0 \Gamma_\gamma / \sigma_m \Gamma)]^{\frac{1}{2}}} \qquad \text{NRIA for homogeneous} \tag{10.165}\star$$

with g_i^∞ given by (10.157). Since the form (10.127) is identical to (10.143) for the heterogeneous system, the result (10.165) may be used to obtain the corresponding expression for the heterogeneous system. Obviously,

$$g_i = \frac{g_i^\infty}{[1 + (\sigma_0 \Gamma_\gamma / \sigma_m^* \Gamma)]^{\frac{1}{2}}} \qquad \text{NRIA for heterogeneous} \tag{10.166}\star$$

with σ_m^* given by (10.141). It should be noted that both results (10.165) and (10.166) in the NRIA approximation are independent of interference effects between resonance and potential scattering, whereas the results for the NR approximation are not; specifically, the quantity in the parentheses of Eq. (10.159) is a correction factor contributed by the interference effect.

The heterogeneous system formulas (10.159) and (10.166) for the contribution to g based on the NR and NRIA approximations may be used to deduce the dependence of the effective resonance integral upon the surface-to-mass ratio (A_F/M_F) of the fuel lump. In the cases wherein the NRIA formula applies, it frequently occurs that $(\sigma_0 \Gamma_\gamma / \sigma_m^* \Gamma) \gg 1$. Then the form (10.166) is well approximated by

$$g_i(\text{NRIA}) \simeq g_i^\infty \left(\frac{\sigma_m^* \Gamma}{\sigma_0 \Gamma_\gamma}\right)^{\frac{1}{2}} \sim \left(\frac{A_F}{M_F}\right)^{\frac{1}{2}} \tag{10.167}$$

since by (10.141)

$$\sigma_m^* = \frac{1}{\bar{s} N_F} = \frac{\rho_F' A_F}{4 N_F M_F} \tag{10.168}$$

where ρ_F' is the density of the fuel-lump material. A similar dependence is found also for the NR approximation. If we write out the various terms in Eq. (10.159), then

$$g_i = g_i^\infty \left\{1 + \frac{\sigma_0/\sigma_p^{(F)}}{1 + \sigma_m^*/\sigma_p^{(F)}} - \frac{g_J \Gamma_n \sigma_0/\sigma_p^{(F)}}{\Gamma[1 + \sigma_m^*/\sigma_p^{(F)}]^2}\right\}^{-\frac{1}{2}}$$

For the two very interesting materials U^{238} and Th^{232}, $\Gamma_n/\Gamma \ll 1$ over the most important part of the resonance range;[1] if we note also that in

[1] Sampson and Chernick, *op. cit.*, Tables 1 and 2.

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general $\sigma_0/\sigma_p^{(F)} \gg 1$, then the above expression may be approximated by

$$g_i(\text{NR}) \simeq g_i^\infty \left[\frac{\sigma_p^{(F)}}{\sigma_0}\right]^{\frac{1}{2}} \left[1 + \frac{\sigma_m^*}{\sigma_p^{(F)}}\right]^{\frac{1}{2}} \tag{10.169}$$

When the fuel lump is thin,

$$\frac{\sigma_m^*}{\sigma_p^{(F)}} = \frac{\lambda_p^{(F)}}{\bar{s}} \gg 1$$

and [cf. Eq. (10.162) and Fig. 10.14]

$$g_i(\text{NR}) \sim \left(\frac{A_F}{M_F}\right)^{\frac{1}{2}} \tag{10.170}$$

Thus for many situations of practical interest the effective resonance integral varies as the square root of the surface-to-mass ratio of the fuel lump in both the NR and NRIA approximations. It is also of interest to note that in mixtures when $\sigma_m/\sigma_p^{(F)} > 10$ both approximations approach the asymptotic value g_i^∞ [see also (10.158)].

g. Effect of Nuclear Motion: The Doppler Broadened Line. The results of the preceding analysis apply strictly to the case of stationary nuclei. More properly we should take into account the relative motion between neutron and nucleus and therefore should use the generalized reaction-rate relation

$$v\sigma(v) = \int_{\mathbf{V}} \sigma^\infty(|\mathbf{v} - \mathbf{V}|)|\mathbf{v} - \mathbf{V}|\mathfrak{M}(\mathbf{V},\theta)\, d\mathbf{V} \tag{4.210}$$

for the cross sections in the resonance integrals in place of the natural line shape given by the Breit-Wigner formula. In the present treatment we base the calculation on the NR approximation and use the Breit-Wigner formulas for the σ^∞ which appears in (4.210) and the Maxwell-Boltzmann distribution (4.213) to describe the motion of the nuclei.

This general problem has been examined by a number of investigators.[1] In the treatment which follows we draw principally from the analysis of L. Dresner.[2] The calculation is developed from an approximate expression for the relative energy between the neutron and the nucleus. For this purpose we require the velocities $\mathbf{w}$ and $\mathbf{W}$ (for neutron and nucleus, respectively) in the center-of-mass system of coordinates (C). These are [cf. Eqs. (4.6)]

$$\mathbf{w} = \frac{A}{1 + A}(\mathbf{v} - \mathbf{V}) \qquad \mathbf{W} = \frac{1}{1 + A}(\mathbf{V} - \mathbf{v}) \tag{10.171}$$

[1] H. A. Bethe and G. Placzek, *Phys. Rev.*, **51**, 450 (1937); E. Creutz, H. Jupnik, T. Snyder, and E. P. Wigner, *J. Appl. Phys.*, **26**, 257–259 (1955); E. P. Wigner, E. Creutz, H. Jupnik, and T. Snyder, *J. Appl. Phys.*, **26**, 260–270 (1955); E. Creutz, H. Jupnik, T. Snyder, and E. P. Wigner, *J. Appl. Phys.*, **26**, 271–275 (1955); E. Creutz, H. Jupnik, and E. P. Wigner, *J. Appl. Phys.*, **26**, 276–279 (1955); G. M. Roe, "The Absorption of Neutrons in Doppler Broadened Resonances," KAPL-1241, 1954.

[2] L. Dresner, *Nuclear Sci. Eng.*, **1**, 68–79 (1956); also, Thesis, *op. cit.*

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Thus the *relative* kinetic energy in system (C) is given by [cf. Eqs. (4.9) and (10.147)]

$$E_{\text{rel}} \equiv \tfrac{1}{2}m\mathbf{w}^2 + \tfrac{1}{2}M\mathbf{W}^2 = \tfrac{1}{2}\mu_*(\mathbf{v} - \mathbf{V})^2 \tag{10.172}$$

where μ_* is the reduced mass defined in Eq. (10.148). Now, if μ_0 denotes the cosine of the angle between the velocity vectors $\mathbf{v}$ and $\mathbf{V}$, then the expansion of (10.172) yields

$$E_{\text{rel}} = \frac{A}{1 + A}(E + \tfrac{1}{2}mV^2 - mvV\mu_0) \tag{10.173}$$

where $E \equiv \frac{1}{2}mv^2$, the neutron energy in system (L). Further, let the relative velocity between the two particles be $\boldsymbol{\varrho} \equiv \mathbf{v} - \mathbf{V}$, where

$$\rho^2 = v^2 + V^2 - 2vV\mu_0 \tag{10.174}$$

Clearly,

$$E_{\text{rel}} = \tfrac{1}{2}\mu_*\rho^2 = \mathcal{E} + \tfrac{1}{2}\mu_*V^2 - (2\mu_*\mathcal{E})^{\frac{1}{2}}V\mu_0 \tag{10.175}$$

with

$$\mathcal{E} = \tfrac{1}{2}\mu_*v^2 = \frac{A}{1 + A}E \tag{10.176}$$

Note that, if the nuclei were stationary, $\mathcal{E}$ would be the relative energy of the neutron. This was mentioned previously in connection with Eqs. (10.144) through (10.147). Now, in most situations,

$$\left(\frac{V}{v}\right)^2 \sim \frac{kT}{A\mathcal{E}} \ll 1 \tag{10.177}$$

Thus (10.175) may be approximated by

$$E_{\text{rel}} \simeq \mathcal{E} - (2\mu_*\mathcal{E})^{\frac{1}{2}}V\mu_0 \tag{10.178}$$

We have also

$$\rho = v\left[1 + \left(\frac{V^2}{v^2} - \frac{2V\mu_0}{v}\right)\right]^{\frac{1}{2}} \simeq v - V\mu_0 \tag{10.179}$$

These results may be used to obtain an estimate of the reaction rate which appears in the integrand of (4.210). If we note that σ^∞ varies rapidly in the resonance range whereas $|\mathbf{v} - \mathbf{V}|$ varies slowly, it seems reasonable to introduce the approximation

$$|\mathbf{v} - \mathbf{V}| \simeq v \tag{10.180}$$

Then, Eq. (4.210) reduces to

$$\sigma(v) \simeq \int_{\mathbf{V}} \sigma^\infty(|\mathbf{v} - \mathbf{V}|)\, \mathfrak{M}(\mathbf{V},\theta)\, d\mathbf{V} \tag{10.181}$$

This result may also be written in terms of the variable $\mathcal{E}$, in place of v. Let us define $\mathbf{V} \equiv \mathbf{V}(V_x,V_y,V_z)$ and take the direction of the component V_z to correspond to $\mathbf{v}$; then $V\mu_0 = V_z$, and

$$E_{\text{rel}} \simeq \mathcal{E} - (2\mu_*\mathcal{E})^{\frac{1}{2}}V_z \tag{10.182}$$

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Since $\rho \equiv |\mathbf{v} - \mathbf{V}|$, and is related to E_{rel} through Eq. (10.175), and to V_x through (10.182), we can write $\sigma^\infty = \sigma^\infty(\rho) = \sigma^\infty(V_x)$. Substitution into Eq. (10.181) and integration over V_x and V_y yield

$$\sigma(v) = \int_{-\infty}^{\infty} \frac{\sigma^\infty(V_x) e^{-V_x^2/4\theta}\, dV_x}{(4\pi\theta)^{\frac{1}{2}}} \tag{10.183}$$

Using (10.182) to replace V_x by E_{rel}, we obtain

$$\sigma(v) \to \sigma(\mathcal{E}) = \left(\frac{M}{4\pi k T \mu_* \mathcal{E}}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \sigma^\infty(E_{\text{rel}}) \exp\left[-\frac{M(\mathcal{E} - E_{\text{rel}})^2}{4kT\mu_* \mathcal{E}}\right] dE_{\text{rel}} \tag{10.184}$$

This result may be used directly in the resonance integral for the various cross sections $\sigma_a^{(F)}$, $\sigma_{sr}^{(F)}$, and $\sigma_{si}^{(F)}$. For the cross sections $\sigma^\infty(E_{\text{rel}})$ which appear in (10.184), we use the single-level Breit-Wigner forms indicated in Eqs. (10.144) through (10.146). In the present application the reduced neutron wavelength is defined [cf. Eq. (10.150)]

$$\lambda\!\!\!^{-} \equiv \frac{h}{2\pi(2\mu_* E_{\text{rel}})^{\frac{1}{2}}} \tag{10.185}$$

These substitutions yield in the case of the absorption cross section the relation

$$\sigma_a(\mathcal{E}) = \left(\frac{M}{4\pi k T \mu_* \mathcal{E}}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(\frac{\sigma_0 \Gamma_\gamma}{\Gamma}\right) \frac{\exp\left[-\dfrac{M(\mathcal{E} - E_{\text{rel}})^2}{4kT\mu_* \mathcal{E}}\right]}{1 + \left(\dfrac{E_{\text{rel}} - E_i}{\Gamma/2}\right)^2} dE_{\text{rel}} \tag{10.186}$$

Generally speaking, the total width Γ of an absorption resonance is appreciably less than E_i. Only when E_{rel} is near E_i is there a significant contribution to the integrand of (10.186); therefore, as before, we assume that $\sigma_0\Gamma_\gamma/\Gamma$ is independent of energy and set its value to be that at the resonance. If we call

$$\begin{aligned} \Delta &\equiv \left(\frac{4kT\mu_* \mathcal{E}}{M}\right)^{\frac{1}{2}} & \vartheta &\equiv \frac{\Gamma}{\Delta} \\ x &\equiv \frac{2}{\Gamma}(\mathcal{E} - E_i) & y &\equiv \frac{2}{\Gamma}(E_{\text{rel}} - E_i) \end{aligned} \tag{10.187}$$

then (10.186) reduces to

$$\frac{\sigma_a(x)}{\sigma_0\Gamma_\gamma/\Gamma} = \frac{\vartheta}{2\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{1}{1 + y^2} \exp\left[-\frac{\vartheta^2}{4}(x - y)^2\right] dy \equiv \psi(\vartheta, x) \tag{10.188}\star$$

This function is the *Doppler broadened* line shape. It is the counterpart of the natural line shape

$$\frac{\sigma_a(x)}{\sigma_0\Gamma_\gamma/\Gamma} = \frac{1}{1 + x^2} \tag{10.189}$$

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given in (10.144) and obtained by taking the limit of $\psi(\vartheta,x)$ as $T \to 0$ $(\vartheta \to \infty)$; thus,

$$\lim_{\vartheta \to \infty} \psi(\vartheta,x) = \psi(\infty,x) = \frac{1}{1+x^2} \tag{10.190}$$

In applying (10.188) it is important to note that ϑ depends on $\mathcal{E}$ through Δ. In the remainder of this analysis it will be assumed that this dependence is unimportant by virtue of the narrowness of the resonance, and $\mathcal{E}$ in Δ is replaced by E_i. An important property[1,2] of $\psi(\vartheta,x)$ is that $\int_{-\infty}^{\infty} \psi(\vartheta,x)\,dx = \pi$. This result leads to the interesting conclusion that in an infinitely dilute system the area under an absorption resonance is independent of temperature, even if one takes into account Doppler broadening.

Expressions of the form (10.188) can also be obtained for the resonance and interference scattering cross sections. These are easily shown to be

$$\frac{\sigma_{sr}(x)}{\sigma_0 \Gamma_n/\Gamma} = \psi(\vartheta,x) \tag{10.191}\star$$

$$\frac{\sigma_{si}^{(F)}(x)}{(\sigma_0 \sigma_p \mathfrak{d})^{\frac{1}{2}}} = \frac{\vartheta}{2\pi^{\frac{1}{2}}} \int_{-\infty}^{\infty} \frac{2y}{1+y^2} \exp\left[-\frac{\vartheta^2}{4}(x-y)^2\right] dy \equiv \chi(\vartheta,x) \tag{10.192}\star$$

Note that

$$\chi(\vartheta,x) = 2x\psi(\vartheta,x) + \frac{4}{\vartheta^2}\frac{\partial}{\partial x}\psi(\vartheta,x) \tag{10.193}$$

and in the limit as $T \to 0$,

$$\lim_{\vartheta \to \infty} \chi(\vartheta,x) = \chi(\infty,x) = \frac{2x}{1+x^2} \tag{10.194}$$

which is the natural line shape given in (10.146).

The functions (10.188), (10.191), and (10.192) may be used to obtain the generalization of the effective resonance integral in both the NR and NRIA approximations to the case of Doppler broadening. The calculation which follows is developed from the resonance-integral formulas for the homogeneous system and does not include the interference between resonance and potential scattering; thus, we take $\mathfrak{d} = 0$. This result will be a good approximation for two situations of practical

[1] The various properties of $\psi(\vartheta,x)$ have been studied by a number of investigators; see, for example, M. Born, "Optik," J. Springer, Berlin, 1933; G. M. Roe, "The Absorption of Neutrons in Doppler Broadened Resonances," KAPL-1241, Oct. 15, 1954; M. E. Rose, W. Muanker, P. Leak, L. Rosenthal, and J. K. Hendrikson, WAPD-SR-506 (1954); and A. T. Monk and J. A. Wheeler, "Effect of Temperature on Shape of Absorption Line," Argonne Metallurgical Laboratory, C-66 (1954), declassified.

[2] See also F. T. Adler, G. W. Hinman, and L. W. Nordheim, "The Quantitative Evaluation of Resonance Integrals," General Atomics GA-350, San Diego, Calif., September, 1958.

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importance, namely, a purely absorbing resonance ($\Gamma_n \ll \Gamma$) or a very dilute mixture [$\sigma_p^{(F)} \ll \sigma_p$]. The latter case occurs most frequently in practice. It is also of interest to mention that there is a third situation wherein the interference term can be neglected, namely, in the case of strong Doppler broadening.[1]

We begin with the NR approximation. The substitution of (10.188) and (10.191) into the formula (10.23) written in terms of the variable $\mathcal{E}$ yields [upon omitting the $\sigma_{si}^{(F)}$ term]

$$g_i = \frac{\sigma_p \Gamma_\gamma}{\Gamma} \int_0^\infty \frac{\psi(\vartheta,x)\, d\mathcal{E}/\mathcal{E}}{\beta + \psi(\vartheta,x)} \tag{10.195}$$

[We have used here also (10.149) for Γ and (10.156) for β.] Now $d\mathcal{E}/\mathcal{E} = \Gamma\, dx/2\mathcal{E}$, and if as before [refer to Eq. (10.155)] we consider narrow resonances and ignore the energy dependence of Γ and $\mathcal{E}$ over the range of integration and simply evaluate them at E_i, then (10.195) may be approximated by

$$g_i \simeq \frac{\sigma_p \Gamma_\gamma}{2E_i} \int_{-2E_i/\Gamma}^\infty \frac{\psi(\vartheta,x)\, dx}{\beta + \psi(\vartheta,x)} \simeq \frac{\sigma_p \Gamma_\gamma}{E_i} \int_0^\infty \frac{\psi(\vartheta,x)\, dx}{\beta + \psi(\vartheta,x)} \tag{10.196}$$

A similar result is obtained for the NRIA approximation. In this case we use the form (10.163) and find

$$g_i = \sigma_m \int_0^\infty \frac{\psi(\vartheta,x)\, d\mathcal{E}/\mathcal{E}}{\delta + \psi(\vartheta,x)} \simeq \frac{\sigma_m \Gamma}{E_i} \int_0^\infty \frac{\psi(\vartheta,x)\, dx}{\delta + \psi(\vartheta,x)} \tag{10.197}$$

where we have applied the same approximations as in (10.196) and called

$$\delta \equiv \frac{\sigma_m \Gamma}{\sigma_0 \Gamma_\gamma} \tag{10.198}$$

The results (10.196) and (10.197) are seen to be of the same form. Therefore, if we define

$$J(\vartheta,\beta) \equiv \int_0^\infty \frac{\psi(\vartheta,s)\, ds}{\beta + \psi(\vartheta,s)} \tag{10.199}\star$$

then

$$\begin{aligned} g_i &= \frac{\sigma_p \Gamma_\gamma}{E_i} J(\vartheta,\beta) \qquad \text{in NR approximation} \\ g_i &= \frac{\sigma_m \Gamma}{E_i} J(\vartheta,\delta) \qquad \text{in NRIA approximation} \end{aligned} \tag{10.200}\star$$

which gives the contribution to the effective resonance integral by the resonance at E_i. These formulas may be used, then, to evaluate the effect of the various resolved resonances on the basis of the resonance parameters and the temperature of the system. The necessary parameters are available in the standard sources.[2]

[1] Dresner, Thesis, *op. cit.*

[2] See, for example, Hughes and Harvey in United States Atomic Energy Commission, "Neutron Cross Sections," McGraw-Hill Book Company, Inc., New York, 1955.

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The function $J(\vartheta,\beta)$ has been tabulated and graphed by Dresner,[1] and analytical expressions are given by Roe.[2,3] Dresner also lists some of the more important properties of the function and various series expansions suitable for computation. Of particular interest here is the limiting form of $J(\vartheta,\beta)$ when $\vartheta \to \infty$, which corresponds to $T \to 0$. This is the case with no Doppler broadening; clearly, ψ then reduces to the expression for the natural line shape (10.189), and J takes the limiting form $J(\vartheta,\beta) \sim \frac{1}{2}\pi[\beta(1+\beta)]^{-\frac{1}{2}}$. Also noteworthy is the fact that, for both limits $\beta \to 0$ and $\beta \to \infty$, J approaches the value for the natural line shape. Note that at either limit $J(\vartheta,\beta)$ is independent of ϑ; thus the resonance integral becomes independent of temperature. This property was previously mentioned in Sec. 6.5g in connection with the temperature coefficient of the resonance-escape probability. Figure 10.15 shows a plot of $J(\vartheta,\beta)$ as a function of β for various values of ϑ. It is of interest to mention that, when $\vartheta = 1$, $J(\vartheta,\beta)$ is again well approximated by the asymptotic form given above for $\vartheta \to \infty$.

h. Evaluation of Resonance Integrals. In the early days of reactor technology the determination of the resonance integral was based primarily upon measurement and on the use of rather approximate analytical models. This approach was favored because at that time the knowledge of neutron cross sections was limited in scope and precision. For homogeneous systems the narrow resonance approximation was considered generally applicable. For heterogeneous systems both the early theory and experiments suggested that the resonance integral could be expressed as the sum of a term proportional to the fuel-lump volume and another term proportional to the surface area. Although it was well recognized that the available theory was in a crude state indeed, the lack of good cross-section data lead soon to the adoption of the so-called two-term "standard formula" which enjoyed general popularity until about the time of the first Conference on Peaceful Uses of Atomic Energy held at Geneva in 1955. The subsequent publication of large amounts of reactor information by both the Western countries and Russia revealed a substantial difference in the analytical formulation of the resonance integral for heterogeneous systems. A careful comparison[4,5] of these formulas showed, however, that over the practical range of fuel-lump sizes these differences were minor. Since then, many new measurements have been made of the detailed resonance structure of the important reactor materials, and it is now possible to calculate the effective resonance integral for homogeneous systems to within a few per cent. Heter-

[1] L. Dresner, *Nuclear Sci. Eng.*, **1**, 68–79 (1956); also Thesis, *op. cit.*
[2] Roe, *op. cit.*
[3] See also Sampson and Chernick, *op. cit.*
[4] Wigner, *op. cit.*, BNL-433.
[5] Sampson and Chernick, *op. cit.*

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ogeneous systems, however, are still analyzed by approximate but generally more accurate models than previously used and by the Monte Carlo method.[1]

The viewpoint and various analytical models derived in the preceding portions of this section represent the essential features of some of the

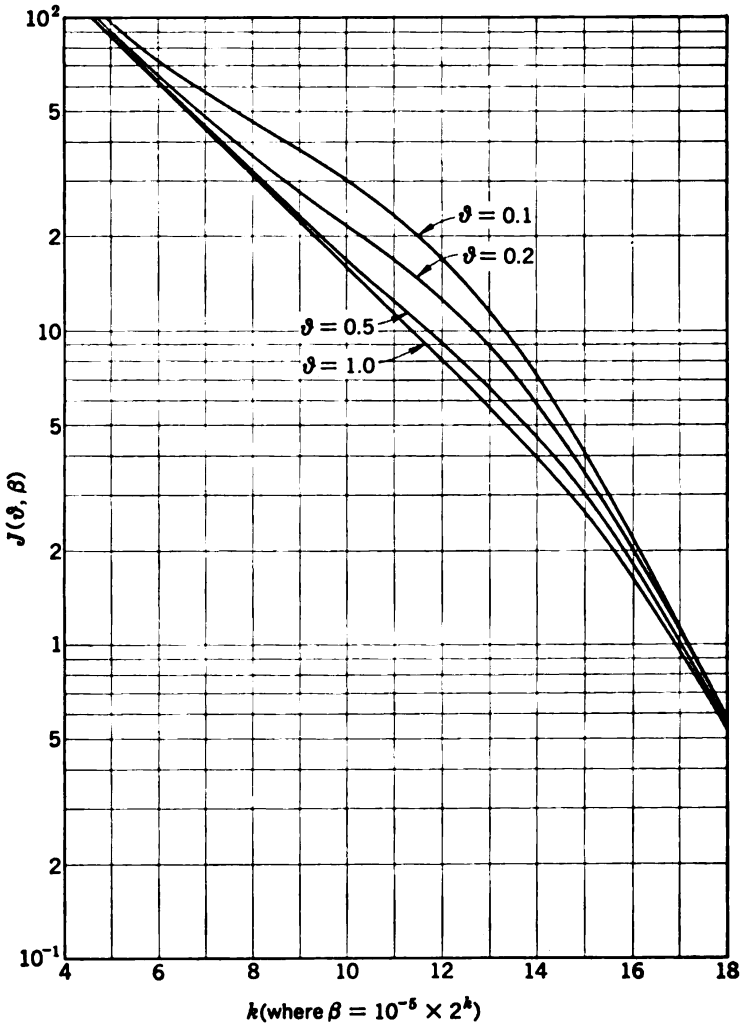

FIG. 10.15 The function $J(\vartheta,\beta)$.

more recent developments in the calculation of resonance integrals. In particular, the so-called NR and NRIA approximations may be applied to obtain first estimates of the contribution to the effective resonance integral from the resolved resonances. In the case of the important fuel materials U^{238} and Th^{232}, the resonance structure up to about 500 ev and 400 ev, respectively, has been resolved. The error due to oversimplifying

[1] R. D. Richtmyer, "Monte Carlo Study of Resonance Absorption in Hexagonal Reactor Lattices," NYO-6479, May 1, 1955.

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the slowing-down problem is minimized if a choice is made between the formulas on the basis of Wigner's *practical width* for the individual resonances. This quantity Γ_r is defined as the span on the energy scale over which the Doppler-broadened resonance cross section including both absorption and scattering is equal to or greater than the background potential-scattering cross section σ_p as defined by (6.177). Note that the practical width therefore depends in general upon the material composition of the system.

Values for the practical width for the principal resonances of U^{238} and Th^{232} are given by Sampson and Chernick.[1,2] The comparison of the maximum energy degradation $\alpha_F E_i$ in a fuel-atom collision with the practical width Γ_r gives a measure of whether a single collision can be expected to remove a neutron from the neighborhood of the resonance. If $\alpha_F E_i$ is considerably greater than Γ_r, it may be expected that the neutron will be removed by such a collision, and the NR approximation is applicable. If, on the other hand, $\alpha_F E_i$ is small compared to Γ_r, the scattered neutron will remain in the region of the resonance, and the NRIA approximation should apply. On this basis Sampson and Chernick show that the NR formula is applicable to all but the lowest several resonances in the case of the pure metals U^{238} and Th^{232}. However, these particular resonances account for a considerable fraction of the total resonance captures, and for these the NRIA formula is preferable. Specifically,[3–5] these are, for U^{238}, the resonances at 6.7, 21, 36.9, 66.3, 103.5, and 192 ev, and for Th^{232}, those at 22.0, 27.7, 70.1, and 174 ev. In the case that the fuel material appears in oxide form, it might be expected that the larger value of $\alpha_{ox} E_i$, where α_{ox} refers to the oxygen atom, would cause the NR approximation to apply for all resonances; however, this does not prove to be the case because well within the practical width the oxygen cross section cannot compete with the resonance cross section, and so the approximations appropriate to the pure metal apply to the oxides as well.[6]

In the absence of strong Doppler broadening, i.e., at relatively low temperatures, the general result (10.155) may be used to estimate the contribution of resolved narrow resonances in homogeneous systems, and (10.165) may be used for the resolved broad resonances. The corresponding formulas (10.159) and (10.166) may be used similarily for heterogeneous systems on the basis that the equivalence relationship is applicable to the case at hand. The accuracy which one can achieve in

[1] Sampson and Chernick, *op. cit.*, Tables 1 and 2.
[2] Adler, Hinman, Nordheim, *op. cit.*, Table I.
[3] *Ibid.*
[4] Chernick and Vernon, *op. cit.*
[5] Hughes and Harvey, *op. cit.*
[6] Sampson and Chernick, *op. cit.*

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using these relatively crude models depends, of course, upon a judicious choice in the application to specific resonances. By exercising some care, it is possible[1] to obtain good estimates for the contributions of the resolved resonances $\sum_{i=1}^{N} g_i$ (where N denotes the number of these for the particular material). More precise computations based on modified forms[2] of the NR and NRIA formulas and various second-order theories[3–6] for the

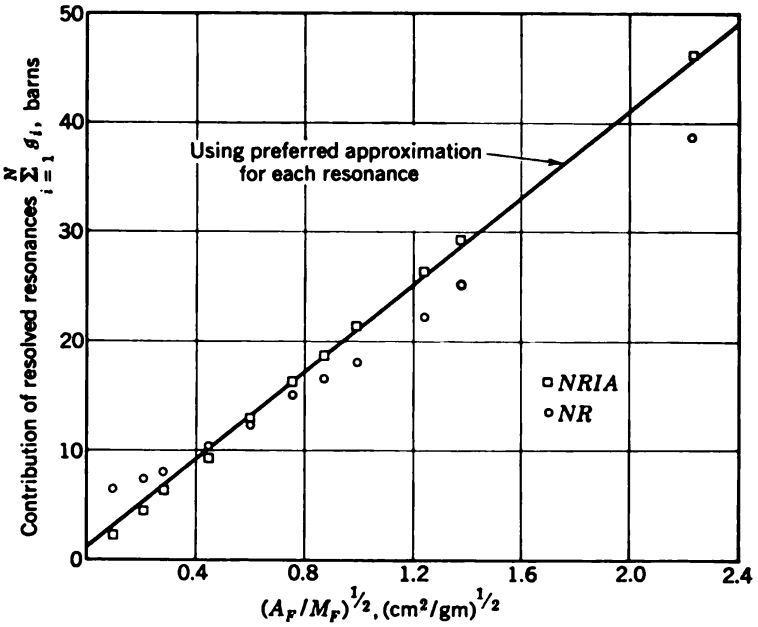

FIG. 10.16 Contribution of resolved resonances to the U^{238} effective resonance integral.

treatment of the slowing-down problem indicate that there is good agreement between the two models except in the case of the very broad, low-energy resonances where the NR formula is less accurate. For all other resonances the more precise versions of the two models are found to yield similar results.[7] Such a comparison is shown in Fig. 10.16 for U^{238}.

[1] Chernick and Vernon, *op. cit.*, Figs. 1 and 2.

[2] K. T. Spinney, "Resonance Absorption in Homogeneous Mixtures," Proceedings of the Brookhaven Conference on Resonance Absorption of Neutrons in Nuclear Reactors, BNL-433, pp. 103–109, 1956.

[3] Sampson and Chernick, *op. cit.*, pp. 236–240.

[4] K. T. Spinney, *J. Nuclear Energy*, **6**, 53 (1957).

[5] Chernick and Vernon, *op. cit.*

[6] A. M. Weinberg and E. P. Wigner, "A Second-order Correction to the Usual Resonance Escape Formula," Proceedings of the Brookhaven Conference on Resonance Absorption of Neutrons in Nuclear Reactors, BNL-433, pp. 125–126, 1956.

[7] W. Rothenstein, *Nuclear Sci. Eng.*, **7**, 162–171 (1960).

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In the event of appreciable Doppler broadening, the general results (10.200) may be used to determine the contributions of particular resonances. It should be noted, however, that the usefulness of these formulas is somewhat limited because of the omission of the scattering-interference term (which is important for thick lumps[1]); when this term is negligible, however, the extensive compilations of the function $J(\vartheta,\beta)$ are generally and directly applicable.

The unresolved resonances in the fuel materials account for roughly one third of the resonance absorption and are therefore generally important. Estimates of the contributions from these higher-lying resonances have been based on statistical methods, using average values for the widths Γ_γ and Γ_n and the resonance level spacing $\mathfrak{D}$ derived from the resonance parameters of the lower-lying resonances. A detailed discussion of the essential ideas involved in such computations is given by Sampson and Chernick,[2] by Chernick and Vernon,[3] and by Dresner.[4] For our present needs it suffices to indicate simply the general form used for this purpose. If E_N denotes the energy of the last resolved resonance, then the contribution $\mathcal{J}_u$ to the effective resonance integral of all resonances above E_N may be computed from the relation

$$\mathcal{J}_u \equiv \frac{\sigma_p\Gamma_\gamma}{\mathfrak{D}} \int_{E_N}^{\infty} J(\vartheta,\beta)\,\frac{dE}{E} \simeq \frac{\sigma_p\Gamma_\gamma}{\mathfrak{D}} \int_{\beta_N}^{\infty} J(\vartheta,\beta)\,\frac{d\beta}{\beta} \tag{10.201}$$

where β_N has the value of σ_p/σ_0 at the resonance E_N; the averaged parameters $\mathfrak{D}$ and Γ_γ are discussed by Dresner.[5] The approximation sign applies when $\Gamma_n \gg \Gamma_\gamma$; in the range of interest, $\Gamma_n \simeq 2\Gamma_\gamma$, or greater. Note also that when $\Gamma_n \gg \Gamma_\gamma$,

$$\beta_N = \frac{2\pi\sigma_p m E_N}{h^2}\left(\frac{\Gamma}{\Gamma_n}\right) \simeq \left(\frac{2\pi\sigma_p m}{h^2}\right) E_N \tag{10.202}$$

Also, it is assumed in Eq. (10.201) that Γ_γ is independent of energy and that σ_p/σ_0 varies in first approximation like E.

In order to complete the theoretical development of the effective resonance integral we require also the contribution of the $1/v$ portion of the resonance range. The appropriate result was obtained previously in Sec. 6.5g. From Eqs. (6.182) and (6.187) we have (refer also to Fig. 6.8)

$$\mathcal{J}_v = \int_{E_{th}}^{E_1} \frac{(c_0/v)\,dE/E}{1 + c_0/v\sigma_p} \simeq 2\sigma_a^{(F)}(v_{th}) \tag{10.203}\star$$

[1] Dresner, *op. cit.*, Thesis.
[2] Sampson and Chernick, *op. cit.*
[3] Chernick and Vernon, *op. cit.*
[4] L. Dresner, *Nuclear Energy*, **2**, 118–127 (1955).
[5] *Ibid.*

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With these results the effective resonance integral may be written

$$\mathcal{I} = 2\sigma_a^{(F)}(v_{th}) + \sum_{i=1}^{N} \mathcal{I}_i + \frac{\sigma_p \Gamma_\gamma}{\mathfrak{D}} \int_{\beta_N}^{\infty} J(\vartheta,\beta)\, \frac{d\beta}{\beta} \qquad (10.204)\star$$

where the function $\mathcal{I}_i$ is to be selected from the two forms given in (10.200) according to the characteristics of the particular resonance represented by the index i. This general form has been used by Dresner to compute the resonance integrals for various homogeneous mixtures of both U^{238} and Th^{232} in moderator. His results are summarized in Table 10.4 for

TABLE 10.4 CALCULATED CONTRIBUTIONS TO THE EFFECTIVE RESONANCE INTEGRALS OF U^{238} AND Th^{232}

Nucleus	T, °K	σ_p, barns	Resolved resonances	Unresolved resonances	$1/v$	Effective resonance integral, barns
U^{238}	0	20	7.9	1.7	1.4	11.0
		200	24.	4.8	1.4	30.
		2,000	74.	11.	1.4	86.
	300	20	8.1	3.0	1.4	12.5
		80	17.	6.6	1.4	25.
		200	27.	9.4	1.4	38.
		2,000	86.	12.	1.4	99.
Th^{232}	0	20	3.7	2.2	3.5	9.4
		200	12.	6.3	3.5	22.
		2,000	33.	14.	3.5	51.
	300	20	4.0	4.0	3.5	11.5
		80	8.9	8.6	3.5	21.0
		200	15.	12.	3.5	31.
		2,000	42.	20.	3.5	66.

system temperatures of $T = 0°K$ and $T = 300°K$. In this particular computation the NR form was used to describe all the resolved resonances. Figure 10.17 shows a plot of these results as functions of $(A_F/M_F)^{\frac{1}{2}}$.

The models developed in this section are especially useful for understanding the principal features of resonance absorption, and by careful choice of formulas, first-order estimates based on the equivalence relationship can be obtained for the effective resonance integral of both homogeneous and heterogeneous systems. In order to achieve greater precision, it is necessary to utilize the higher-order approximations and the more sophisticated models mentioned previously. Additional refinements of the theory may be achieved by introducing the Dancoff-Ginsburg and Pershagen-Carlvik corrections; the former[1] accounts for the mutual

[1] Dancoff and Ginsburg, *op. cit.*; J. A. Thie, *Nuclear Sci. Eng.*, **5**, 75–77 (1959).

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shielding of fuel lumps in close-packed arrays and amounts, essentially, to an effective increase in the average chord length $\bar{s}$ for the fuel lump; the latter correction[1] is used in treating complicated fuel-element geometries, including clustered configurations.[2]

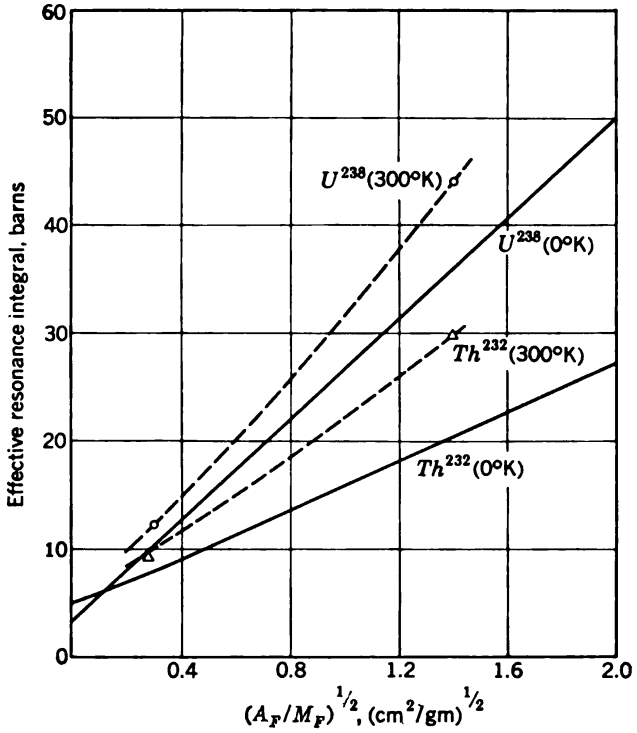

FIG. 10.17 Doppler corrected resonance integrals for U^{238} and Th^{232}.

i. The Standard Formula. The preceding analyses constitute a systematic and consistent development from a common basis of useful expressions for the resonance absorption in both homogeneous and heterogeneous systems. An essential consideration in this approach was the derivation of the equivalence relationships and their use in the analysis of heterogeneous systems. Important limitations in this treatment were the flat flux assumption and the application of the rational approximation for the fuel-lump escape probability. This general treatment represents the more recent approach to the problem of evaluating resonance integrals. It is expected that in time, as more precise measurements are obtained of the resonance parameters, there will be greater incentive for further development of these general methods and models in order to achieve wider applicability and accuracy.

Although these recent calculational methods have proved to be useful

[1] I. Carlvik and B. Pershagen, "Calculation of Effective Surfaces for Resonance Absorption in Tubes and Parallel Plates" (Aktiebolaget Atomenergi), AEF-49, 1955.
[2] Hellstrand, *op. cit.*

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in understanding the phenomenon of resonance absorption and moreover show promise of high reliability, it is still of considerable interest to compare the present models with the standard formula. The comparison is drawn with the aid of the narrow resonance approximation for the collision density. For this case the resonance integral is given by

$$\mathcal{I} = \int_0^{u_{th}} \frac{\sigma_a^{(F)}}{\sigma_t^{(F)}} \{\sigma_p^{(F)} + [\sigma_t^{(F)} - \sigma_p^{(F)}]P_{esc}\}\, du \qquad (10.135)$$

If we separate the integrand into its two terms as written above, then comparison with Eq. (10.37) reveals that we can identify the first integral as

$$\int_0^{u_{th}} a(u)\, du \equiv \int_0^{u_{th}} \frac{\sigma_a^{(F)}\sigma_p^{(F)}\, du}{\sigma_t^{(F)}} \qquad (10.205)$$

and the second as

$$\int_0^{u_{th}} b(u)\, du \equiv \frac{\rho'_F}{4N_F} \int_0^{u_{th}} \frac{\sigma_a^{(F)}[\sigma_t^{(F)} - \sigma_p^{(F)}]\mathcal{P}\, du}{[\sigma_t^{(F)}]^2} \qquad (10.206)$$

We have used here Eq. (10.131) for P_{esc}, and $M_F \equiv V_F\rho'_F$. When these expressions are compared with (10.38) and (10.39), it is seen that the NR approximation for $\mathcal{I}$ leads rather easily to a statement in the standard form. Note, however, that because of the flat flux assumption the present derivation does not contain the resonance disadvantage factor[1,2] ζ_r. This quantity is customarily computed using a one-velocity model to represent the entire fast-neutron population. It is well recognized that this point of view is crude and somewhat unclear. On the one hand, when used with the NR approximation, it may be observed that only one collision is required to remove a resonance neutron from the vicinity of a resonance; thus the spatial distribution would be very nearly uniform and isotropic. On the other hand, if the NRIA approximation is valid, then the absorptions are necessarily very strong and the use of a disadvantage factor based on diffusion theory is not well justified. For these reasons it has been omitted in this treatment as an unwarranted refinement not in keeping with the precision of the over-all model.[3]

Another point of immediate interest concerns the nature of the integral (10.206); this term in the standard formula is customarily called the surface absorption. It is seen to depend upon the geometry of the lump

[1] A discussion of the resonance disadvantage factor and its use in estimating resonance absorption is given by A. M. Weinberg and E. P. Wigner, "The Physical Theory of Neutron Chain Reactors," pp. 663–668, University of Chicago Press, Chicago, 1958.

[2] See also S. Glasstone and M. C. Edlund, "The Elements of Nuclear Reactor Theory," pp. 258–263, D. Van Nostrand Company, Inc., Princeton, N.J., 1952; and Sampson and Chernick, *op. cit.*

[3] Chernick and Vernon, *op. cit.*

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both through the factor A_F/M_F and through $\mathcal{P}$; thus the surface term is not really proportional to A_F/M_F, as implied in the construction of the standard formula. Although this dependency was known in the early days of the technology, the $\mathcal{P}$ factor fell into disuse; it was reintroduced more recently by Wigner.[1]

The physical interpretation of the standard formula is that resonance absorption consists of two parts, each arising from a different mechanism for the supply of resonance neutrons to the fuel lump. The first term in (10.205), called volume (or mass) absorption, is due to fast neutrons which diffuse into the lump at energies between resonances and then through a series of scattering collisions with fuel nuclei, are eventually slowed to an energy interval overlapping a resonance (i.e., the practical width Γ_r), and subsequently captured. The second, or *surface*, term (10.206) represents resonance captures arising from neutrons having energies that already overlap the practical width of a resonance and which have appeared in the lump from direct transport through its surface. This viewpoint of resonance capture originated with Wigner[2] and Dancoff and Ginsburg.[3]

The interpretation is given in terms of the energy spectrum of the neutron flux. Consider first the moderator region. Throughout the moderator, and at all energies, the flux is assumed to have the usual asymptotic form. In the fuel lump, between resonances, the spectrum is again $1/E$. This flux serves as an internal source within the lump of neutrons fed into the resonances and leads, therefore, to the volume term in the standard formula. When the resonances are narrow and far apart (so that there is no mutual interaction), their effect on the spectrum is highly localized (see Fig. 4.21), so that the supply of neutrons to each resonance is essentially $1/E$ and spatially uniform. This gives rise to a uniform distribution of fast-neutron sources in the lump, and the corresponding term is therefore independent of lump geometry. The surface term, on the other hand, must be dependent on the geometry since it represents absorptions in the lump arising from an inward current. In contrast to the spectrum in the lump, this current is an undepleted $1/E$ over the entire resonance range. Thus, when the lump is thin, this resonance flux from the moderator is essentially $1/E$ throughout, and when the lump is thick, it is $1/E$ at least within the outer layers of the lump.

[1] E. P. Wigner, *J. Appl. Phys.*, **26**, 255–279 (1955); see also the work of J. Chernick, "The Theory of Uranium Water Lattices," Proceedings of the International Conference on the Peaceful Uses of Atomic Energy, Physics of Reactor Design, vol. 5, P/603, pp. 215–228, 1956.

[2] E. P. Wigner, "Resonance Absorption of Neutrons by Spheres," Argonne Metallurgical Laboratory, C-4; and *J. Appl. Phys.*, **26**, 255–279 (1955).

[3] Dancoff and Ginsburg, *op. cit.*, Argonne Metallurgical Laboratory, CP-1589.

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This point of view[1,2] places emphasis on the volume term, and treats the surface contributions as a secondary effect. Although this approach is useful for handling most of the resonances in the important fuels, it is inadequate when dealing with the low-lying broad resonances wherein the volume term actually makes a relatively small contribution. This point was recognized by the early investigators, but its importance was underestimated because the rough resonance parameter measurements available at that time gave low estimates of the practical widths of these resonances. It is expected that the use of the NRIA approximation in the present analysis will yield a somewhat more realistic representation.

We conclude the section with a last remark on the approach used by the Russian investigators in their treatment of the resonance absorption problem. These analyses, in contrast to the Western approach, place emphasis on the surface absorption phenomenon and treat the volume absorption as a secondary effect. The calculations are based on a neutron-flight-path analysis, and in the case of thin lumps, scattering and slowing down within the lump are neglected.[3] Thus this viewpoint bears some resemblance to the NRIA model, and it is not surprising therefore that both yield expressions for the effective resonance integral which show a square-root dependence on A_F/M_F [see Eqs. (10.167) and (10.47) and the discussions of Sampson and Chernick and of Wigner and Weinberg, who compare the Russian and Western work].

10.4 Fast Effect

a. Significance of the Fast Effect. The role of the fast-fission effect in a neutron chain reaction was first demonstrated in connection with the design and operation of heterogeneous natural uranium reactors. Since the multiplication constants of these systems are not much different from unity, the few per cent increase in k to be realized by taking the fast effect into account was quickly recognized. This factor is important, of course, not only to the design of marginal systems, but to any system which is dependent on the fast-fission phenomenon as a principal source of neutrons. Good examples are converter and breeder reactors which maintain their chain reactions on fast fissions alone.

Nonthermal fissions are due to two different sources. There are first the resonance fissions which occur in the thermally fissionable fuel (such

[1] For further discussion of the arguments leading to the use of the standard formula, see Weinberg and Wigner, "The Physical Theory of Neutron Chain Reactors," pp. 681–689.

[2] Also, Sampson and Chernick, *op. cit.*

[3] Gurevich and Pomeranchouk, *op. cit.*; see also Sampson and Chernick, *op. cit.*, p. 247.

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as U^{233}, U^{235}, and Pu^{239}), and second there are the fast fissions which occur in materials such as U^{238} and Th^{232}, which have fission thresholds in the Mev range. Once the fast-fission threshold[1] has been exceeded, these sources can become an important factor in the over-all fission rate. In the case of natural uranium the relative abundance of U^{238} is some two orders of magnitude greater than that of U^{235}; thus, the fissions in U^{238} represent the bulk of the fast-fission reactions in these systems.

The calculation of the fast effect in homogeneous systems was discussed in Sec. 6.3a. The method outlined is generally applicable to any system which satisfies the requirements of the Fermi age model. When the system is multiregioned, the methods of multigroup analysis may be employed, as discussed in Sec. 8.8. Finally, one can also modify the standard two-group model to take fast multiplication into account. In any case, the approach is straightforward, and the analytical machinery previously developed is adequate for the task. The calculation of the fast effect for heterogeneous systems, however, requires some additional effort. The method generally used consists of two parts: (1) There is the calculation of the probability that a fast neutron born in the fuel lump makes a collision before escaping. (2) There is the calculation of the contribution this neutron makes to the fast effect, as well as the contributions of subsequent generations of neutrons produced by cascade effects.

b. Cascade Effects on the Fast-fission Factor. Fast neutrons appear in the fuel lump as a result of both fast and thermal fissions. This supply is continually depleted, however, by two processes which reduce the energy of the fast neutrons to values below the fast-fission threshold. Once so degraded, a neutron can no longer contribute to the fast effect. The first removal process is the inelastic scattering in the fuel lump. Since the inelastic-scattering cross section is a sizable fraction of the total cross section in the Mev range, this process is a powerful loss mechanism. The second process is the leakage into the moderator. This is a removal process since a fast neutron which reaches the moderator has small likelihood of returning to the lump with sufficient kinetic energy to excite a fast fission.

The supply of fast neutrons from thermal fissions occurs at a steady rate, and the source is distributed within the fuel lump in the shape of the thermal flux. The supply due to fast fissions, however, varies from one generation to the next because of cascade effects. This phenomenon can give rise to a slight augmentation of the fast-neutron population, depending on the properties of the fuel lump. The cascade effect may be described in terms of the collision probabilities and the fast-

[1] Approximately 0.9 Mev for U^{238}.

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neutron cross sections of the fuel-lump material. For this purpose we introduce a quantity $\mathcal{P}_n$, where

$$\mathcal{P}_n \equiv \begin{array}{l}\text{probability that a fast neutron produced in}\\ \text{the } n\text{th generation fast fission makes a col-}\\ \text{lision before escaping from the fuel lump}\end{array} \tag{10.207}$$

It is understood that the sequence of n fast-fission generations was initiated by a single fast neutron produced by a thermal fission. With the function $\mathcal{P}_n$, and a suitable set of *fast-neutron cross sections*, we can define several pertinent processes which describe alternate histories for each neutron appearing in the nth generation. These are:

(1) Probability of escaping from the fuel lump: $1 - \mathcal{P}_n$

(2) Probability of making a collision of type i: $\dfrac{\Sigma_i}{\Sigma}\mathcal{P}_n$

Here, the subscript i denotes any one of the following nuclear reactions: inelastic scattering Σ_{in}, radiative capture Σ_γ, fission Σ_f, and elastic scattering Σ_s. The symbol Σ denotes an appropriate "total" cross section for the fast neutrons; it is defined by [cf. Eq. (2.50)]

$$\Sigma \equiv \Sigma_\gamma + \Sigma_f + \Sigma_{in} + \Sigma_s \tag{10.208}$$

A suitable approximation for Σ is given in Sec. 10.4c.

Consider now the history of neutrons produced in the first generation of fast fission. The number $\mathcal{P}_1$ gives the fraction of these which make some kind of collision in the lump before escaping. For each such collision a certain fraction yields additional fast neutrons; thus, for each first-generation neutron

$$\left(\frac{\nu\Sigma_f + \Sigma_s}{\Sigma}\right)\mathcal{P}_1 \equiv c\mathcal{P}_1 \tag{10.209}$$

fast neutrons are produced; note that this expression implies that an elastic-scattering reaction in the lump has a negligible effect in reducing the energy of the fast neutron. The quantity c is defined [cf. Eq. (7.236)]

$$c \equiv \frac{\nu\Sigma_f + \Sigma_s}{\Sigma} \tag{10.210}$$

The first generation also suffers some losses by escapes, inelastic collisions, and radiative captures, as noted above. Of these, a certain fraction enter the slowing-down process:

$$1 - \mathcal{P}_1 + \frac{\Sigma_{in}}{\Sigma}\mathcal{P}_1 \equiv \begin{array}{l}\text{fraction of first generation neutrons}\\ \text{entering slowing-down process}\end{array} \tag{10.211}$$

The histories of subsequent generations may be traced in the same way. Some of these are indicated in Table 10.5. The neutrons entering the

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slowing-down process may still produce fast fissions in the thermally fissionable materials; however, they cannot, once slowed below the fission threshold, produce further neutrons from such materials as U^{238} and Th^{232}. The contribution of the thermally fissionable materials to the fast-fission process is neglected in the present analysis, for the reason previously cited.

TABLE 10.5 FAST-EFFECT CASCADE HISTORIES

Generation	Fast neutrons available	Neutrons participating in cascade	Neutrons entering slowing-down process
1	1	$c\mathcal{P}_1$	$1 - \mathcal{P}_1 + \frac{\Sigma_{in}}{\Sigma}\mathcal{P}_1$
2	$c\mathcal{P}_1$	$c^2\mathcal{P}_1\mathcal{P}_2$	$c\mathcal{P}_1\left[1 - \mathcal{P}_2 + \frac{\Sigma_{in}}{\Sigma}\mathcal{P}_2\right]$
3	$c^2\mathcal{P}_1\mathcal{P}_2$	$c^3\mathcal{P}_1\mathcal{P}_2\mathcal{P}_3$	$c^2\mathcal{P}_1\mathcal{P}_2\left[1 - \mathcal{P}_3 + \frac{\Sigma_{in}}{\Sigma}\mathcal{P}_3\right]$
.	.	.	.
.	.	.	.
.	.	.	.
n	$c^{n-1}\mathcal{P}_1 \cdots \mathcal{P}_{n-1}$	$c^n\mathcal{P}_1 \cdots \mathcal{P}_n$	$c^{n-1}\mathcal{P}_1 \cdots \mathcal{P}_{n-1}\left[1 - \mathcal{P}_n + \frac{\Sigma_{in}}{\Sigma}\mathcal{P}_n\right]$

The calculation of the fast-fission factor ϵ proceeds directly from results given in Table 10.5. In order to obtain ϵ we need simply sum the number of neutrons supplied to the slowing-down process from each fast-fission generation. These are the terms indicated in the last column. Thus we obtain

$$\epsilon \equiv \left[1 - \mathcal{P}_1 + \frac{\Sigma_{in}}{\Sigma}\mathcal{P}_1\right] + c\mathcal{P}_1\left[1 - \mathcal{P}_2 + \frac{\Sigma_{in}}{\Sigma}\mathcal{P}_2\right] + \cdots + c^{n-1}\mathcal{P}_1 \cdots \mathcal{P}_{n-1}\left[1 - \mathcal{P}_n + \frac{\Sigma_{in}}{\Sigma}\mathcal{P}_n\right] + \cdots \quad (10.212)$$

The collision probabilities $\mathcal{P}_n$ given here vary from one generation to the next since the spatial distribution of the fast-fission neutrons changes with successive cascades. The original, or source, distribution $F_0(\mathbf{r})$ of the first generation varies as the thermal flux, which is depressed toward the center of the fuel lump. The spatial form of the next generation $F_1(\mathbf{r})$ is given by the distribution of first collisions experienced by the neutrons originating from the source $F_0(\mathbf{r})$. The next generation in turn is obtained from $F_1(\mathbf{r})$, and so on. With each succeeding generation, the distribution becomes more peaked toward the center of the lump. The asymptotic form corresponding to $n \to \infty$ does not differ much, however,

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from the forms for $n = 1$ or 2. Figure 10.18 shows a typical sketch of the spatial forms F_n of the first few fast-fission generations. It may be concluded from these remarks that $\mathcal{P}_1 < \mathcal{P}_n < 1$, but because of the small variation in the F_n for $n > 0$, a reasonable approximation is to take $\mathcal{P}_n \equiv \mathcal{P}_0$, some fixed number independent of n. Conceivably it might be better to take the actual $\mathcal{P}_1$ for the first generation and then some other number for all $n > 1$; but this much care may not be entirely justified.

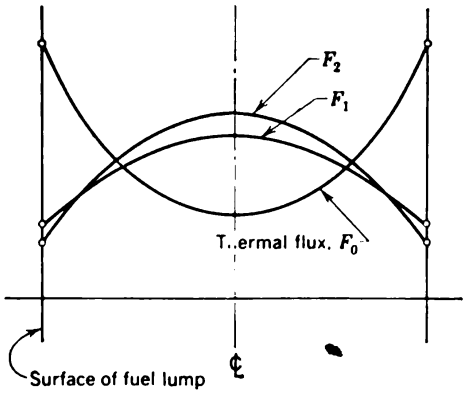

FIG. 10.18 Spatial distribution of successive generations of fast-fission neutrons.

If we assume the same number $\mathcal{P}_0$ for all n, then Eq. (10.212) is readily summed, giving

$$\epsilon = 1 + \left(\nu - 1 - \frac{\Sigma_\gamma}{\Sigma_f}\right) \frac{\Sigma_f \mathcal{P}_0}{\Sigma(1 - c\mathcal{P}_0)} \tag{10.213}\star$$

It is clear that this sum exists only if $c\mathcal{P}_0 < 1$.

c. Computation of Fast-neutron Cross Sections and Collision Probabilities. The cross sections to be used in the relations derived above should be averaged over the normalized fission spectrum $\chi(u)$. This is required also for the collision probability $\mathcal{P}_0$ since this function too is energy dependent. Thus we take

$$\mathcal{P}_0 \equiv \int \mathcal{P}_0(u)\, \chi(u)\, du \tag{10.214}$$

where it is understood that the integration extends down to the fast-fission threshold. A consistent definition for the various cross sections Σ_i is then

$$\frac{\Sigma_i}{\Sigma} \equiv \frac{1}{\mathcal{P}_0} \int \frac{\mathcal{P}_0(u)\, \Sigma_i(u)}{\Sigma(u)}\, \chi(u)\, du \tag{10.215}$$

If it is assumed that $\mathcal{P}_0$ is energy independent, then

$$\frac{\Sigma_i}{\Sigma} = \int \frac{\Sigma_i(u)}{\Sigma(u)}\, \chi(u)\, du \tag{10.216}$$

This definition of the Σ_t is useful for estimating the fast-neutron radiative capture and fission cross sections. It is not so useful for determining the corresponding inelastic cross section because of the dearth of information on the details of the actual inelastic cross sections and of the spectrum of the neutrons coming out of these reactions. This quantity is best determined from the measured value of the *reaction cross section*,[1] Σ_{react}; thus,

$$\Sigma_{in} = \Sigma_{react} - (\Sigma_f + \Sigma_\gamma) \tag{10.217}$$

[1] Values for σ_{react} are given by Hughes and Harvey, *op. cit.*

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For the total cross section Σ of these fast neutrons we take the transport cross section Σ_{tr}. This choice is suggested by certain physical features of the problem, the most important being the strong forward bias of scattered fast neutrons. Because of this effect, the fast neutrons have a greater tendency to move outward from their point of origin. The net result is as though the supporting medium had a mean free path somewhat larger than its actual value. Moreover, in calculating the functions F_n and $\mathcal{P}_n$ we will assume that the scattering process, as well as the others, is isotropic. By combining this assumption with the use of the transport cross section based on a value of $\overline{\mu_0}$ appropriate to the forward-scattering tendency of the nucleus in question,[1] it is expected that a fairly good representation of the fast-neutron "diffusion" in the fuel lump can be achieved. Thus we take

$$\Sigma \equiv \Sigma_{tr} = \Sigma_\gamma + \Sigma_f + \Sigma_{in} + \Sigma_s \tag{10.218}$$

Note that if the relation (10.217) is used to obtain Σ_{in} then Eq. (10.218), along with a computed Σ_{tr}, can be used to determine Σ_s. The approach outlined here for estimating the fast-neutron cross sections is found to give reasonably good agreement with experiment in the case of U^{238}. Table 10.6 lists the various cross sections which have been used in the past to relate the theory to actual measurements on the fast effect.

TABLE 10.6 U^{238} CROSS SECTIONS FOR FAST-EFFECT COMPUTATION

Reaction	σ barns
Radiative capture σ_γ	0.04
Fission σ_f	0.29
Inelastic scattering σ_{in}	2.47
Elastic scattering σ_s	1.5
"Total" σ_{tr}	4.3

The corresponding value for ν to be used with these data is 2.5.

A formal definition of the collision probabilities $\mathcal{P}_n$ may be given in terms of the transport kernel and the normalized spatial distributions $F_n(\mathbf{r})$. Thus to compute the number of first collisions which occur in $d\mathbf{r}$ about $\mathbf{r}$ due to the $F_n(\mathbf{r}')\,d\mathbf{r}'$ nth generation neutrons produced in $d\mathbf{r}'$ about $\mathbf{r}'$, we take

$$\Sigma_{tr}\,d\mathbf{r}\,\frac{e^{-\Sigma_{tr}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|^2}\,F_n(\mathbf{r}')\,d\mathbf{r}'$$

Since F_n is normalized, to compute $\mathcal{P}_{n+1}$ we simply integrate the above expression over all $\mathbf{r}$ and $\mathbf{r}'$; thus

$$\mathcal{P}_{n+1} = \frac{\Sigma_{tr}}{4\pi}\int_{V_P} d\mathbf{r}\int_{V_P}\frac{e^{-\Sigma_{tr}|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|^2}\,F_n(\mathbf{r}')\,d\mathbf{r}' \tag{10.219}$$

[1] D. J. Hughes and R. S. Carter, "Neutron Cross Sections; Angular Distributions," Brookhaven National Laboratory, BNL-400, June, 1956.

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It follows that

$$F_{n+1}(\mathbf{r}) = \frac{\Sigma_{tr}}{4\pi\mathcal{P}_{n+1}} \int_{V_P} \frac{F_n(\mathbf{r}')e^{-\Sigma_{tr}|\mathbf{r}-\mathbf{r}'|}\,d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^2} \tag{10.220}$$

The functions F_n may be developed successively from a knowledge of F_0, which is obtained from the shape of the thermal flux. Some of these functions have been computed for the standard fuel-lump shapes.[1] It is of interest to mention in this connection that, when the F_n are uniform, the collision probabilities $\mathcal{P}_n$ are simply the quantities P_c defined previously in Eq. (10.130).

10.5 Effects of Cavities and Fuel Lumps on Migration Area

The first-order correction to the migration area to account for the effects of heterogeneity appears through the diffusion length. A simple relationship was given previously in Eq. (10.14) for computing L^2 in a system consisting only of fuel lumps and moderator. When the nature of the heterogeneity becomes more complex, additional corrections are required. Some of these were considered in Sec. 10.2e in connection with the calculation of the disadvantage factor and the thermal utilization. From the standpoint of its effect on the migration area, the most important correction is due to the presence of gas passages and cavities. Of particular interest, from an engineering standpoint, are power-reactor designs utilizing parallel arrays of cooling holes or pebble beds and research reactors with irradiation holes and tubes. In each case, the presence of the cavity, whether regularly or randomly distributed, serves, effectively, to lengthen the total path traversed by a neutron from point of birth to point of capture (or escape from the reactor). The problem of computing this increase in path length is further complicated by considerations of anisotropy, as occur in designs involving parallel rows of holes. In these situations the average path length transverse to the holes will be noticeably shorter than that in the direction parallel to the holes, since a neutron entering a hole at a shallow angle will enjoy a long free ride before encountering any solid materials.

An elementary approach frequently used for estimating the increase in the average path length consists of applying a uniform correction factor to all macroscopic cross sections to account for the effective reduction in density due to the presence of the holes. On this basis, it is easy to show that the corrected diffusion length L_H (with holes) is related to the diffu-

[1] See, for example, G. W. Anthony, "The Average Escape Probabilities for Once- and Many-times-scattered Neutrons for a Slab," Hanford Laboratories Operation, HW-49188, Mar. 22, 1957.

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sion length L of the reactor materials [computed from Eq. (10.14)] without the holes through the equation

$$L_H^2 = L^2(1 + v_H)^2 \tag{10.221}$$

where

$$v_H \equiv \frac{V_H}{V_S} \tag{10.222}$$

and V_H denotes the total volume of holes in the reactor and V_S the volume occupied by the solid materials.

It has been demonstrated by D. J. Behrens[1] that the estimate given by (10.221) is the first-order isotropic correction term for small holes in a more general result which takes into account the distribution and shape of the holes. The results obtained by Behrens for the diffusion length are summarized in the work which follows, and extension is made to the calculation of the age, using a procedure suggested by the work of Kaplan and Chernick.[2] Finally, the estimates of the age and diffusion length obtained in this way are used to compute the migration area and thence the multiplication constant of the reactor.

a. Corrections to the Diffusion Length. The general relations obtained by Behrens for computing the diffusion length in a system containing a uniform distribution of holes are summarized by the expression

$$\frac{L_H^2}{L^2} = 1 + 2v_H + v_H^2\left[\frac{2r_H/v_H\lambda_M}{\exp(2r_H/v_H\lambda_M) - 1}\right] + \frac{Gr_Hv_H}{\lambda_M} \tag{10.223}\star$$

where v_H, L_H, and L have the definitions given above, and r_H is the hydraulic radius of the holes (twice the volume of the hole divided by the surface area), λ_M is the mean free path in the moderator,[3] and $G \equiv \overline{s^2}/(\bar{s})^2$ for the holes; here $\bar{s}$ denotes the average passage (chord) length through a hole and $\overline{s^2}$ the average square passage length. The quantity G has been computed by Behrens for various shapes of holes, and the results are given in Table 10.7. It is of interest to note that the limiting case $x = 1$ for the annular geometry corresponds to a two-dimensional gap in the reactor. This is an unfavorable arrangement from the standpoint of neutron leakage and should in general be avoided. Such a gap, however, has been incorporated in the design of the Brookhaven graphite reactor as a part of the gas cooling circuit. The effect of such gaps on criticality is discussed by Kaplan and Chernick in the reference paper.

[1] D. J. Behrens, *Proc. Phys. Soc.*, **62**, A, 607 (1949).

[2] I. Kaplan and J. Chernick, "Uranium Graphite Lattices—The Brookhaven Reactor," *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, 1955, P/606, pp. 295–308.

[3] May be estimated from $\lambda_M \equiv \frac{1}{\Sigma_t^{(M)}}$.

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When both the size and spacing of the holes are small compared to the mean free path in the moderator, the expression (10.223) is well approximated by (10.221); moreover, it may be shown that in any event

TABLE 10.7 THE QUANTITY G FOR VARIOUS CAVITY SHAPES

Geometry	G
Sphere	1.125
Infinite cylindrical hole with:	
Circular cross section	1.333
Hexagonal cross section	1.397
Square cross section	1.487
Equilateral-triangle cross section	1.648
Cylindrical annulus	$1.333F_c(x)$‡
Spherical annulus	$1.125F_s(x)$‡

‡ $x \equiv$ ratio of inner radius to outer radius.

x	$F_s(x)$	$F_c(x)$	x	$F_s(x)$	$F_c(x)$
0	1	1	0.6	1.11	1.2023
0.1	1.001	1.0258	0.7	1.18	1.2604
0.2	1.005	1.0536	0.8	1.27	1.3422
0.3	1.016	1.0839	0.9	1.43	1.53
0.4	1.04	1.1178	1.0	∞	∞
0.5	1.07	1.1567			

the value of L_H^2/L^2 obtained from Eq. (10.223) is at least as large as that from Eq. (10.221). Thus, properly

$$\frac{L_H^2}{L^2} \geq (1 + v_H)^2 \tag{10.224}$$

In the event that the holes are far apart, Eq. (10.223) may be approximated by

$$\frac{L_H^2}{L^2} \simeq (1 + v_H)^2 + \frac{G r_H v_H}{\lambda_M} \tag{10.225}$$

Some representative values of L_H^2/L^2 are given in Table 10.8 for various hole sizes and spacings so as to display the relative accuracy of Eq. (10.221).

When the distribution of cavities in the reactor is directionally nonuniform, some modification must be made in the general result (10.223). The only term in this expression which is affected by the directional properties of the cavities is the last, and this dependence appears through the shape factor G. A particular type of anisotropy which is frequently encountered in reactor design is in the form of parallel passages of cooling holes. In this situation the corrected diffusion length in the direction

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of the passage axes will be different from that in the direction perpendicular to the passages. Behrens has shown that in this case the appropriate expressions for the diffusion lengths in the two directions are given by

$$\left(\frac{1}{3}\right)\frac{L^2_{\text{axial}}}{L^2} = 1 + 2v_H + v_H^2\left[\frac{2r_H/v_H\lambda_M}{\exp\,(2r_H/v_H\lambda_M) - 1}\right] + \frac{3Gr_Hv_H}{2\lambda_M} \tag{10.226}$$

$$\left(\frac{1}{3}\right)\frac{L^2_{\text{trans}}}{L^2} = 1 + 2v_H + v_H^2\left[\frac{2r_H/v_H\lambda_M}{\exp\,(2r_H/v_H\lambda_M) - 1}\right] + \frac{3Gr_Hv_H}{4\lambda_M} \tag{10.227}$$

These relations apply for any cross-sectional shape. The quantity G is obtained from Table 10.7.

TABLE 10.8 TYPICAL VALUES OF L_H^2/L^2 FOR SQUARE ARRAY OF CYLINDRICAL HOLES

Hole diameter \ Lattice spacing	10	5	3	2
$r_H/\lambda_M \ll 1$	1.016 1.000[a]	1.066 1.000	1.200 1.000	1.548 1.000
$r_H/\lambda_M = \frac{1}{3}$	1.019 1.003	1.079 1.012	1.234 1.028	1.609 1.039
$r_H/\lambda_M = 1$	1.026 1.010	1.108 1.039	1.319 1.099	1.815 1.172
$r_H/\lambda_M = 3$	1.047 1.031	1.194 1.066	1.574 1.312	2.466 1.593

[a] Lower figures computed from $(1 + v_H)^2$.

Finally, it should be mentioned that calculations of the diffusion length for systems with holes formed as interstices between random arrangements of solid bodies in point or line contact have also been carried out by Behrens and are reported in the reference paper. These results may be applied in the study of reactor core designs utilizing pebble beds and other similar configurations.

b. Corrections to the Age. The corrections to the diffusion length outlined above may be extended to account for the streaming effect on the neutron age. The correction consists of two parts, one to account for the effect of cavities (as in the preceding analyses) and another to account for the effect of the fuel lumps. The latter correction is necessary because of the difference in the slowing-down properties of the fuel and moderator materials. In first approximation the effect of the fuel lumps is to give a neutron a "free ride" in the slowing-down process, since a collision with a fuel nucleus produces no significant change in the

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neutron energy. A second effect of the fuel lump on the age is to lower the initial energy of a fast neutron entering the slowing-down process; this reduction results from inelastic-scattering collisions in the lump.

An estimate of the combined effect on the age of fuel lumps and cavities in the form of parallel passages may be obtained from the following relations:

$$\tau_{\text{axial}} = [(1 - \mathcal{P}_{\text{in}})\tau_M(u_{\text{th}};0) + \mathcal{P}_{\text{in}}\tau_M(u_{\text{th}};u_{\text{in}})]\left(1 + \frac{2V_F}{V_M}\right)\left[1 + \left(2 + \frac{3Gr_H}{2\lambda_M}\right)v_H\right] \quad (10.228)$$

$$\tau_{\text{trans}} = [(1 - \mathcal{P}_{\text{in}})\tau_M(u_{\text{th}};0) + \mathcal{P}_{\text{in}}\tau_M(u_{\text{th}};u_{\text{in}})]\left(1 + \frac{2V_F}{V_M}\right)\left[1 + \left(2 + \frac{3Gr_H}{4\lambda_M}\right)v_H\right] \quad (10.229)$$

where $\tau_M(u_{\text{th}};0)$ denotes the age to thermal of fission neutrons in pure moderator and $\tau_M(u_{\text{th}};u_{\text{in}})$ the age from lethargy u_{in} to thermal in pure moderator. The symbol u_{in} denotes the average lethargy of a fission neutron after an inelastic-scattering collision in fuel material. The quantity $\mathcal{P}_{\text{in}}$ is the probability that a fission neutron makes an inelastic collision before escaping from the fuel lump; it may be estimated by

$$\mathcal{P}_{\text{in}} \equiv (1 - P_{\text{esc}})\frac{\Sigma_{\text{in}}^{(F)}}{\Sigma_t^{(F)}} \quad (10.230)$$

with P_{esc} given by the usual relation (7.234). On the basis of these definitions it is seen that the first bracket in Eqs. (10.228) and (10.229) gives a weighted age for fission neutrons in the moderator which accounts for the fact that not all neutrons enter the slowing-down process at $u = 0$. The first term in the bracket is a population-weighted age for those neutrons which do start at $u = 0$. The second term, on the other hand, accounts for the age of fission neutrons which begin the slowing-down process in the moderator after they have first suffered an inelastic-scattering collision in the fuel lumps. These substantially lower-energy neutrons enter the slowing-down process at $u = u_{\text{in}}$ and thereby acquire ages $\tau_M(u_{\text{th}};u_{\text{in}})$ different from $\tau_M(u_{\text{th}};0)$. Note that the age $\tau_M(u_{\text{th}};u_{\text{in}})$ is weighted by the fraction of neutrons $\mathcal{P}_{\text{in}}$ which have experienced inelastic-scattering collisions.

The second bracket in Eqs. (10.228) and (10.229) gives a first-order correction to the moderator density to account for the free ride acquired by the neutron in traversing a fuel lump. This factor corresponds to the isotropic term (10.221) derived for cavities. The last bracket in the two expressions for τ takes into account the presence of holes in the form of parallel passages and includes the directional effects discussed

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previously. These factors are approximations to expressions (10.226) and (10.227) given in the form of (10.225).

c. Migration Area and Multiplication Constant. Estimates for the migration area which include streaming-effect corrections for cavities and fuel lumps on the thermal-diffusion length and the age may be obtained from the general relations (10.226) through (10.229). The appropriate expressions for M^2 are seen to be

$$\begin{aligned} M^2_{\text{axial}} &\equiv \tau_{\text{axial}} + L^2_{\text{axial}} \\ M^2_{\text{trans}} &\equiv \tau_{\text{trans}} + L^2_{\text{trans}} \end{aligned} \tag{10.231}$$

It is evident from these relations that the buckling parameters required in the criticality calculation will differ for the various axes of symmetry of the reactor. An idea as to how this anisotropy might be included in the calculation of the multiplication constant is suggested by the migration-area model of the homogeneous reactor. The neutron-balance relation for this system is

$$-D_{\text{th}}\nabla^2\phi_{\text{th}} + \Sigma_a^{\text{th}}\phi_{\text{th}} = k_\infty \Sigma_a^{\text{th}} g_{\text{th}}\phi_{\text{th}} \tag{6.87}$$

or

$$\nabla^2\phi_{\text{th}} + B^2\phi_{\text{th}} = 0$$

where [using Eq. (8.84)]

$$B^2 \equiv \frac{1}{L^2}(k_\infty g_{\text{th}} - 1) \simeq \frac{1}{L^2}\left(\frac{k_\infty}{1 + B^2\tau_{\text{th}}} - 1\right)$$

which yields for B^2

$$B^2 \simeq \frac{k_\infty - 1}{M^2} \qquad \text{with } M^2 \equiv \tau_{\text{th}} + L^2$$

Substitution into the diffusion equation yields

$$M^2\nabla^2\phi_{\text{th}} + (k_\infty - 1)\phi_{\text{th}} = 0 \tag{10.232}$$

which applies for the isotropic system.

On this basis we take, for the anisotropic system,

$$M^2_{\text{axial}}\frac{\partial^2\phi_{\text{th}}}{\partial x^2} + M^2_{\text{trans}}\left(\frac{\partial^2\phi_{\text{th}}}{\partial y^2} + \frac{\partial^2\phi_{\text{th}}}{\partial z^2}\right) + (k_\infty - 1)\phi_{\text{th}} = 0 \tag{10.233}\star$$

The multiplication constant for this system is seen to be

$$k = \frac{k_\infty}{1 + M^2_{\text{axial}}B_x^2 + M^2_{\text{trans}}(B_y^2 + B_z^2)} \tag{10.234}$$

where B_x, B_y, and B_z are the bucklings in the x, y, and z directions, with x denoting the axis parallel to the gas passages and y and z the axes perpendicular to the passages.

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10.6 Feinberg-Galanin Method of Heterogeneous-reactor Calculations

a. Principles of the Heterogeneous Method. If a heterogeneous reactor contains a small number of fuel lumps, its critical size cannot be well computed, in general, by means of the unit-cell model of Wigner and Seitz. In these cases the detailed arrangement of the fuel lumps is an essential feature of the nuclear configuration which must be included in the criticality considerations. Thus, it becomes necessary to discard the "homogenized" model based on the gross properties of the unit cell in favor of a more detailed picture which accounts for the discrete form and distribution of the neutron sources.

Such a model[1] has been developed by Feinberg[2] and Galanin, and in this approach the general heterogeneous reactor is treated as a collection of line or point sources in a matrix of moderator material. The fast (fission) neutrons produced by these sources slow down in the moderator, and it is assumed that the resulting spatial distribution of thermal neutrons may be represented by the Fermi age solution appropriate to the source geometry. The usual one-velocity diffusion equation is then used to describe the distribution of the thermal group in the moderator. The source term in this relation is obtained by superimposing the contributions of all the fuel lumps in the reactor. It is necessary, however, to provide an additional absorption term which accounts for the fact that each fuel lump acts also as a sink for thermal (as well as fast) neutrons. The results obtained from diffusion theory for line and point sources are used for this purpose. Throughout these calculations the properties of the fuel lumps as neutron sources and sinks are described in terms of the so-called *thermal constant* (effectively, the logarithmic derivative of the flux at the surface of the lump).

The heterogeneous method is especially suited for analyzing reactors with small numbers of fuel lumps.[3] Because of the general formulation of this model it is also useful for determining the effects on criticality of various lattice geometries. Feinberg has treated specifically the cases of the square, rectangular, rhombic, and hexagonal arrays. He has further used the method to study the effect of superimposing several lattices of fuel lumps, each lattice having a different module and size and nuclear property of the fuel lumps. The practical interest in this application is, of course, the use of "seeded cores" in heterogeneous

[1] First described by E. Teller, "Influence of the Lattice Structure on the Exponential Pile Experiment," Argonne Metallurgical Laboratory, CP-165, 1942.

[2] S. M. Feinberg, "Heterogeneous Methods for Calculating Reactors: Survey of Results and Comparison with Experiment," *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, vol. 5, 1955, P/669, pp. 484–500.

[3] A. D. Galanin, "Critical Size of Heterogeneous Reactor with Small Number of Rods," *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, vol. 5, 1955, P/663, pp. 462–465.

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arrays. This general approach has also been applied by Galanin to determine the effectiveness of control rods.[1] It is perhaps worth while to point out that the analytical approach used in treating these problems could be extended to study the effect of other kinds of heterogeneity as well, for example, cavities.

The present treatment of the heterogeneous method develops the principal concepts obtained in the general theory. The analysis begins with a derivation of the neutron-balance equations. These results are then used to obtain the criticality condition and to demonstrate the basis of the homogenization method used with the unit-cell model. The section is concluded with an application of the method to a reactor with four fuel rods and a numerical example.

b. Neutron-balance Equation. The general theory of the heterogeneous method is developed on the basis of the following assumptions: (1) The thermal-neutron flux in the vicinity of a fuel lump possesses spherical or axial symmetry so that the individual lumps may be treated as point or line sources and sinks. (2) Elementary diffusion theory applies for the thermal flux in the moderator regions between the lumps. (3) The thermal-absorption properties of the fuel lumps are adequately described by means of the logarithmic derivative (thermal constant) of the thermal flux at the surface of the lump. (4) There is no resonance absorption in the fuel lumps.

The first of these assumptions is valid if the distance between lumps is much larger than their dimensions. Because of the symmetric configuration of most reactor lattices, this condition tends to be favored and the theory is applicable even in the case of quite large lumps. The second assumption requires that the absorption in the moderator be low and that the diffusion length for the moderator material be small in comparison to the lattice spacing. The third assumption is essentially a computational convenience, and its applicability is determined primarily by the accuracy of the model by which the thermal constant is computed. The last assumption is made to simplify the analysis. A treatment in which resonance absorption is included is given by Galanin.[2]

With the above assumptions as a model, consider an infinite heterogeneous lattice composed of cylindrical fuel rods of infinite length (i.e., a two-dimensional configuration). Let the radius of these rods be ρ_F and their neutron yield per thermal-neutron absorption be η. The thermal constant γ_h for the fuel lumps is defined as

$$\gamma_h \equiv \begin{array}{l}\text{ratio of total net current of thermal} \\ \text{neutrons into the lump to the value of the} \\ \text{thermal flux at the surface}\end{array} \qquad (10.235)$$

[1] A. D. Galanin, "The Thermal Reactor Regulator's Efficiency," *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, vol. 5, 1955, P/668; pp. 511–514 and P/663.
[2] *Ibid.*

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On the basis of the diffusion-theory model the thermal-neutron flux at a field point $\mathbf{r}$ in the moderator is given by

$$-D_M\nabla^2\phi_M(\mathbf{r}) + \Sigma_a^{(M)}\phi_M(\mathbf{r}) = S(\mathbf{r}) - A_{\text{rod}}(\mathbf{r} \to \mathbf{r}_k) \qquad (10.236)$$

where the vector $\mathbf{r}_k$ denotes the position of the kth rod. The two functions at the right-hand side are defined by

$S(\mathbf{r}) \equiv$ number of thermal neutrons appearing at $\mathbf{r}$ per unit volume per unit time from all sources in the system

$A_{\text{rod}}(\mathbf{r} \to \mathbf{r}_k) \equiv$ number of thermal-neutron absorptions per unit time due to the fuel rod at $\mathbf{r}_k$, if the field point $\mathbf{r}$ is located at one of the rod positions

For the present, the effect of the rod size is ignored and all rods are treated as line sources and sinks.

The thermal-source function S is easily computed in terms of the thermal constant γ_h. If $J(\mathbf{r}_k)$ is the net current of thermal neutrons at the surface of the rod at $\mathbf{r}_k$ (taken as positive when directed outward), then

$$S_f(\mathbf{r}_k) \equiv -\eta 2\pi\rho_F J(\mathbf{r}_k) \qquad (10.237)$$

is the total number of fast neutrons produced per unit time per unit length of rod from fissions in rod $\mathbf{r}_k$. But, by (10.235),

$$\gamma_h = -\frac{2\pi\rho_F J(\mathbf{r}_k)}{\phi_M(\mathbf{r}_k)} \qquad (10.238)\star$$

where $\phi_M(\mathbf{r}_k)$ denotes the thermal flux at the surface of the rod. Note that this expression is for unit length of rod and that the proper units of γ_h are length squared. Substitution into (10.237) yields

$$S_f(\mathbf{r}_k) = \eta\gamma_h\phi_M(\mathbf{r}_k) \qquad (10.239)$$

which gives the source strength of fast neutrons in terms of the local thermal flux. To obtain the thermal source at field point $\mathbf{r}$ due to this source of fast neutrons at $\mathbf{r}_k$, we simply multiply (10.239) by $q(\mathbf{r};\mathbf{r}_k)$, the number of neutrons slowing down to thermal at $\mathbf{r}$ per unit time per unit volume due to a fast source producing one neutron per unit time at $\mathbf{r}_k$. The function $S(\mathbf{r})$ is then simply the sum of such contributions from all the rods; thus, we compute

$$S(\mathbf{r}) \equiv \eta\gamma_h \sum_{\mathbf{k}} q(\mathbf{r};\mathbf{r}_k)\ \phi_M(\mathbf{r}_k) \qquad (10.240)$$

noting that γ_h and η are the same for all rods. The sum extends over the two-dimensional matrix of rod locations represented by the two coordinates of the vector $\mathbf{k}$.

The rod absorption (sink) term may also be expressed in terms of γ_h.

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An appropriate relation is seen to be (cf. Prob. 6.4)

$$A_{\text{rod}}(\mathbf{r} \to \mathbf{r}_k) \equiv -\sum_{\mathbf{k}} 2\pi\rho_F J(\mathbf{r}_k)\,\delta(\mathbf{r}-\mathbf{r}_k) = \gamma_h \sum_{\mathbf{k}} \delta(\mathbf{r}-\mathbf{r}_k)\,\phi_M(\mathbf{r}_k) \tag{10.241}$$

where $\delta(\mathbf{r}-\mathbf{r}_k)$ denotes the Dirac delta function. The substitution of this result, along with (10.240), into (10.236) gives

$$-D_M\nabla^2\phi_M(\mathbf{r}) + \Sigma_a^{(M)}\phi_M(\mathbf{r}) = \eta\gamma_h \sum_{\mathbf{k}} q(\mathbf{r};\mathbf{r}_k)\,\phi_M(\mathbf{r}_k) - \gamma_h \sum_{\mathbf{k}} \delta(\mathbf{r}-\mathbf{r}_k)\,\phi_M(\mathbf{r}_k) \tag{10.242}\star$$

for the *neutron-balance* equation at field point $\mathbf{r}$. This equation defines the thermal flux in the moderator, given a suitable expression for the function $q(\mathbf{r};\mathbf{r}_k)$. On the basis of the Fermi age model, the appropriate relation for the line source is obtained from (6.37). In the present application, this takes the form

$$q(\mathbf{r};\mathbf{r}_k) \equiv \frac{e^{-(\mathbf{r}-\mathbf{r}_k)^2/4\tau}}{4\pi\tau} \tag{10.243}$$

where τ denotes the age to thermal in the moderator material.

The solution to Eq. (10.242) may be obtained with the aid of the two-dimensional Fourier transform. In the case of the flux we define this to be

$$\mathfrak{F}\{\phi_M(\mathbf{r})\} \equiv \Phi(\mathbf{s}) \equiv \frac{1}{2\pi}\iint_{-\infty}^{\infty} \phi_M(\mathbf{r})e^{i\mathbf{s}\cdot\mathbf{r}}\,d\mathbf{r} \tag{10.244}$$

in accord with previous work [cf. Eq. (4.216)]. With this general form we compute the Fourier transform of Eq. (10.242) and solve for the transform function $\Phi(\mathbf{s})$. The result is found to be (where $\kappa^2 \equiv \Sigma_a^{(M)}/D_M$)

$$\Phi(\mathbf{s}) = \frac{\gamma_h}{2\pi D_M}\sum_{\mathbf{k}} \frac{\phi_M(\mathbf{r}_k)e^{i\mathbf{s}\cdot\mathbf{r}_k}}{s^2+\kappa_M^2}\,(\eta e^{-s^2\tau}-1) \tag{10.245}$$

where we have used Eq. (4.229). The inversion of this result yields the thermal flux in the moderator. For the inversion formula we use the two-dimensional form of Eq. (4.226). The first term in (10.245) yields

$$\frac{\eta}{2\pi}\iint_{-\infty}^{\infty} \frac{\exp[-s^2\tau + i\mathbf{s}\cdot(\mathbf{r}_k-\mathbf{r})]\,d\mathbf{s}}{s^2+\kappa_M^2} = \frac{\eta}{2\pi}\int_0^{\infty}\frac{se^{-s^2\tau}\,ds}{s^2+\kappa_M^2}\int_0^{2\pi} e^{i\mathbf{s}\cdot(\mathbf{r}_k-\mathbf{r})}\,d\theta \tag{10.246}$$

where $\mathbf{s} \equiv \mathbf{s}(s,\theta)$ and θ is the angle between the vectors $\mathbf{s}$ and $\mathbf{z}$, with $\mathbf{z} \equiv \mathbf{r}_k - \mathbf{r}$. The θ integration is recognized as the definition of the

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zeroth-order Bessel function[1] of the first kind and yields

$$\int_0^{2\pi} e^{is\cdot(r_k-r)}\,d\theta = \int_0^{2\pi} e^{isz\cos\theta}\,d\theta = 2\pi J_0(sz) \tag{10.247}$$

The inversion of the second term leads to a similar result; thus,

$$\frac{1}{2\pi}\iint_{-\infty}^{\infty} \frac{e^{is\cdot(r_k-r)}\,ds}{s^2+\kappa_M^2} = \int_0^{\infty} \frac{sJ_0(sz)\,ds}{s^2+\kappa_M^2} = K_0(z\kappa_M) \tag{10.248}$$

the last step follows from an integral expression for the K_0 function.[2] If (10.247) is substituted into (10.246) and the result, along with (10.248), is used in the inversion of (10.245), the solution to the flux is found to be

$$\phi_M(r) = \frac{\gamma_h}{2\pi D_M}\sum_k \phi_M(r_k)\left[\eta\int_0^{\infty}\frac{se^{-s^2\tau}J_0(s|r-r_k|)\,ds}{s^2+\kappa_M^2} - K_0(\kappa_M|r-r_k|)\right] \tag{10.249}$$

It will be convenient for use in subsequent analyses to write the above result in a more symmetric form. For this purpose we define a function $G(\tau)$ such that

$$G(\tau) \equiv \int_0^{\infty}\frac{se^{-s^2\tau}J_0(sz)\,ds}{s^2+\kappa_M^2} \tag{10.250}$$

It follows then that

$$\frac{\partial G}{\partial\tau} - \kappa_M^2 G = -\frac{1}{2\tau}e^{-z^2/4\tau}$$

The solution to this differential equation yields (with $\beta \equiv \tau\kappa_M^2$)

$$G(\tau) = e^{\beta}\int_{\beta}^{\infty}\exp\left[-\left(s+\frac{\kappa_M^2 z^2}{4s}\right)\right]\frac{ds}{2s} \tag{10.251}$$

where we have used the fact that

$$K_0(\kappa_M z) \equiv \int_0^{\infty}\exp\left[-\left(s+\frac{\kappa_M^2 z^2}{4s}\right)\right]\frac{ds}{2s} \tag{10.252}$$

and noted that $G(0) = K_0(\kappa_M z)$. The substitution of these results into (10.249) yields

$$\phi_M(r) = \frac{\gamma_h}{2\pi D_M}\sum_k \phi_M(r_k)\left\{\eta e^{\beta}\int_{\beta}^{\infty} - \int_0^{\infty}\right\}\exp\left[-(s+|r-r_k|^2/4sL_M^2)\right]\frac{ds}{2s} \tag{10.253}$$

[1] Copson, *op. cit.*, p. 319.

[2] E. T. Whittaker and G. N. Watson, "A Course in Modern Analysis," 4th ed., p. 383, example 36, Cambridge University Press, New York, 1950.

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with the application of the relations $z = |\mathbf{r} - \mathbf{r}_k|^2$ and $L_M^2 \equiv D_M/\Sigma_a^{(M)}$. The present notation may be further simplified if we call

$$H(z) \equiv \frac{V_{\text{cell}}}{2\pi L_M^2}\left\{\eta e^{\beta}\int_{\beta}^{\infty} e^{-\left(s+\frac{\beta z^2}{4\tau s}\right)}\frac{ds}{2s} - \int_0^{\infty} e^{-\left(s+\frac{\beta z^2}{4\tau s}\right)}\frac{ds}{2s}\right\} \tag{10.254}$$

where V_{cell} denotes the total area of a "unit cell." Equation (10.253) may then be written in the form

$$c_0\phi_M(\mathbf{r}) = \sum_{\mathbf{k}} H(|\mathbf{r} - \mathbf{r}_k|)\phi_M(\mathbf{r}_k) \tag{10.255}\star$$

with

$$c_0 \equiv \frac{V_{\text{cell}}\Sigma_a^{(M)}}{\gamma_h} \tag{10.256}$$

c. Thermal Constant. It was noted that the thermal constant could be computed with various degrees of accuracy, ranging from an estimate based on diffusion theory to one based on the transport relations. Since the diffusion-theory estimate will suffice in many instances, the appropriate relations will be developed on the basis of this model. The computation is straightforward and utilizes the unit-cell concept in the usual way. In general the relations (5.349) through (5.354) will be required. By applying the continuity conditions for both the net current and the flux at the surface of the fuel rod, it is necessary to consider only the flux distribution on the inside. This is given by the usual differential equation (5.350). In cylindrical geometry the solution to this equation is seen to be

$$\phi_F(\rho) \equiv A I_0(\varkappa_F\rho) \tag{10.257}$$

wherein we have applied the requirement that the flux be finite within the rod (5.354). The net current is obtained from (10.257); thus

$$J(\rho_F) \equiv -D_F\phi_F'(\rho_F) = -A D_F\varkappa_F I_0'(\varkappa_F\rho_F)$$

Substitution of these expressions into (10.238) yields

$$\gamma_h = \frac{2\pi\rho_F\Sigma_a^{(F)}I_1(\varkappa_F\rho_F)}{I_0(\varkappa_F\rho_F)\varkappa_F} \tag{10.258}$$

It is of interest to mention that the thermal constant is related to two other quantities previously defined; these are γ_F and F. The relationship is

$$\gamma_h = \pi\rho_F\gamma_F = \frac{\pi\rho_F^2\Sigma_a^{(F)}}{F} \tag{10.259}$$

where γ_F is given by (10.80) and F by (10.88).

d. Criticality Condition. In the case of the infinite heterogeneous lattice it is clear that there can be no gross spatial variations in the neutron flux, only the local variations between lumps. These local vari-

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ations are of course repeated with the periodicity of the lattice. The flux at each fuel lump must be the same, however, for a completely symmetric and singly periodic lattice. Then, the flux at any rod $\mathbf{r}_n$ must be some constant, and the neutron-balance relation (10.255) reduces to the simple form

$$c_0 = \sum_{\mathbf{k}} H(r_k) \tag{10.260}$$

where r_k now denotes the distance between some reference rod and any other rod in the lattice. This result is in fact the criticality condition for the infinite lattice, since given any three of the four basic parameters η, the lattice spacing, the rod size, and the reactor materials, Eq. (10.260) will yield the value of the fourth which will make the system critical.

The result (10.255) may also be used to obtain the criticality condition for the reactor with finite fuel lattice but with infinite reflector (composed of the same moderator material). It is necessary to impose the latter condition because otherwise expression (10.243) for the slowing-down density would no longer be valid and new functions satisfying boundary conditions at the surface of the reflector would be required. The appropriate neutron-balance relation in terms of the flux at the various fuel-rod locations is then given by

$$c_0\phi_M(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\mu}} H(|\boldsymbol{\lambda} - \boldsymbol{\mu}|)\phi_M(\boldsymbol{\mu}) \tag{10.261}$$

The components of the vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ denote the coordinates of any two fuel rods in the finite array, as located from some arbitrary reference point. The result (10.261) is a system of homogeneous equations for the thermal fluxes at the surfaces of the fuel rods $\phi_M(\boldsymbol{\lambda})$. A solution to this system exists if the determinant is identically zero. By imposing this requirement the determinant reduces to an algebraic equation, and the roots of this equation yield the condition for criticality. As in the case of the infinite lattice, all four of the basic parameters will be involved in this equation, and by specifying any three, the determinant will yield the fourth. The calculational procedure involved here is demonstrated in a subsequent analysis, and an example of a four-rod reactor is computed in detail.

e. Proof of the Homogenization Method. In as much as the homogenization method based on the unit-cell model is entirely adequate for computing most heterogeneous-reactor configurations, it is of interest to determine the error acquired in using the method and to delineate the conditions under which it is valid. To study this problem we postulate a reactor with a finite array of cylindrical fuel rods and an infinite reflector composed of the same material as used for the moderator in the lattice. We further specify that the lattice is a square array with spacing a.

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The general solution for the thermal flux in this reactor may be given as a linear combination of the elementary solutions $e^{i\mathbf{B}\cdot\mathbf{r}}$ (cf. Sec. 7.4d). Thus we take for the flux at the surface of the rod at $\mathbf{r}$

$$\phi_M(\mathbf{r}) \equiv e^{i\mathbf{B}\cdot\mathbf{r}} \tag{10.262}$$

The substitution of this relation into the thermal-flux equation (10.255) yields

$$c_0 = \sum_{\mathbf{k}} H(|\mathbf{r} - \mathbf{r}_k|)e^{i\mathbf{B}\cdot(\mathbf{r}_k - \mathbf{r})} \tag{10.263}$$

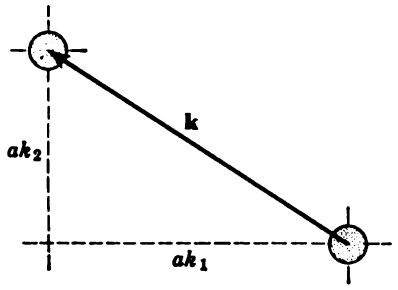

FIG. 10.19 Position vector of fuel rod.

In order to demonstrate the relationship between the present model and the homogenization method, it is necessary to reduce Eq. (10.263) to a form which is readily identifiable with the established results of the latter. This is accomplished by deriving suitable approximations to the sums indicated in (10.263). We begin with the first term S_1, as given by the definition of H in Eq. (10.254):

$$S_1 = \frac{\eta V_{\text{cell}} e^{\beta}}{2\pi L_M^2} \int_{\beta}^{\infty} \frac{e^{-s}\, ds}{2s} \sum_{\mathbf{k}} \exp\left\{ iB_1(ak_1 - x) + iB_2(ak_2 - y) - [(ak_1 - x)^2 + (ak_2 - y)^2] \frac{\kappa_M^2}{4s} \right\} \tag{10.264}$$

wherein we have defined the components of the vectors $\mathbf{r}$, $\mathbf{r}_k$, and $\mathbf{B}$ by (refer to Fig. 10.19)

$$\mathbf{r} \equiv \mathbf{r}(x,y) \qquad \mathbf{r}_k \equiv \mathbf{r}_k(ak_1, ak_2) \qquad \mathbf{B} \equiv \mathbf{B}(B_1, B_2) \tag{10.265}$$

The series given in (10.264) may be summed with the aid of the Poisson summation formula,[1] which is given by

$$\sum_{l=-\infty}^{\infty} f(l) = (2\pi)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} \mathfrak{F}\{f\} \equiv (2\pi)^{\frac{1}{2}} \sum_{m=-\infty}^{\infty} F(2\pi m) \tag{10.266}$$

where $f(l)$ denotes some function of the summation index l and F is its Fourier transform. In the present case, there are two summation variables k_1 and k_2, and for k_1 we have

$$f(k_1) \equiv \exp\left[iB_1(ak_1 - x) - \frac{(ak_1 - x)^2 \kappa_M^2}{4s} \right]$$

[1] Morse and Feshbach, *op. cit.*, p. 467.

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The application of the summation formula yields

$$\sum_{k_1} f(k_1) = \frac{1}{2\pi} \sum_{m_1} \int_{-\infty}^{\infty} \exp\left[iw_1 m_1 + iB_1 \left(\frac{aw_1}{2\pi} - x \right) - \left(\frac{aw_1}{2\pi} - x \right)^2 \frac{\kappa_M^2}{4s} \right] dw_1$$

and it follows that the double sum in (10.264) may be written in the form

$$\sum_{\mathbf{k}} = \frac{1}{(2\pi)^2} \sum_{\mathbf{m}} \iint_{-\infty}^{\infty} \exp\left[i\mathbf{w} \cdot \mathbf{m} + i\mathbf{B} \cdot \left(\frac{a\mathbf{w}}{2\pi} - \mathbf{r} \right) - \left(\frac{a\mathbf{w}}{2\pi} - \mathbf{r} \right)^2 \frac{\kappa_M^2}{4s} \right] d\mathbf{w} \tag{10.267}$$

where $\mathbf{w} \equiv \mathbf{w}(w_1, w_2)$. The integration is easily carried out by introducing the new variable

$$\mathbf{u} \equiv \frac{a\mathbf{w}}{2\pi} - \mathbf{r} \qquad \text{with } \mathbf{u} \equiv \mathbf{u}(u_1, u_2) \tag{10.268}$$

and completing the square in the exponential. In this way we obtain for the u_1 component

$$\int_{-\infty}^{\infty} \exp\left[-\frac{\kappa_M^2 u_1^2}{4s} + iu_1 \left(B_1 + \frac{2\pi m_1}{a} \right) \right] du_1 = 2L_M (s\pi)^{\frac{1}{2}} \exp\left[-\left(B_1 + \frac{2\pi m_1}{a} \right)^2 sL_M^2 \right]$$

The integral involving u_2 is evaluated in the same way. If these expressions for the integrals are used in (10.267), the double sum takes the form

$$\sum_{\mathbf{k}} \exp\left[i\mathbf{B} \cdot (\mathbf{r}_k - \mathbf{r}) - (\mathbf{r}_k - \mathbf{r})^2 \frac{\kappa_M^2}{4s} \right] = \frac{4s\pi L_M^2}{a^2} \sum_{\mathbf{m}} \exp\left[i \left(\frac{2\pi}{a} \right) \mathbf{m} \cdot \mathbf{r} - s \left(\frac{2\pi L_M}{a} \right)^2 \left(\mathbf{m} + \frac{a\mathbf{B}}{2\pi} \right)^2 \right] \tag{10.269}$$

The substitution of this result into Eq. (10.264), followed by the integration over the range of the variable s, yields the following double sum for S_1:

$$S_1 = \eta \sum_{\mathbf{m}} \frac{\exp\left[i \left(\frac{2\pi}{a} \right) \mathbf{m} \cdot \mathbf{r} - \frac{4\tau\pi^2}{a^2} \left(\mathbf{m} + \frac{a\mathbf{B}}{2\pi} \right)^2 \right]}{1 + \frac{4\pi^2 L_M^2}{a^2} \left(\mathbf{m} + \frac{a\mathbf{B}}{2\pi} \right)^2} \tag{10.270}$$

where we have used $V_{\text{cell}} = a^2$ for the square lattice.

The result (10.270) takes on an especially simple form if the following

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conditions are satisfied:

$$4e^{-4\tau\pi^2/a^2} \ll 1 \qquad \text{and} \qquad \frac{aB}{2\pi} \ll 1 \tag{10.271}$$

The first is satisfied if the lattice spacing is small compared to the slowing-down length in the moderator. It is interesting to note that under these conditions the Gaussian distribution of the thermal neutrons is broad compared to the unit-cell dimension (refer to Fig. 10.4). It is a good approximation, then, to assume that the thermal source in the moderator is uniform; of course, this is precisely the assumption made in the unit-cell calculation. Thus we have established one condition which must be satisfied if the homogenization method is to be valid. The second condition in (10.271) simply requires that the reactor be large in comparison to the lattice spacing. If these conditions are satisfied, then a good approximation to the sum S_1 is obtained by taking the first term only ($m_1 = m_2 = 0$); thus

$$S_1 \simeq \frac{\eta e^{-\tau B^2}}{1 + L_M^2 B^2} \tag{10.272}$$

which leads to the interesting observation that in first approximation S_1 is independent of $\mathbf{r}$ and the lattice geometry (i.e., shape of the unit cell). Thus in the present computation it would have sufficed to take the field point $\mathbf{r}$ at any rod location $\mathbf{r}_\lambda$; this would have reduced (10.263) to the form [cf. Eq. (10.261)]

$$c_0 = \sum_{\boldsymbol{\mu}} H(|\boldsymbol{\lambda} - \boldsymbol{\mu}|) e^{i\mathbf{B}\cdot(\boldsymbol{\mu}-\boldsymbol{\lambda})} = \sum_{\boldsymbol{\kappa}} H(|\boldsymbol{\kappa}|) e^{i\mathbf{B}\cdot\boldsymbol{\kappa}a} \tag{10.273}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ denote two rod positions and $\boldsymbol{\mu} - \boldsymbol{\lambda} = a\boldsymbol{\kappa}$; note that $\mathbf{r}_\lambda \equiv \boldsymbol{\lambda}$. The evaluation of (10.273) is a somewhat easier task than the evaluation of (10.263).

The second sum S_2 which appears in (10.263) is

$$S_2 = -\frac{V_{\text{cell}}}{2\pi L_M^2} \int_0^\infty \frac{e^{-s}\, ds}{2s} \sum_{\mathbf{k}} \exp\left[i\mathbf{B}\cdot(\mathbf{r}_k - \mathbf{r}) - (\mathbf{r}_k - \mathbf{r})^2 \frac{\varkappa_M^2}{4s} \right] \tag{10.274}$$

A comparison of this function with (10.264) reveals that S_2 is simply S_1 divided by minus η and evaluated at $\tau = 0$. Thus we obtain from (10.270)

$$S_2 = -\sum_{\mathbf{m}} \frac{e^{i(2\pi/a)\mathbf{m}\cdot\mathbf{r}}}{1 + (2\pi L_M/a)^2[\mathbf{m} + (a\mathbf{B}/2\pi)]^2} \tag{10.275}$$

As in the S_1 computation, the above result can be reduced to a simpler form by imposing some restrictions on the system. If we require that

$$\left(\frac{2\pi L_M}{a}\right)^2 \gg 1 \qquad \text{and} \qquad \frac{aB}{2\pi} \ll 1 \tag{10.276}$$

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then the series (10.275) is well approximated by

$$S_2 \simeq -\frac{1}{1 + L_M^2 B^2} - c_1 \tag{10.277}$$

where

$$c_1 \equiv \left(\frac{a}{2\pi L_M}\right)^2 \sum_{\mathbf{m}}{}' \frac{\cos\left[(2\pi/a)\mathbf{m} \cdot \mathbf{r}\right]}{m_1^2 + m_2^2} \tag{10.278}$$

with $\mathbf{m} \equiv \mathbf{m}(m_1,m_2)$; the prime means that the first term ($m_1 = m_2 = 0$) is omitted.

The comparison with the homogenization method is to be made on the basis of the criticality condition. To obtain the appropriate relation for the present method, we require that the field point $\mathbf{r}$ be taken at $\mathbf{r}_k$, the fuel-rod locations. This corresponds to the procedure outlined in connection with Eq. (10.261). Then, collecting the results (10.272) and (10.277), we obtain for (10.263)

$$c_0 = \frac{\eta e^{-\tau B^2} - 1}{1 + L_M^2 B^2} - c_1(0) \tag{10.279}$$

where $c_1(0)$ is (10.278) for the case $\mathbf{r} \to \mathbf{r}_k$. Finally, if we call

$$\mathfrak{c} \equiv c_0 + c_1(0) \tag{10.280}$$

and

$$f \equiv \frac{1}{1 + \mathfrak{c}} \tag{10.281}$$

we obtain from (10.279) the relation

$$1 = \frac{\eta f e^{-B^2\tau}}{1 + L^2 B^2}$$

where L^2 is defined by

$$L^2 \equiv L_M^2(1 - f)$$

Thus, if we identify f as the thermal utilization and L as the diffusion length for the equivalent homogeneous system, the relation (10.263) yields the usual criticality condition (without resonance absorption) obtained in the homogenization method using the unit-cell concept. But to obtain this result we introduced several restrictions on the system, (10.271) and (10.276); these are the conditions then which must be fulfilled in order that the homogenization method yield reliable results.

It should be noted that the function $c_1(0)$ accounts for the nonuniformity in the thermal-neutron distribution through the unit cell. Galanin[1] has shown that for a square lattice

$$c_1(0) = \frac{a^2}{4\pi L_M^2}\left(\ln \frac{a^2}{\pi \rho_F^2} + \frac{\pi \rho_F^2}{a^2} - \ln 4\pi + \frac{\pi}{3}\right) \tag{10.282}$$

where $\ln 4\pi - \pi/3 = 1.48$. A careful calculation using the Wigner-Seitz

[1] Galanin, *op. cit.*, P/666.

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model leads to the same result, but with $\frac{8}{3}$ in place of the 1.48. Other lattice geometries have also been computed by Galanin, and the appropriate results for $c_1(0)$ are given in the reference paper. Another detail that should be mentioned is that the computation of c_0 should take into account the finite size of the fuel lumps; therefore, in applying Eq. (10.256), one should use in place of V_{cell}, $V_M = V_{\text{cell}} - \pi\rho_F^2$.

Finally, it is of interest to note that the quantity $\mathfrak{c}$ defined by Eq. (10.280) is essentially the disadvantage factor ζ used in unit-cell theory. Since,

$$\frac{1}{f} - 1 = \left[\frac{V_M\Sigma_a^{(M)}}{V_F\Sigma_a^{(F)}}\right]\zeta \tag{10.283}$$

we obtain from (10.281)

$$\mathfrak{c} = \left[\frac{V_M\Sigma_a^{(M)}}{V_F\Sigma_a^{(F)}}\right]\zeta \tag{10.284}$$

f. Reactors with Small Number of Fuel Lumps. Inasmuch as the Feinberg-Galanin method is especially useful for analyzing reactors with small numbers of fuel lumps, it is of interest to develop the criticality relations in a form suitable for computation. The analysis which follows is carried out for the case of a square lattice geometry with spacing a which constitutes the core region; the core is surrounded by an infinite reflector. The fuel lumps are cylindrical rods of radius ρ_F and infinite length. As in the preceding analyses, the resonance absorption in the fuel is omitted in this calculation.

The criticality condition is obtained from Eq. (10.261), with the function H given by (10.254). For the present application it is convenient to write this function in the form

$$H(|\boldsymbol{\lambda} - \boldsymbol{\mu}|) \equiv \frac{a^2}{2\pi L_M^2}\left[\eta e^{\beta}\int_{\beta}^{\infty}\exp\left(-s - |\boldsymbol{\lambda} - \boldsymbol{\mu}|^2\frac{\kappa_M^2}{4s}\right)\frac{ds}{2s} - K_0(\kappa_M|\boldsymbol{\lambda} - \boldsymbol{\mu}|)\right] \tag{10.285}$$

where $\boldsymbol{\mu}$ and $\boldsymbol{\lambda}$ denote the position of various fuel rods. The notation of function H is somewhat simplified by introducing the definitions

$$\begin{aligned} |\boldsymbol{\lambda} - \boldsymbol{\mu}| &\equiv d_{\lambda\mu} && \text{when } \lambda \neq \mu \\ &\equiv d_{\lambda\lambda} \equiv \rho_F && \lambda = \mu \end{aligned} \tag{10.286}$$

Then

$$H(d_{\lambda\mu}) = \frac{a^2}{2\pi L_M^2}\left[\eta e^{\beta}\int_{\beta}^{\infty}\exp\left(-s - \frac{d_{\lambda\mu}^2\kappa_M^2}{4s}\right)\frac{ds}{2s} - K_0(d_{\lambda\mu}\kappa_M)\right] \tag{10.287}$$

A useful approximation to this expression can be obtained in the case that $\tau\kappa_M^2 \equiv \beta \ll 1$, which will apply in many practical situations. To

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derive this form we write the first integral in (10.287) as

$$\int_\beta^\infty \exp\left(-s - \frac{d_{\lambda\mu}^2 \kappa_M^2}{4s}\right)\frac{ds}{2s} \equiv K_0(d_{\lambda\mu}\kappa_M) - A(d_{\lambda\mu}) \qquad (10.288)$$

where

$$A(d_{\lambda\mu}) \equiv \int_0^\beta \exp\left(-s - \frac{d_{\lambda\mu}^2 \kappa_M^2}{4s}\right)\frac{ds}{2s} \qquad (10.289)$$

Next, the factor e^{-s} is expanded in a Taylor series and the integration carried out term by term; the first two terms yield the approximation

$$A(d_{\lambda\mu}) \simeq \frac{1}{2}\left[\left(1 + \frac{d_{\lambda\mu}^2 \kappa_M^2}{4}\right) E_1\left(\frac{d_{\lambda\mu}^2}{4\tau}\right) - \beta e^{-d_{\lambda\mu}^2/4\tau}\right] \qquad (10.290)$$

wherein we have used Eq. (5.306) and the recursion relation

$$E_n(x) = \frac{1}{n-1}\left[e^{-x} - xE_{n-1}(x)\right] \qquad n > 1 \qquad (10.291)$$

Finally

$$H(d_{\lambda\mu}) \simeq \frac{a^2}{2\pi L_M^2}\left[(\eta e^\beta - 1)K_0(d_{\lambda\mu}\kappa_M) - \eta e^\beta A(d_{\lambda\mu})\right] \qquad (10.292)\star$$

This result will be used in the present calculation, with $A(d_{\lambda\mu})$ given by (10.290) and the $d_{\lambda\mu}$ defined according to (10.286). In the event that $d_{\lambda\mu}\kappa_M \ll 1$, the Bessel function may be computed by

$$K_0(x) \simeq -\left(0.5772 + \ln\frac{x}{2}\right) \qquad \text{with } x \ll 1 \qquad (10.293)$$

The criticality condition for the reactor is derived from Eq. (10.261), which, to simplify notation, we write in the form [with $H_{\lambda\mu} \equiv H(d_{\lambda\mu})$]

$$c_0\phi_\lambda = \sum_\mu H_{\lambda\mu}\phi_\mu \qquad \text{or} \qquad \sum_\mu (H_{\lambda\mu} - c_0\delta_{\lambda\mu})\phi_\mu = 0 \qquad (10.294)$$

Here, $\phi_\mu \equiv \phi_M(\mathbf{y})$, and for convenience we have labeled each rod with a single index. As previously indicated, (10.294) is a set of homogeneous equations in the unknowns ϕ_μ. A nontrivial solution exists only if the determinant of this system is identically zero, that is,

$$|H_{\lambda\mu} - c_0\delta_{\lambda\mu}| \equiv 0 \qquad (10.295)$$

This equation gives the criticality condition. In many cases of practical interest, the lattice configuration will exhibit several degrees of symmetry. As a result, various elements of the determinant will be repeated, and this in turn can lead to high orders of degeneracy in the determinantal (characteristic) equation. Thus the complicated equation symbolized by (10.295) will in actual practice reduce to considerably simpler forms. This point is well demonstrated in the example which follows.

To illustrate the computational procedure, we consider a heterogeneous reactor consisting of four fuel rods. Each rod is at the corner of a square

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with sides of length a (see Fig. 10.20). The distances $d_{\lambda\mu}$ between the various rods are

$$\begin{gathered} d_{12} = d_{23} = d_{34} = d_{14} = a \qquad d_{13} = d_{24} = \sqrt{2}\, a \\ d_{\lambda\lambda} = \rho_F \qquad d_{\lambda\mu} = d_{\mu\lambda} \end{gathered} \tag{10.296}$$

and we find that the determinant (10.295) can be written in terms of three numbers, namely, b_{11}, b_{12}, and b_{13}, where

$$b_{\lambda\mu} \equiv H_{\lambda\mu} - c_0\delta_{\lambda\mu} \tag{10.297}$$

The appropriate values of $H_{\lambda\mu}$ are [with $A_{\lambda\mu} \equiv A(d_{\lambda\mu})$]

$$\begin{aligned} H_{11} &= \frac{a^2}{2\pi L_M^2}\left[(\eta e^{\beta} - 1)K_0(\rho_F \varkappa_M) - \eta e^{\beta} A_{11}\right] \\ H_{12} &= \frac{a^2}{2\pi L_M^2}\left[(\eta e^{\beta} - 1)K_0(a\varkappa_M) - \eta e^{\beta} A_{12}\right] \\ H_{13} &= \frac{a^2}{2\pi L_M^2}\left[(\eta e^{\beta} - 1)K_0(\sqrt{2}\, a\varkappa_M) - \eta e^{\beta} A_{13}\right] \end{aligned} \tag{10.298}$$

and A_{11}, A_{12}, and A_{13} may be computed from (10.290).

The relation (10.295) is a simple 4 by 4 determinant which is easily shown to yield the equation

$$(b_{11} - b_{13})^2[(b_{11} + b_{13})^2 - 4b_{12}^2] = 0 \tag{10.299}$$

which has four roots:

$$\begin{gathered} b_{11} = b_{13} \qquad \text{a double root} \\ b_{11} + b_{13} \pm 2b_{12} = 0 \end{gathered} \tag{10.300}$$

Only one of these roots is physically acceptable, however. The first two are rejected since they require that $\phi_2 = -\phi_4$, an impossible situation since all the ϕ_μ must be identical by symmetry requirements. Likewise the third root, corresponding to the negative sign in the second equation, must be rejected since it would require that $\phi_1 = -\phi_2$. Only the root corresponding to the positive sign is acceptable inasmuch as it allows the solution ϕ_μ = constant. This relation is in fact the criticality condition. As previously noted, this result can also be obtained by imposing the symmetry requirements directly onto the determinant. Since in the present case we have ϕ_μ = constant, then each of the four equations implied in (10.294)

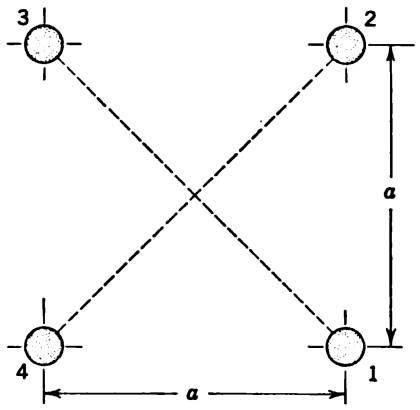

FIG. 10.20 Four-rod heterogeneous reactor.

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must be identical; moreover, they must each yield the relation

$$H_{11} + H_{13} + 2H_{12} = c_0 \tag{10.301}$$

which is precisely the second of equations (10.300), using the plus sign. For computational convenience we write (10.301) in the form

$$\eta(\psi_{11} + \psi_{13} + 2\psi_{12}) = \chi_{11} + \chi_{13} + 2\chi_{12} + c_0 \tag{10.302}$$

where

$$H_{\lambda\mu} \equiv \eta\psi_{\lambda\mu} - \chi_{\lambda\mu} \tag{10.303}$$

and

$$\psi_{\lambda\mu} \equiv \frac{a^2 e^{\beta}}{2\pi L_M^2}\,[K_0(d_{\lambda\mu}\kappa_M) - A_{\lambda\mu}] \qquad \chi_{\lambda\mu} \equiv \frac{a^2}{2\pi L_M^2}\,K_0(d_{\lambda\mu}\kappa_M) \tag{10.304}$$

g. Numerical Example. The results obtained in the preceding analyses are used to determine the criticality requirements of a four-rod reactor using enriched uranium and a graphite moderator and (infinite) reflector. The present results are presented in the form of curves of the multiplication constant versus rod spacing for various U^{235} enrichments. A single rod size is used throughout. The computed multiplication constant for two different heterogeneous arrangements is then compared to the values predicted by the homogenization method using the two-group model. The nuclear data and other parameters used in these computations are as follows:

$L_M = 52$ cm	$\tau = 364$ cm^2	$\Sigma_a^{(M)} = 0.000355$ cm^{-1}
$\rho_F = 3$ cm	$\eta_{25} = 2.08$	$\sigma_a^{25} = 687$ barns
$\sigma_a^{28} = 2.75$ barns	$N_U = 0.0482 \times 10^{24}$ nuclei/cm^3	$D_F = 0.702$ cm

where (28) refers to U^{238} and (25) to U^{235}.

The multiplication constant for the four-rod reactor is obtained from (10.302). Thus if η is the average number of neutrons produced per absorption in the fuel rod (taking into account the concentration of U^{235} and U^{238}), and η_c is the usual fictitious yield required to achieve steady state, then

$$k = \frac{\eta}{\eta_c} = \frac{\eta(\psi_{11} + \psi_{13} + 2\psi_{12})}{\chi_{11} + \chi_{13} + 2\chi_{12} + c_0} \tag{10.305}$$

For c_0 we use [cf. Eq. (10.256) and discussion following Eq. (10.282)]

$$c_0 = \frac{(a^2 - \pi\rho_F^2) I_0(\rho_F\kappa_F)\Sigma_a^{(M)}}{2\pi\rho_F I_1(\rho_F\kappa_F)\Sigma_a^{(F)} L_F} \tag{10.306}$$

Equation (10.305) has been applied for the enrichment ratios $N_{25}/N_U = 0.05, 0.10, 0.20$ as well as for natural uranium $N_{25}/N_U = 0.0071$. The results are shown in Fig. 10.21.

Since the four-rod lattice is a near minimum heterogeneous configuration, it is of some interest to determine the error which is incurred in analyzing the system by the homogenization method. To obtain a com-

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parison we define an equivalent homogeneous reactor which has the same moderator and reflector as the heterogeneous configuration. Since the heterogeneous system has an axial symmetry we define the core of the

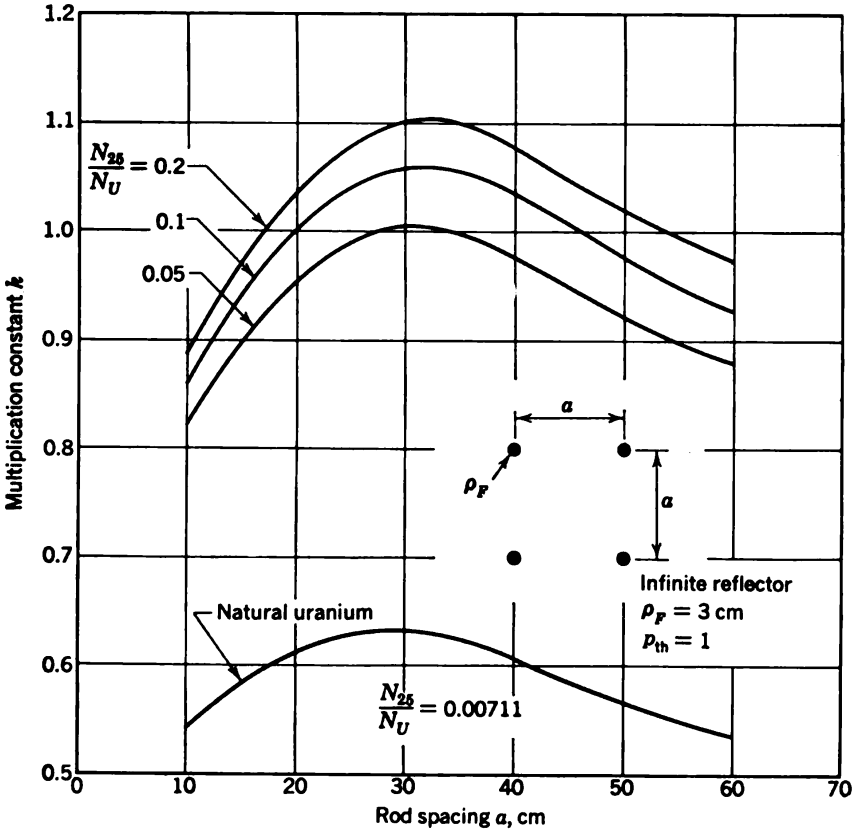

FIG. 10.21 Multiplication constant as a function of rod spacing in a four-rod graphite reactor.

homogeneous reactor to be a cylinder of infinite length and radius:

$$R_{\text{core}} = \left(\frac{na^2}{\pi}\right)^{\frac{1}{2}} \tag{10.307}$$

where n is the number of fuel rods. The nuclear characteristics of this system are computed with the aid of the two-group model of Sec. 8.4. The principal parameters required for this calculation are the infinite-medium multiplication constant k_∞^{het}, the roots of the characteristic equation (8.171), and the constants a_1 and a_2 (8.175), as well as the cylindrical functions listed in Table 8.3. The infinite-medium multiplication constant is computed from

$$k_\infty^{\text{het}} = \eta f = \frac{\eta}{1 + \mathfrak{c}} \tag{10.308}$$

using Eqs. (10.280), (10.282), and (10.306). For the roots of the characteristic equation we use the approximations (8.226) and (8.227). In the

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present case μ^2 is given by

$$\mu^2 \simeq \frac{1}{M^2}\left(\frac{k_\infty^{\text{het}}}{k_{\text{hom}}} - 1\right)\left[1 - \frac{\tau L^2}{M^4}\left(\frac{k_\infty^{\text{het}}}{k_{\text{hom}}} - 1\right)\right] \tag{10.309}$$

where the L^2 is computed from $L^2 \equiv L_M^2(1 - f)$. The symbol k_{hom} denotes the multiplication constant of the equivalent homogeneous system. Finally, the coupling constants are obtained from

$$a_1 \equiv \frac{\Sigma_R^{(1)}(1 + \tau\mu^2)}{\Sigma_a^{(2)} k_\infty^{\text{het}}} \qquad a_2 \equiv \frac{\Sigma_R^{(1)}(1 - \tau\lambda^2)}{\Sigma_a^{(2)} k_\infty^{\text{het}}} \tag{10.310}$$

and $\Sigma_a^{(2)} = D_2/L^2$.

This general scheme has been used to compute the three configurations indicated in Table 10.9. All three reactors have been taken from the

TABLE 10.9 COMPARISON OF THE FEINBERG-GALANIN METHOD WITH THE HOMOGENIZATION METHOD

a, cm	R_{core}, cm	f	k_∞^{het}	k_{hom}	k_{het}	Error
20	22.5	0.978	1.890	1.198	0.954	+26
30	33.8	0.934	1.802	1.220	1.005	21
40	45.1	0.868	1.680	1.060	0.978	8.4

results given in Fig. 10.21 for the 3-cm fuel rods with 5 per cent enrichment. The three cases correspond to $a = 20$, 30, and 40 cm. The parameters common to the three systems are:

$$\eta = 1.935 \qquad D_1 = D_2 = D_3 = D_4 = 0.909 \text{ cm}$$
$$\frac{N_{25}}{N_U} = 0.05 \qquad \Sigma_R^{(1)} = \Sigma_R^{(3)} = 0.00249 \text{ cm}^{-1}$$

along with the nuclear constants given previously.

Similar calculations performed by Galanin[1] for a number of lattice schemes indicate that the error due to the use of the homogenization method is generally a few per cent, the actual magnitude depending strongly upon the nuclear properties of the system, as well as on the geometry and number of rods.

[1] Galanin, *op. cit.*, P/663.

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CHAPTER 11

CONTROL-ROD THEORY

11.1 Central Rod in One-velocity Model

The present treatment of the problem of computing the effectiveness of a control rod is carried out on the basis of two analytical models. In the application of the first, the one-velocity model, attention is directed toward displaying some of the elementary physical ideas involved in the general problem. The model is used here primarily for purposes of illustration; it is not intended that it be considered an accurate tool for computing control-rod effectiveness. The second method utilizes the two-group model, and it is expected that the results obtained from this formulation will be useful for the solution of practical problems.

In order that the present analyses remain relatively unencumbered of mathematical complications a single elementary reactor configuration has been selected, and all control-rod calculations are carried out for that system alone. The geometry used is the bare cylindrical reactor. It should be recognized, however, that this choice does not necessarily limit the results obtained here to this particular configuration. If the actual reactor of interest has a markedly different shape, it is still possible to determine an "equivalent" bare cylindrical configuration by matching bucklings. Also, if the actual reactor is reflected, a suitable bare cylinder can be computed by applying reflector savings corrections to the various reflected surfaces. Some care should be exercised in dealing with this latter situation, however, if the control-rod locations are near the core-reflector interface. Under these circumstances the bare-reactor calculations will tend to underestimate the rod effectiveness in certain cases since in the actual reflected reactor the rods may be exposed to substantially higher thermal fluxes than in the bare model because of possible peaking in the reflector (see Fig. 8.23). In any case, it is expected that estimates based on the bare cylindrical reactor will give a reliable indication of the characteristics of a design configuration, very possibly within the limitations of the two-group model.

In the various control-rod problems considered here it is always assumed that the rods have circular cross sections and that they are inserted with axes parallel to the axis of the reactor. Calculations of rod effectiveness are carried out only for the case of the fully inserted

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rod. Although methods have been developed for treating the problem of the partially inserted rod, these either require more sophisticated analytical techniques (perturbation theory) than available here or are semiempirical, and calculations must be correlated with measurements (the "window-shade" method[1] used by H. L. Garabedian).

Finally, it is pointed out that, unless otherwise stated, control-rod effectiveness is measured in the present work by the reactivity of the reactor without the specified rod configuration, but with the same fuel loading. In performing these calculations it is assumed that the reactor with fully inserted rods is just critical; the reactor without rods contains no "holes" corresponding to the rod locations.

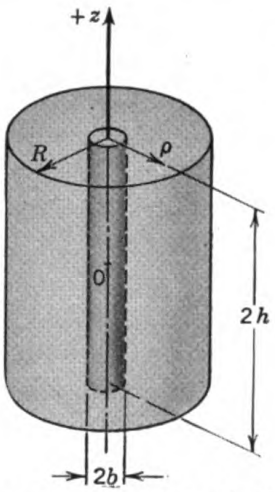

FIG. 11.1 Cylindrical reactor with central control rod.

a. Cylindrical Reactor with Gray Control Rod. As a first elementary example of a control-rod calculation consider the case of a bare cylindrical thermal reactor operating at steady state with a fully inserted central gray control rod. By "gray rod" we mean a rod composed of a material which, although a reasonably good neutron absorber, is not so strong an absorber that it cannot support some neutron flux. Also, let us assume that diffusion theory is applicable within the interior of this rod. It should be noted that these assumptions are not unlike those imposed for fuel lumps in the calculation of the thermal utilization for heterogeneous lattices.

The general geometric configuration of this reactor with control rod is shown in Fig. 11.1. The radius of the reactor is R and its height $2h$. The radius of the rod is b. At steady state the appropriate one-velocity neutron-flux equations are

$$\begin{aligned} -D_C\nabla^2\phi_C(\mathbf{r}) + \Sigma_a^{(C)}\phi_C(\mathbf{r}) &= \nu\Sigma_f p_{\text{th}} g_{\text{th}}\phi_C(\mathbf{r}) \\ -D_B\nabla^2\phi_B(\mathbf{r}) + \Sigma_a^{(B)}\phi_B(\mathbf{r}) &= 0 \end{aligned} \tag{11.1}$$

where C refers to the reactor and B to the control rod and the remaining symbols have the usual definitions. The neutron fluxes given by the above equations are to satisfy the following boundary conditions:

(1) $\phi_C(\tilde{R},z) = \phi_C(\rho,\pm\tilde{h}) = \phi_B(\rho,\pm\tilde{h}) = 0$
(2) ϕ_B finite for all $\mathbf{r}$
(3) The flux and net current are continuous at the rod-reactor interface $\rho = b$ (11.2)

[1] H. L. Garabedian, "Boundary Conditions in Multi-region Reactor Theory," Westinghouse Atomic Power Division, Pittsburgh, Pa. WAPD-TN-516, July, 1956.

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The solution to these equations may be obtained by the method of separation of variables. It is easy to show that, with the application of the first two conditions (11.2), the results may be written

$$\begin{aligned} \phi_C(\rho,z) &= A[Y_0(\lambda_1\tilde{R})\, J_0(\lambda_1\rho) - J_0(\lambda_1\tilde{R})\, Y_0(\lambda_1\rho)] \cos\frac{\pi z}{2\tilde{h}} \\ \phi_B(\rho,z) &= CI_0(\lambda_2\rho) \cos\frac{\pi z}{2\tilde{h}} \end{aligned} \tag{11.3}$$

where

$$\begin{aligned} B^2 &\equiv \frac{k_\infty g_{\text{th}} - 1}{L^2} & \kappa_B^2 &\equiv \frac{\Sigma_a^{(B)}}{D_B} \\ \lambda_1^2 &= B^2 - \left(\frac{\pi}{2\tilde{h}}\right)^2 & \lambda_2^2 &= \kappa_B^2 + \left(\frac{\pi}{2\tilde{h}}\right)^2 \end{aligned} \tag{11.4}$$

The criticality condition for this system is obtained by the application of the interface condition (3), in (11.2), above. This is readily found to be

$$\frac{D_C\lambda_1[Y_0(\lambda_1\tilde{R})\, J_1(\lambda_1 b) - J_0(\lambda_1\tilde{R})\, Y_1(\lambda_1 b)]}{Y_0(\lambda_1\tilde{R})\, J_0(\lambda_1 b) - J_0(\lambda_1\tilde{R})\, Y_0(\lambda_1 b)} = \frac{D_B\lambda_2 I_1(\lambda_2 b)}{I_0(\lambda_2 b)}$$

The usual procedure in carrying out control-rod calculations is to determine the rod radius required to overcome a given amount of excess multiplication. Thus the reactor configuration and nuclear parameters, including the fuel loading, are generally known, as are also those of the rod material (in the case of the gray rod). This means that all quantities in the above equation are determined, except the rod radius b. This equation is used, then, to compute b; since the relation is transcendental, a trial-and-error procedure is required and the usual practice is to select various values of b and plot the b-dependent portion of the function against the R-dependent portion. For this purpose the relation given above is more conveniently written in the form

$$\frac{Y_0(\lambda_1\tilde{R})}{J_0(\lambda_1\tilde{R})} = \frac{Y_1(\lambda_1 b) - D_B\lambda_2 I_1(\lambda_2 b)\, Y_0(\lambda_1 b)/D_C\lambda_1 I_0(\lambda_2 b)}{J_1(\lambda_1 b) - D_B\lambda_2 I_1(\lambda_2 b)\, J_0(\lambda_1 b)/D_C\lambda_1 I_0(\lambda_2 b)} \tag{11.5}\star$$

The effectiveness of the control rod determined by this equation is measured in terms of the excess multiplication of the reactor prior to the insertion of the rod. If we assume that the omission of the rod leaves no "hole" (e.g., a fluid core reactor), then the multiplication constant of the reactor is obtained from Eq. (6.81). As a rule it is useful to describe the rod effectiveness by the per cent reactivity it can suppress, that is, by $\omega \equiv \delta k/k$.

b. Cylindrical Reactor with Black Control Rod. In the present example we consider the reactor configuration of the preceding analysis, but now containing a black control rod. The "black" rod differs from the

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gray rod in that it can support no neutron flux at all. Thus the black rod serves effectively as an "absolute" sink of neutrons. From this standpoint the black rod behaves just like the space surrounding the outer surface of a reactor; any neutrons which enter this region are permanently lost to the chain reaction. The analogy is in fact precise, and one is easily lead to the use of the extrapolation-distance concept for describing the characteristics of the rod.

Figure 11.2 shows the local flux distribution in the vicinity of a centrally located black control rod. The dimension b is the actual radius of the rod, and ϵ is the extrapolation distance *into* the rod (cf. Fig. 5.10). The dimension b_* defined by the relation

$$b_* \equiv b - \epsilon \tag{11.6}$$

is the *effective* radius of the rod. As in the case of the extrapolation distance for the outer surface of the reactor, ϵ is computed in terms of the nuclear properties of the reactor materials (and not of the rod). However, inasmuch as b is frequently of the same order of magnitude as the transport mean free path in the reactor $\lambda_{tr}^{(c)}$, or very much less, the usual relation (5.71) is not generally applicable. In fact, in the limit for very small values of b, a proper estimate for ϵ is $4\lambda_{tr}^{(c)}/3$. For intermediate values[1] of the rod radius, ϵ may be determined from Fig. 11.3.

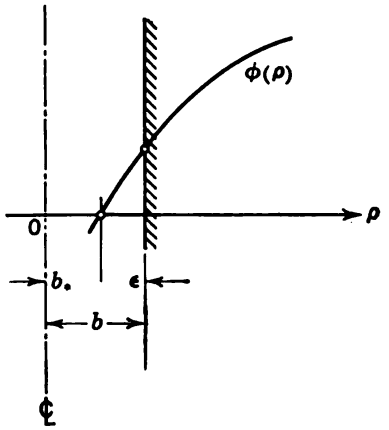

FIG. 11.2 Extrapolation distance for black control rod.

The neutron flux in the vicinity of the rod may be estimated with the aid of the extrapolation length from the relation [cf. Eq. (5.65)]

$$\phi(b_*) \simeq \phi(b) - \epsilon\phi'(b) \tag{11.7}$$

If we specify

$$\phi(b_*) \equiv 0 \tag{11.8}\star$$

then [cf. Eq. (5.66)]

$$\frac{\phi(b)}{\phi'(b)} = \epsilon \tag{11.9}\star$$

In most of the subsequent analyses we will use the extrapolation-length condition (11.8). For the present problem it is more suitable to use the relation (11.9); the latter is the form in which it was first used by Scalettar and Nordheim.[2]

[1] B. Davison and S. Kushneriuk, "Linear Extrapolation Length for a Black Sphere and a Black Cylinder," MT-214, Mar. 30, 1946.

[2] R. Scalettar and L. W. Nordheim, "Theory of Pile Control Rods," MDDC-42, June 17, 1946.

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The appropriate differential equation for this system is given by the first of the set (11.1), and the boundary conditions by (1) in (11.2) and by (11.9). The general solution for the flux ϕ_C which satisfies condition

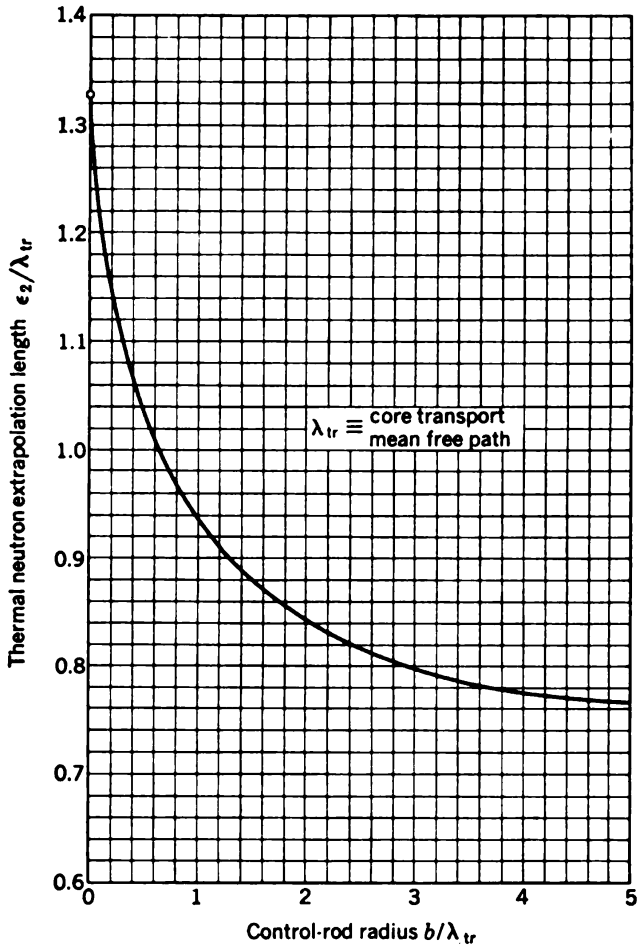

FIG. 11.3 Extrapolation length for black cylindrical control rods.

(1) is given in (11.3). The application of Eq. (11.9) yields the criticality condition.

$$\frac{Y_0(\lambda_1 \tilde{R})}{J_0(\lambda_1 \tilde{R})} = \frac{Y_0(\lambda_1 b) + \epsilon\lambda_1 Y_1(\lambda_1 b)}{J_0(\lambda_1 b) + \epsilon\lambda_1 J_1(\lambda_1 b)} \qquad (11.10)\star$$

This equation may be solved for the required rod size b for a given set of reactor parameters by the procedure described previously.

c. Cylindrical Reactor with Central Hole. In some practical situations (solid-core reactors) the withdrawal of a control rod leaves a hole in the reactor. The present example demonstrates a procedure for computing the effectiveness of a control rod in such systems; however, in order to simplify the calculation we assume that the reactor is of infinite length.

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The flux distribution for this system is given by the differential equation (11.1). In addition to the extrapolation boundary condition for the outer surface, (1) in (11.2), we require for the rod-reactor interface[1]

$$\left[\frac{\partial}{\partial \rho}\,\phi_C(\rho,z)\right]_{\rho=b} = 0 \tag{11.11}$$

With the application of the outer boundary condition, the solution is given by the form (11.3), which we now write in terms of λ_3; thus

$$\phi_C(\rho,z) = A[Y_0(\lambda_3\tilde{R})\,J_0(\lambda_3\rho) - J_0(\lambda_3\tilde{R})\,Y_0(\lambda_3\rho)]\cos\frac{\pi z}{2\tilde{h}} \tag{11.12}$$

with

$$\lambda_3^2 \equiv B^2 - \left(\frac{\pi}{2\tilde{h}}\right)^2 \tag{11.13}$$

and B^2 as given by Eq. (11.4). The use of condition (11.11) yields the criticality relation

$$\frac{Y_0(\lambda_3\tilde{R})}{J_0(\lambda_3\tilde{R})} = \frac{Y_1(\lambda_3 b)}{J_1(\lambda_3 b)} \tag{11.14}$$

In order to compute the effectiveness of a gray rod we apply Eq. (11.14) to determine first the critical fuel concentration for the reactor with hole, given the core size and rod radius. [Note that the solution of (11.14) yields λ_3, which may then be used in (11.13) to compute the concentration.] The next step is to use this concentration, along with the other nuclear parameters and the values of $\tilde{R}$ and b, in Eq. (11.5) to determine a value for λ_1. The final step is to compare the values of ν required to keep each system at steady state. Since the reactor without rod is just critical, we have

$$1 = \frac{\nu\Sigma_f p_{\text{th}} g_{\text{th}}}{\Sigma_a^{(C)} + D_C[\lambda_3^2 + (\pi/2\tilde{h})^2]} \tag{11.15}$$

The reactor with rod inserted, and the same fuel concentration, is then subcritical, and for steady-state operation we require some $\nu_c > \nu$; thus,

$$1 = \frac{\nu_c\Sigma_f p_{\text{th}} g_{\text{th}}}{\Sigma_a^{(C)} + D_C[\lambda_1^2 + (\pi/2\tilde{h})^2]} \tag{11.16}$$

The reactivity, or *worth*, of the rod is then given by the relation

$$\omega = \frac{\delta k}{k} = \frac{\nu}{\nu_c} - 1 = \frac{\lambda_3^2 - \lambda_1^2}{\kappa_C^2 + \lambda_1^2 + (\pi/2\tilde{h})^2} \tag{11.17}$$

[1] The appropriate interior boundary condition for a finite cylindrical reactor with central hole is discussed by E. L. Zimmerman, "Boundary Values for the Inner Radius of a Cylindrical Annular Reactor," Oak Ridge National Laboratory, ORNL-2484, June 5, 1958.

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It is of interest to mention that one of the important effects of a control rod is the increased leakage it produces at the outer surface of the reactor and that this, rather than the extra neutrons absorbed, may be the predominant factor in determining the worth of the rod.

A similar computation to that given above may be carried out to determine the reactivity of a black control rod which leaves a hole in the reactor when withdrawn. For this case Eq. (11.10) is used in place of Eq. (11.5).

11.2 Central Rod in Two-group Model

a. General Solution. The one-velocity model is generally inadequate for obtaining reliable estimates of control-rod effectiveness. A more accurate method which is customarily used in design practice is based on the standard two-group model. The application of this model to the calculation of the effectiveness of a single centrally located round rod in a bare cylindrical reactor is a straightforward extension of the general results obtained in Sec. 8.4 for the cylindrical reactor with side reflector (refer to Table 8.3). The only modification required for this application is a restatement of the rod-reactor interface condition in terms of the particular properties of control-rod materials.

The primary purpose of the present section is to exhibit the various interface conditions that have been proposed for use in control-rod calculations. It is of interest to mention that the accuracy of the entire computation is very much dependent on the proper choice of the interface condition for the fast flux. In the analyses which follow we consider a number of such choices and show how the appropriate criticality relation is derived for each case for both a solid and a hollow control rod. This section is concluded with a numerical example which compares the effectiveness of these two types of rods when fully inserted in the center of a bare cylindrical reactor.

We begin the analysis with a summary of the general solutions derived for the two-group model. The appropriate differential equations for the fast flux (ϕ_1) and the thermal flux (ϕ_2) in the multiplying region of the reactor are given by (8.147). In the present problem it is convenient to write the solutions for these functions in the form [cf. Eqs. (8.164)]

$$\begin{aligned} \phi_1(\mathbf{r}) &= Z(\mathbf{r}) + W(\mathbf{r}) \\ \phi_2(\mathbf{r}) &= a_1 Z(\mathbf{r}) + a_2 W(\mathbf{r}) \end{aligned} \tag{11.18}$$

These must satisfy the boundary condition on the outer surface of the reactor:

$$\phi_1(\tilde{\mathbf{R}}) = \phi_2(\tilde{\mathbf{R}}) = 0 \tag{11.19}$$

The functions Z and W are solutions to the differential equations

$$\nabla^2 Z(\mathbf{r}) + \mu^2 Z(\mathbf{r}) = 0 \qquad \nabla^2 W(\mathbf{r}) - \lambda^2 W(\mathbf{r}) = 0 \tag{8.174}$$

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and the parameters μ^2 and λ^2 are the two roots given by Eq. (8.171). The coupling constants a_1 and a_2 are defined by Eqs. (8.175). For the reactor with a single central rod, the flux distributions are axially symmetric and we may define

$$Z(\mathbf{r}) \equiv H(z)\, R(\rho) \qquad W(\mathbf{r}) \equiv H(z)\, S(\rho) \tag{11.20}$$

If these relations are used in Eqs. (8.174), it is readily shown that for the bare cylindrical reactor the indicated functions are given by the expressions (refer to Fig. 11.1)

$$H(z) = \cos\left(\frac{\pi z}{2\tilde{h}}\right) \tag{11.21}$$

and

$$\begin{aligned} R(\rho) &= \mathfrak{A} J_0(\gamma\rho) + \mathfrak{B} Y_0(\gamma\rho) \\ S(\rho) &= \mathfrak{M} I_0(\beta\rho) + \mathfrak{N} K_0(\beta\rho) \end{aligned} \tag{11.22}$$

where

$$\gamma^2 \equiv \mu^2 - \left(\frac{\pi}{2\tilde{h}}\right)^2 \qquad \beta^2 \equiv \lambda^2 + \left(\frac{\pi}{2\tilde{h}}\right)^2 \tag{11.23}$$

The neutron-flux relations (11.18) may then be written

$$\begin{aligned} \phi_1(\rho,z) &= H(z)\{\mathfrak{A} J_0(\gamma\rho) + \mathfrak{B} Y_0(\gamma\rho) + \mathfrak{M} I_0(\beta\rho) + \mathfrak{N} K_0(\beta\rho)\} \\ \phi_2(\rho,z) &= H(z)\{a_1[\mathfrak{A} J_0(\gamma\rho) + \mathfrak{B} Y_0(\gamma\rho)] + a_2[\mathfrak{M} I_0(\beta\rho) + \mathfrak{N} K_0(\beta\rho)]\} \end{aligned} \tag{11.24}$$

The coefficients $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{M}$, and $\mathfrak{N}$ are evaluated by the application of the rod-reactor interface condition.

b. Solid-absorber Control Rod. For the first example we consider the case of a fully inserted central control rod which is black to thermal neutrons only; thus the thermal-neutron properties of the rod material are similar to those specified for the rod in the one-velocity problem of Sec. 11.1b. For the fast-neutron properties of this rod we allow two different possibilities: (1) the fast flux remains finite everywhere within the rod, and (2) the net current of fast neutrons is zero at the surface of the rod.

Consider first case 1. Specifically, we will impose the conditions:

$$\begin{aligned} &(1)\ \frac{\mathfrak{B}}{\mathfrak{N}} = \frac{\pi}{2} \\ &(2)\ \phi_2(b_*,z) = 0 \qquad \text{with } b_* \equiv b - \epsilon_2 \end{aligned} \tag{11.25}$$

The first of these follows from the fact that the expression for the fast flux ϕ_1 contains two functions Y_0 and K_0, which are singular at $\rho = 0$. Since we have that

$$\lim_{x\to 0} \frac{Y_0(x)}{K_0(x)} = -\frac{2}{\pi} \tag{11.26}$$

it is clear that the choice given in (1) of (11.25) provides that the fast

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flux remains finite for all values of ρ inside the rod. The second condition (2) requires that the thermal flux vanish at the effective radius of the rod b_*, which is determined from the actual radius b by the application of an extrapolation length correction based on the thermal-group properties of the core [cf. Eq. (11.6)].

The combination of control-rod boundary conditions (11.25) is based on the notion of representing a control rod by a line singularity. It should be recognized that condition (1) in (11.25) serves essentially to keep ϕ_1 finite, and this is achieved by relinquishing control over other physical features of the system. As will be seen shortly, this condition leads to a large net current of fast neutrons into the rod, which is inconsistent with the premise that the rod is a strong absorber of thermal neutrons only. This discrepancy can result in serious overestimates of rod effectiveness, and for this reason the method is not recommended for general usage. It has been included here primarily for reasons of historical interest, being one of the earliest formulations of the control-rod problem.

The application of condition (1) to the general solutions (11.24) yields

$$\begin{aligned}\phi_1(\rho,z) &= H(z)\left\{\mathfrak{A}J_0(\gamma\rho) + \frac{\pi}{2}\mathfrak{N}Y_0(\gamma\rho) + \mathfrak{M}I_0(\beta\rho) + \mathfrak{N}K_0(\beta\rho)\right\}\\ \phi_2(\rho,z) &= H(z)\left\{a_1\left[\mathfrak{A}J_0(\gamma\rho) + \frac{\pi}{2}\mathfrak{N}Y_0(\gamma\rho)\right] + a_2[\mathfrak{M}I_0(\beta\rho) + \mathfrak{N}K_0(\beta\rho)]\right\}\end{aligned} \tag{11.27}$$

A relation is obtained between the coefficients $\mathfrak{M}$ and $\mathfrak{N}$ and between $\mathfrak{A}$ and $\mathfrak{N}$ if we use next the boundary condition for the outer extrapolated surface of the reactor (11.19). These are

$$\mathfrak{M} = -\mathfrak{N}\frac{K_0(\beta\tilde{R})}{I_0(\beta\tilde{R})} \qquad \text{and} \qquad \mathfrak{A} = -\frac{\pi}{2}\mathfrak{N}\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} \tag{11.28}$$

The substitution of these results into the expression for ϕ_2 yields

$$\begin{aligned}\phi_2(\rho,z) = \mathfrak{N}H(z)\left\{\frac{\pi}{2}a_1\left[-\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})}J_0(\gamma\rho) + Y_0(\gamma\rho)\right]\right.\\ \left. + a_2\left[-\frac{K_0(\beta\tilde{R})}{I_0(\beta\tilde{R})}I_0(\beta\rho) + K_0(\beta\rho)\right]\right\}\end{aligned} \tag{11.29}$$

The application of condition (2) from (11.25) to this equation gives the criticality condition for the reactor. This is found to be

$$\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} = \frac{1}{J_0(\gamma b_*)}\left\{Y_0(\gamma b_*) + \frac{2a_2}{\pi a_1}\left[K_0(\beta b_*) - \frac{K_0(\beta\tilde{R})}{I_0(\beta\tilde{R})}I_0(\beta b_*)\right]\right\} \tag{11.30}$$

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In general $\beta\tilde{R} > 1$, and $K_0(\beta\tilde{R})/I_0(\beta\tilde{R}) \ll 1$; then, a good approximation to the above result is

$$\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} \simeq \frac{1}{J_0(\gamma b_*)}\left[Y_0(\gamma b_*) + \frac{2a_2}{\pi a_1} K_0(\beta b_*)\right] \qquad (11.31)\star$$

It is of some interest to note that this result may also be written in terms of the extrapolation distance ϵ_2 in place of the effective rod radius b_*. For this purpose we require the Taylor series expansions about $\rho = b$ for the various Bessel functions involving b_*. In general, we can apply the functional form (11.7); then, for example,

$$J_0(\gamma b_*) \simeq J_0(\gamma b) + \epsilon_2\gamma J_1(\gamma b)$$

and so on. The use of this form in (11.31) yields the alternative expression

$$\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} \simeq \frac{Y_0(\gamma b) + \epsilon_2\gamma Y_1(\gamma b) + \dfrac{2a_2}{\pi a_1}[K_0(\beta b) + \epsilon_2\beta K_1(\beta b)]}{J_0(\gamma b) + \epsilon_2\gamma J_1(\gamma b)} \qquad (11.32)$$

Some authors prefer this form to (11.31). From an analytical standpoint this implies that the thermal-flux condition of the type (11.9) was used instead of (11.8).

Criticality analyses may be performed with the aid of Eq. (11.31) and the procedure outlined for the one-velocity model of the preceding section. As before, the criticality relation may be used to compute the effective rod size b_* required to maintain a reactor of given size and fuel concentration at steady state. The rod effectiveness may then be determined by computing the multiplication constant of the reactor for the same fuel loading, but without the rod (and without a residual "hole").

Figure 11.4 shows the local flux distribution in the vicinity of a central control rod as determined from the condition that the fast flux be finite inside the rod. Note that, although $\phi_1(0,z)$ is finite as required, it dips to large negative values inside the rod and produces a large gradient at the rod-reactor interface.

For case 2 we take the following pair of boundary conditions:

$$(3)\ \left[\frac{\partial}{\partial\rho}\phi_1(\rho,z)\right]_{\rho=b} = 0 \qquad (4)\ \phi_2(b_*,z) = 0 \qquad (11.33)$$

The coefficients in the general solutions (11.24) are evaluated by first applying the boundary condition for the outer surface of the reactor. From the condition for the surface $\rho = \tilde{R}$ we obtain the relations

$$\frac{\mathfrak{A}}{\mathfrak{B}} = -\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} \quad \text{and} \quad \frac{\mathfrak{M}}{\mathfrak{N}} = -\frac{K_0(\beta\tilde{R})}{I_0(\beta\tilde{R})} \qquad (11.34)$$

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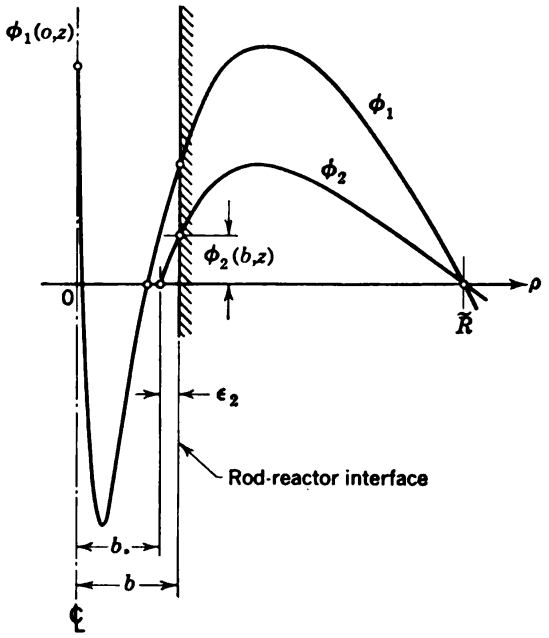

FIG. 11.4 Flux distribution around a central rod using the condition that fast flux is finite everywhere in the rod.

If these results are used in Eqs. (11.24) and conditions (11.33) applied, we obtain two equations relating, say, $\mathfrak{a}$ and $\mathfrak{M}$. From condition (4) we have, for example,

$$\mathfrak{a}\left[J_0(\gamma b_*) - \frac{J_0(\gamma\tilde{R})}{Y_0(\gamma\tilde{R})}\,Y_0(\gamma b_*)\right] = -\frac{a_2}{a_1}\left[I_0(\beta b_*) - \frac{I_0(\beta\tilde{R})}{K_0(\beta\tilde{R})}\,K_0(\beta b_*)\right]\mathfrak{M} \tag{11.35}$$

If we use also condition (3) and divide the resulting expression by (11.35), the following criticality equation is obtained:

$$\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} = \frac{\frac{\beta}{\gamma}\,Y_0(\gamma b_*)c_3(b) + (a_2/a_1)Y_1(\gamma b)c_1(b_*)}{\frac{\beta}{\gamma}\,J_0(\gamma b_*)c_3(b) + (a_2/a_1)J_1(\gamma b)c_1(b_*)} \tag{11.36}$$

where

$$\begin{aligned} c_1(x) &\equiv I_0(\beta x) - \frac{I_0(\beta\tilde{R})}{K_0(\beta\tilde{R})}\,K_0(\beta x) \simeq -\frac{I_0(\beta\tilde{R})K_0(\beta x)}{K_0(\beta\tilde{R})} \\ c_3(x) &\equiv I_1(\beta x) + \frac{I_0(\beta\tilde{R})}{K_0(\beta\tilde{R})}\,K_1(\beta x) \simeq +\frac{I_0(\beta\tilde{R})K_1(\beta x)}{K_0(\beta\tilde{R})} \end{aligned} \tag{11.37}$$

The indicated approximations apply for $\tilde{R} \gg x$. For most practical

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situations $\tilde{R} \gg b$, and a good approximation to (11.36) is

$$\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} \simeq \frac{Y_0(\gamma b_*)\left[\frac{\beta}{\gamma}\left(\frac{a_1}{a_2}\right)\frac{K_1(\beta b)}{K_0(\beta b_*)}\right] - Y_1(\gamma b)}{J_0(\gamma b_*)\left[\frac{\beta}{\gamma}\left(\frac{a_1}{a_2}\right)\frac{K_1(\beta b)}{K_0(\beta b_*)}\right] - J_1(\gamma b)} \qquad (11.38)\star$$

The flux distribution for this case is shown in Fig. 11.5.

c. Hollow Control Rod. In most practical situations control rods are not fabricated from solid sections of high thermal cross section materials. Since the mean free paths in such materials are very short, it is clear that very few thermal neutrons can ever penetrate far inside a rod of several centimeters width. Nearly all the neutron captures occur close to the surface of the rod, and the interior serves no real purpose. Thus the surface area, or cross-sectional perimeter, of the rod is a rough measure of its effectiveness as a neutron sink. In practice one is therefore led to the use of cross-sectional configurations having large perimeters, such as blades, shells, and tubes.

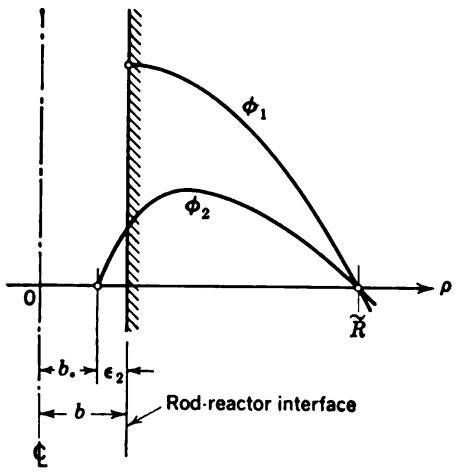

FIG. 11.5 Flux distribution around a central rod using the condition of zero net current of fast neutrons at the rod surface.

In the treatment which follows we analyze the effectiveness of control rods in the form of hollow tubes of absorber (thermally black) material. Although the present study is carried out in detail for the round tube, the results may be applied in first approximation to tubes of any cross section by simply matching perimeters.

Two cases of considerable practical interest are considered: (1) the hollow "control tube" and (2) the control tube containing moderator material. The first case is essentially the same as an absorber tube filled with high-atomic-mass material. In both situations the tube stops the passage of all thermal neutrons. Fast neutrons, however, easily pass through the tube (since by assumption the tube is black to thermal neutrons only) and, moreover, are little affected by the interior of the rod, whether it be empty or filled with heavy nuclei. In either case the fast neutrons emerge with much the same angular distribution as that with which they entered and without any significant degradation in energy. From an analytical viewpoint both problems are equivalent to the solid-control-rod problem with zero fast-neutron net current at the surface. Thus the results (11.36) and (11.38) apply directly to case 1.

In case 2 the moderator material within the control element serves

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effectively as a sink of fast neutrons. Since the tube itself is transparent to fast neutrons, these have access to the interior and some are removed from the fast group by being scattered to lower energies by the moderator material. Configurations of this type are frequently encountered in practice. A good example is the control-rod design appropriate to the nuclear package power reactor of Sec. 8.5. The core of this reactor consists of a lattice of MTR-type fuel elements immersed in water. The control "rods" are simply hollow rectangular tubes of absorber material which fit into regular fuel lattice points (see Figs. 5.27 and 5.28). When the rods are inserted, the interior regions fill with water, and the resulting fast-neutron moderation within the rod yields a substantial increase in the worth of the rod by virtue of its properties as a fast-neutron sink. (Note that thermal neutrons produced by this moderation cannot escape from the rod and thereby contribute to the chain reaction in the core.)

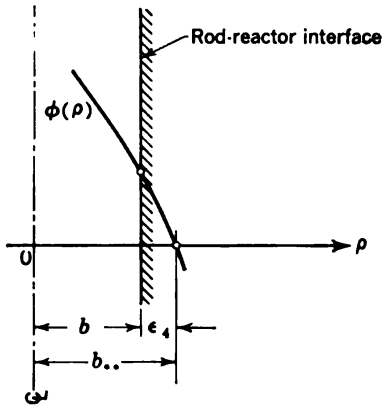

FIG. 11.6 Extrapolation distance for the interior of a hollow control rod.

A first estimate of the effectiveness of a moderator-filled control element can be obtained from the application of the diffusion theory to the interior region of the rod. Thus in the present calculation we treat the rod as though it were an internal reflector, from the viewpoint of the fast-neutron group. From the viewpoint of the thermal neutrons, the absorber tube acts as a curtain for neutrons passing both in and out.

The appropriate two-group equations for the flux distribution within the rod are given by (8.149). In the present application ϕ_1 and ϕ_2 denote the fast and thermal fluxes, respectively, in the reactor core, and ϕ_3 and ϕ_4, those in the control rod. The boundary conditions to be satisfied by the fast flux ϕ_3 are the usual requirements of continuity and finiteness:

$$\begin{aligned} &(1)\ \phi_3(b,z) = \phi_1(b,z) \\ &(2)\ D_3\left[\frac{\partial}{\partial\rho}\phi_3(\rho,z)\right]_{\rho=b} = D_1\left[\frac{\partial}{\partial\rho}\phi_1(\rho,z)\right]_{\rho=b} \\ &(3)\ \phi_3(\rho,z) \text{ finite within rod} \end{aligned} \tag{11.39}$$

and for the thermal flux ϕ_4:

$$\phi_4(b_{**},z) = 0 \qquad \text{or} \qquad \phi_4(b,z) = -\epsilon_4\left[\frac{\partial}{\partial\rho}\phi_4(\rho,z)\right]_{\rho=b} \tag{11.40}$$

where ϵ_4 is the extrapolation distance based on the transport mean free path of the material contained in the rod and (refer to Figs. 11.3 and 11.6)

$$b_{**} \equiv b + \epsilon_4 \tag{11.41}$$

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As before, we require that ϕ_2 satisfy Eq. (11.33), condition (4). The solution for the fast flux can be written in the form

$$\phi_3(\rho,z) = H(z)\, T(\rho) \tag{11.42}$$

where $H(z)$ is given by Eq. (11.21). The function $T(\rho)$, which satisfies condition (3) in (11.39), is

$$T(\rho) = \mathfrak{C} I_0(\kappa_3\rho) \tag{11.43}$$

with

$$\kappa_3^2 \equiv \frac{\Sigma_R^{(3)}}{D_3} + \left(\frac{\pi}{2\tilde{h}}\right)^2 \tag{11.44}$$

Note that the first term in this relation may be approximated by $1/\tau$, where τ is the age to thermal for the central region of the rod.

These results may be used to determine the flux distribution within the core and the criticality condition for the reactor. The general solutions for the fluxes in the core are given by Eqs. (11.24). If we apply boundary condition (11.19), i.e., the relations (11.34), the solutions may be written

$$\begin{aligned}
\phi_1(\rho,z) &= H(z)\left\{\mathfrak{A}\left[J_0(\gamma\rho) - \frac{J_0(\gamma\tilde{R})}{Y_0(\gamma\tilde{R})}\, Y_0(\gamma\rho)\right]\right. \\
&\qquad \left. + \mathfrak{M}\left[I_0(\beta\rho) - \frac{I_0(\beta\tilde{R})}{K_0(\beta\tilde{R})}\, K_0(\beta\rho)\right]\right\} \\
\phi_2(\rho,z) &= H(z)\left\{a_1\mathfrak{A}\left[J_0(\gamma\rho) - \frac{J_0(\gamma\tilde{R})}{Y_0(\gamma\tilde{R})}\, Y_0(\gamma\rho)\right]\right. \\
&\qquad \left. + a_2\mathfrak{M}\left[I_0(\beta\rho) - \frac{I_0(\beta\tilde{R})}{K_0(\beta\tilde{R})}\, K_0(\beta\rho)\right]\right\}
\end{aligned} \tag{11.45}$$

The expression for the fast flux must satisfy the interface conditions (1) and (2) in (11.39), and the thermal flux, condition (4) of (11.33). The relations for the fast flux we write in the form

$$\frac{\phi_1(b,z)}{\left[\dfrac{\partial}{\partial\rho}\,\phi_1(\rho,z)\right]_{\rho=b}} = \epsilon_1 = \frac{D_1 I_0(\kappa_3 b)}{\kappa_3 D_3 I_1(\kappa_3 b)} \tag{11.46}$$

where ϵ_1 denotes the extrapolation distance for the fast group in the core. Note that this quantity may be computed in terms of the fast-group properties of the interior of the rod by virtue of the continuity relations (11.39).

Equation (11.46) together with (4) from (11.33) constitute the criticality condition for this system. From the latter, we obtain Eq. (11.35)

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relating $\mathfrak{a}$ and $\mathfrak{M}$. A second relation is obtained from (11.46):

$$\mathfrak{a}\left[J_0(\gamma b) - \frac{J_0(\gamma\tilde{R})}{Y_0(\gamma\tilde{R})}\,Y_0(\gamma b)\right] + \mathfrak{M}\left[I_0(\beta b) - \frac{I_0(\beta\tilde{R})}{K_0(\beta\tilde{R})}\,K_0(\beta b)\right]$$
$$= \epsilon_1\left\{\gamma\mathfrak{a}\left[-J_1(\gamma b) + \frac{J_0(\gamma\tilde{R})}{Y_0(\gamma\tilde{R})}\,Y_1(\gamma b)\right]\right.$$
$$\left. + \beta\mathfrak{M}\left[I_1(\beta b) + \frac{I_0(\beta\tilde{R})}{K_0(\beta\tilde{R})}\,K_1(\beta b)\right]\right\} \tag{11.47}$$

Equation (11.35) may be used in this expression to eliminate $\mathfrak{a}$ and $\mathfrak{M}$. The resulting expression is the criticality condition, which we write in the form

$$\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} = \frac{(a_2/a_1)c_1(b_*)c_4(b) + Y_0(\gamma b_*)[\beta\epsilon_1 c_3(b) - c_1(b)]}{(a_2/a_1)c_1(b_*)c_2(b) + J_0(\gamma b_*)[\beta\epsilon_1 c_3(b) - c_1(b)]} \tag{11.48}$$

with c_1 and c_3 defined by (11.37) and

$$\begin{aligned} c_2(b) &\equiv J_0(\gamma b) + \gamma\epsilon_1 J_1(\gamma b) \\ c_4(b) &\equiv Y_0(\gamma b) + \gamma\epsilon_1 Y_1(\gamma b) \end{aligned} \tag{11.49}$$

When $\tilde{R} \gg b$, a good approximation to (11.48) is given by

$$\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} \simeq \frac{Y_0(\gamma b_*)\,M(b) - c_4(b)}{J_0(\gamma b_*)\,M(b) - c_2(b)} \tag{11.50$\star$}$$

wherein we have used the approximations from (11.37) and defined

$$M(b) \equiv \frac{a_1}{a_2}\left[\frac{\beta\epsilon_1 K_1(\beta b) + K_0(\beta b)}{K_0(\beta b_*)}\right] \tag{11.51}$$

The result (11.50) has also been developed[1] in terms of the extrapolation distance ϵ_2, where

$$\phi_2(b,z) = \epsilon_2\left[\frac{\partial}{\partial\rho}\,\phi_2(\rho,z)\right]_{\rho\to b} \tag{11.52}$$

It is of interest to determine the limiting form of the criticality condition (11.50) for very small rods ($b \to 0$). The application of the appropriate forms for the various Bessel functions in the case of vanishing argument gives the approximations $M(b) \simeq \beta a_1\epsilon_1 K_1(\beta b)/a_2 K_0(\beta b_*)$ and $c_4 \simeq \gamma\epsilon_1 Y_1(\gamma b)$. Also since c_2 approaches a finite limit as $b \to 0$, we may neglect it in relation to the M term in the denominator. Then, using also the limit

$$\lim_{b\to 0}\frac{\beta K_1(\beta b)}{\gamma Y_1(\gamma b)} = -\frac{\pi}{2} \tag{11.53}$$

[1] R. L. Murray and J. W. Niestlie, *Nucleonics*, **13**(2), 18–22 (1955).

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it is easily shown that the result (11.50) reduces to the form

$$\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} \simeq Y_0(\gamma b_*) + \frac{2}{\pi}\left(\frac{a_2}{a_1}\right) K_0(\beta b_*) \qquad \text{for } b \to 0 \tag{11.54}$$

This approximation may be checked by comparing it to the limiting form of the relation (11.36) or (11.38), the criticality condition for the solid rod with zero fast current at the rod surface. The comparison is easily drawn with the aid of the approximations mentioned above for $b \to 0$. It is then seen that (11.38) reduces to (11.54), as is to be expected, since for the very small moderator-filled control tube, the effect of the moderator as a fast-neutron sink becomes negligibly small and the tube behaves like a solid rod.

The thermal-flux distribution within the moderator contained in the control rod (tube) is obtained from the differential equation

$$-D_4\nabla^2\phi_4(\mathbf{r}) + \Sigma_a^{(4)}\phi_4(\mathbf{r}) = p_3\Sigma_R^{(3)}\phi_3(\mathbf{r}) \tag{11.55}$$

where p_3 denotes the resonance-escape probability for the moderator material inside the rod. The solution for ϕ_4 we take in the form

$$\phi_4(\rho,z) = H(z)\,[\mathcal{E}U(\rho) - a_3T(\rho)] \tag{11.56}$$

with H and T as given in (11.42). The function HU denotes the solution to the homogeneous equation

$$\nabla^2\psi - \kappa_4^2\psi = 0 \qquad \text{with } \kappa_4^2 \equiv \frac{\Sigma_a^{(4)}}{D_4} \tag{11.57}$$

The function $H(z)$ was previously given by (11.21), and

$$U(\rho) = I_0(\kappa_4\rho) \qquad \text{where } \kappa_4^2 \equiv \kappa_4^2 + \left(\frac{\pi}{2\tilde{h}}\right)^2 \tag{11.58}$$

We have used here the requirement that the solution remain finite inside the rod and that Eq. (11.19) be satisfied. If we use (11.56) in (11.55), along with (11.42), we obtain

$$a_3[\nabla^2(HT) - \kappa_4^2HT] = \frac{p_3\Sigma_R^{(3)}}{D_4}HT \tag{11.59}$$

wherein we have used (11.57). But, by Eq. (8.149) for ϕ_3, we note that

$$\nabla^2(HT) - \kappa_3^2HT = 0 \qquad \text{with } \kappa_3^2 \equiv \frac{\Sigma_R^{(3)}}{D_3} \tag{11.60}$$

Thus (11.59) yields the relation [cf. Eq. (8.184)]

$$a_3 = \frac{p_3\Sigma_R^{(3)}}{D_4(\kappa_3^2 - \kappa_4^2)} \tag{11.61}$$

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The constant $\mathcal{E}$ indicated in (11.56) may be related to $\mathcal{C}$ from (11.43) by applying the boundary condition (11.40). The result is

$$\mathcal{E} = \left[\frac{a_3 I_0(\kappa_3 b_{**})}{I_0(\kappa_4 b_{**})}\right] \mathcal{C}$$

If this expression is used in (11.56), along with (11.61), we obtain for the thermal flux inside the rod

$$\phi_4(\rho,z) = \frac{p_3 \Sigma_R^{(3)} \mathcal{C}}{D_4(\kappa_3^2 - \kappa_4^2)} \left[\frac{I_0(\kappa_3 b_{**})}{I_0(\kappa_4 b_{**})} I_0(\kappa_4 \rho) - I_0(\kappa_3 \rho)\right] H(z) \quad (11.62)$$

It should be noted that the constant $\mathcal{C}$ may be related to $\mathfrak{A}$ and $\mathfrak{M}$ through (2) in (11.39).

Figure 11.7 shows the flux distribution around a tubular control rod containing moderator material.

d. Numerical Example. The results developed in the preceding analyses are used to determine the reactivity of a single centrally located control rod for both the solid absorber and moderator-filled tube configurations. For the sample reactor we use the package power plant of Sec. 8.5. The appropriate two-group constants for the core region of this reactor are listed in Table 8.5. For the present example we select the fuel volume fraction $v_F = 0.00622$, which gives $k_\infty = 1.312$. Since the actual configuration is a completely reflected square cylinder, it will be necessary to determine the size of an equivalent square unreflected cylinder with the same fuel loading. For this purpose we use Eq. (8.224) with $k \equiv 1$, and obtain $B^2 = 0.006125$ cm^{-2}, which corresponds to a square cylinder with $\tilde{R} = 36.7$ cm.

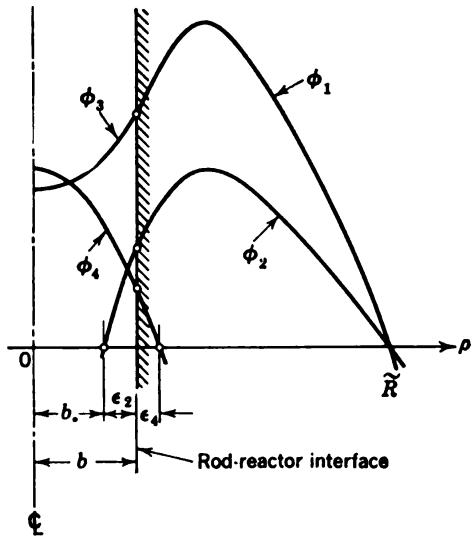

FIG. 11.7 Flux distribution around a control rod containing moderator material.

Some indication of the relative effectiveness of solid absorber and water-filled control rods is obtained by computing $\delta k/k$ as a function of rod size for the 36.7-cm-radius bare cylinder. The results of such a computation are shown in Fig. 11.8. For both types of rods, the problem was solved by selecting various fuel loadings (corresponding to various values of $\delta k/k$) and computing the required rod radius to make the system just critical. The zero fast current criticality equation (11.38) was used for the solid absorber rod, and the approximation (11.50) was used for the water-filled rod. It is of interest to note that the com-

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puted results indicate that the water-filled rod is approximately twice as effective as the solid rod in this application.[1,2] These results should be most reliable for the larger rods, since the requirements of the slowing-down diffusion model, as represented by the two-group equations, are

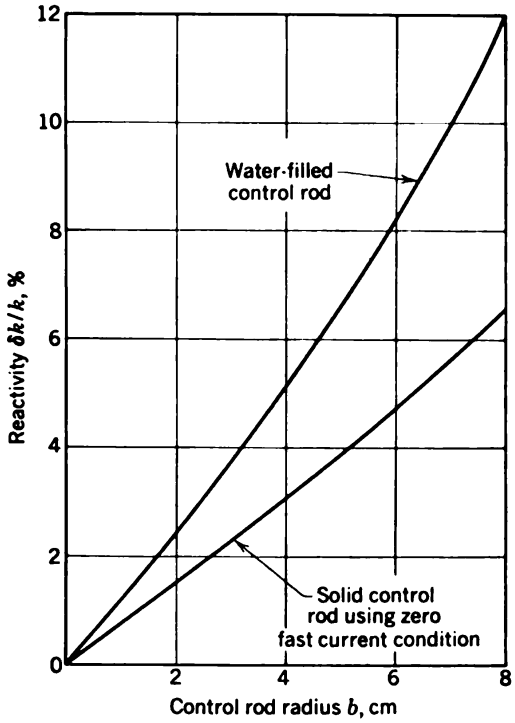

FIG. 11.8 Comparison of water-filled and solid-absorber control rods.

not well satisfied when the characteristic dimensions of the rod are in the order of one diffusion length.

11.3 Eccentric Control Rod in Two-group Model

a. General Solution for the Two-group Equations. The calculation of the effectiveness of a single eccentric control rod in a bare cylindrical reactor on the basis of the two-group model requires the general solution to Eqs. (8.147) which includes the azimuthal dependence. The formal solution for the fast and thermal fluxes ϕ_1 and ϕ_2, respectively, we write in the form (11.18). In the present application, however, we define Z and W by

$$Z(\mathbf{r}) \equiv H(z)\,G(\rho,\varphi) \qquad W(\mathbf{r}) \equiv H(z)\,F(\rho,\varphi) \tag{11.63}$$

where ρ, z, and φ denote the usual cylindrical coordinates. If these

[1] Some additional examples and experimental data are reported by Murray and Niestlie, *op. cit.*

[2] See also P. F. Gast, "Experimental Checks of Control Rod Theory," *Proc. Intern. Conf. Peaceful Uses of Atomic Energy, Geneva*, 1955, vol. 5, P/612, pp. 389–392.

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expressions are used in Eqs. (8.174), it is easily shown that the function $H(z)$ is given by Eq. (11.21), for the bare cylinder of height $2h$, and the functions G and F by

$$\frac{\partial^2 G}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial G}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 G}{\partial \varphi^2} + \gamma^2 G = 0 \tag{11.64}$$

$$\frac{\partial^2 F}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial F}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 F}{\partial \varphi^2} - \beta^2 F = 0 \tag{11.65}$$

with γ and β defined as before (11.23). We seek solutions to these equations in the form[1]

$$G(\rho,\varphi) \equiv \Phi(\varphi)\, R(\rho) \qquad F(\rho,\varphi) \equiv \Phi(\varphi)\, S(\rho) \tag{11.66}$$

Substitution into the differential equations yields the eigenfunctions

$$\Phi(\varphi) \to \Phi_n(\varphi) \equiv e^{in\varphi} \qquad n = 0, \pm 1, \pm 2, \ldots \tag{11.67}$$

and [cf. Eqs. (11.22)]

$$R(\rho) \to R_n(\rho) \equiv \mathfrak{A}_n J_n(\gamma\rho) + \mathfrak{B}_n Y_n(\gamma\rho) \tag{11.68}$$

$$S(\rho) \to S_n(\rho) \equiv \mathfrak{M}_n I_n(\beta\rho) + \mathfrak{N}_n K_n(\beta\rho) \tag{11.69}$$

The general solutions are then

$$G(\rho,\varphi) \equiv \sum_{n=-\infty}^{\infty} [\mathfrak{A}_n J_n(\gamma\rho) + \mathfrak{B}_n Y_n(\gamma\rho)] e^{in\varphi} \tag{11.70}$$

$$F(\rho,\varphi) \equiv \sum_{n=-\infty}^{\infty} [\mathfrak{M}_n I_n(\beta\rho) + \mathfrak{N}_n K_n(\beta\rho)] e^{in\varphi} \tag{11.71}$$

The substitution of these results, along with (11.63), into (11.18) yields for the fluxes in the core:

$$\phi_1(\rho,\varphi,z) = H(z) \sum_{n=-\infty}^{\infty} \{[\mathfrak{A}_n J_n(\gamma\rho) + \mathfrak{B}_n Y_n(\gamma\rho)] + [\mathfrak{M}_n I_n(\beta\rho) + \mathfrak{N}_n K_n(\beta\rho)]\} e^{in\varphi} \tag{11.72}\star$$

$$\phi_2(\rho,\varphi,z) = H(z) \sum_{n=-\infty}^{\infty} \{a_1[\mathfrak{A}_n J_n(\gamma\rho) + \mathfrak{B}_n Y_n(\gamma\rho)] + a_2[\mathfrak{M}_n I_n(\beta\rho) + \mathfrak{N}_n K_n(\beta\rho)]\} e^{in\varphi} \tag{11.73}\star$$

The coefficients $\mathfrak{A}_n$, $\mathfrak{B}_n$, $\mathfrak{M}_n$, and $\mathfrak{N}_n$ are arbitrary constants to be determined from the boundary conditions at the outer surface of the reactor and at the rod-reactor interface. The application of these requirements will also yield a criticality condition for the reactor with inserted rod, as in the analyses of the preceding section.

[1] See also H. Margenau and G. M. Murphy, "The Mathematics of Physics and Chemistry," p. 226, D. Van Nostrand Company, Inc., Princeton, N.J., 1943.

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b. Application to Solid-absorber Rod. The general solutions (11.72) and (11.73) are used here to derive the criticality condition for the eccentric solid absorber control rod. For this analysis we apply the following set of boundary conditions (refer to Fig. 11.9):

$$\begin{aligned} &(1)\ \phi_2 = 0 && \text{at } d_1 = b_* \\ &(2)\ \frac{\partial \phi_1}{\partial d_1} = 0 && \text{at } d_1 = b \end{aligned} \tag{11.74}$$

along with (11.19). As before, b_* denotes the effective rod radius [refer to Eqs. (11.25)], and d_1 is the radial distance out from the center of the rod. It is clear that the above conditions are essentially the set (11.33).

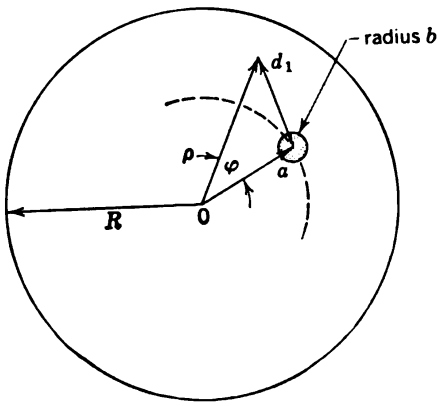

FIG. 11.9 Eccentric control rod.

The interface conditions (11.74) apply strictly only when the flux distributions around the rod are symmetrical, and this will not be true for the case of the eccentric rod or a ring of rods. However, it has been shown[1,2] that to good approximation the expressions for the neutron fluxes can be divided into two parts, one containing two functions which vary strongly near the rod but are symmetric about the rod, and the other being a function which varies slowly near the rod but is not symmetric about it. It will be assumed in the analysis which follows that the radial derivative at the rod of the slowly varying function can be neglected by comparison with the derivatives of the two strongly varying functions. Hence in evaluating the derivatives with respect to d_1, we shall consider only the strongly varying symmetric functions. Also, when applying (1) of (11.74), we evaluate the strongly varying functions at $d_1 = b_*$ and the slowly varying functions at the center of the rod.

On the basis of the assumptions and postulates set forth above, we select the following pair of functions to describe the rapid variation in the flux distribution in the vicinity of the control rod:

$$\mathcal{E}_0 Y_0(\gamma d_1) + \mathcal{F}_0 K_0(\beta d_1) \tag{11.75}$$

But these functions are symmetric about the center of the rod; therefore, they cannot satisfy the boundary condition (11.19) at $\rho = \tilde{R}$ in the

[1] F. T. Adler, "Efficiency of Control Rods as a Function of Their Position in a Cylindrical Pile in the One Group Picture," Chalk River Laboratory, MT-222, Feb. 12, 1947.

[2] J. Codd and C. A. Rennie, "Two Group Theory of Control Rods in a Thermal Reactor," Harwell, A.E.R.E., R/R 818, 1952.

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neighborhood of the rod. Thus we must add a suitable general solution for ϕ_2, in the nature of a correction, and determine the arbitrary constants in such a way as to satisfy all four conditions given by (11.19) and (11.74). We select, therefore,

$$\phi_2(\rho,\varphi,z) \equiv H(z)\left\{a_1\mathcal{E}_0 Y_0(\gamma d_1) + a_2\mathcal{F}_0 K_0(\beta d_1) + \sum_{n=-\infty}^{\infty}[a_1\mathcal{C}_n J_n(\gamma\rho) + a_2\mathcal{M}_n I_n(\beta\rho)]e^{in\varphi}\right\} \quad (11.76)$$

Note that the terms in $\mathcal{C}_n Y_n(\gamma\rho)$ and $\mathcal{M}_n K_n(\beta\rho)$ have been omitted so as to avoid the singularity at $\rho = 0$ [cf. Eq. (11.73)]. For the fast flux we take the corresponding form

$$\phi_1(\rho,\varphi,z) \equiv H(z)\left\{\mathcal{E}_0 Y_0(\gamma d_1) + \mathcal{F}_0 K_0(\beta d_1) + \sum_{n=-\infty}^{\infty}[\mathcal{C}_n J_n(\gamma\rho) + \mathcal{M}_n I_n(\beta\rho)]e^{in\varphi}\right\} \quad (11.77)$$

In working with these expressions for the fluxes it will be convenient to use series expansions for the singular functions. With the aid of the addition theorem for Bessel functions[1] we can write (refer to Fig. 11.9)

$$\begin{aligned} Y_0(\gamma d_1) &\equiv \sum_{m=-\infty}^{\infty} J_m(\gamma a)\, Y_m(\gamma\rho)e^{im\varphi} \\ K_0(\beta d_1) &\equiv \sum_{m=-\infty}^{\infty} I_m(\beta a)\, K_m(\beta\rho)e^{im\varphi} \end{aligned} \quad (11.78)$$

where $d_1^2 = a^2 + \rho^2 - 2a\rho\cos\varphi$ and φ is measured counterclockwise from the radial line to the center of the rod. If these expressions are used in (11.76) and (11.77), the result may be written

$$\phi_1(\rho,\varphi,z) = H(z)\sum_{m=-\infty}^{\infty}\{\mathcal{E}_0 J_m(\gamma a)\, Y_m(\gamma\rho) + \mathcal{F}_0 I_m(\beta a)\, K_m(\beta\rho) + \mathcal{C}_m J_m(\gamma\rho) + \mathcal{M}_m I_m(\beta\rho)\}e^{im\varphi} \quad (11.79)$$

$$\phi_2(\rho,\varphi,z) = H(z)\sum_{m=-\infty}^{\infty}\{a_1[\mathcal{E}_0 J_m(\gamma a)\, Y_m(\gamma\rho) + \mathcal{C}_m J_m(\gamma\rho)] + a_2[\mathcal{F}_0 I_m(\beta a)\, K_m(\beta\rho) + \mathcal{M}_m I_m(\beta\rho)]\}e^{im\varphi} \quad (11.80)$$

In order to satisfy the boundary conditions at the outer surface of the reactor (11.19), we must equate to zero the coefficients of $e^{im\varphi}$ for all m, in both ϕ_1 and ϕ_2 evaluated at $\rho = \tilde{R}$. The simultaneous solution of the resulting expressions yields the following set of relations between the

[1] See, for example, G. N. Watson, "Theory of Bessel Functions," chap. XI, Cambridge University Press, New York, 1944.

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coefficients:

$$\mathfrak{A}_m = -\mathcal{E}_0 J_m(\gamma a)\frac{Y_m(\gamma\tilde{R})}{J_m(\gamma\tilde{R})} \tag{11.81}$$

$$\mathfrak{M}_m = -\mathcal{F}_0 I_m(\beta a)\frac{K_m(\beta\tilde{R})}{I_m(\beta\tilde{R})} \tag{11.82}$$

which apply for all m. In applying the interface conditions (11.74), it will be convenient to denote the series indicated in (11.76) and (11.77) by the functions

$$P(\rho,\varphi) \equiv \sum_{m=-\infty}^{\infty} [\mathfrak{A}_m J_m(\gamma\rho) + \mathfrak{M}_m I_m(\beta\rho)]e^{im\varphi} \tag{11.83}$$

$$Q(\rho,\varphi) \equiv \sum_{m=-\infty}^{\infty} [a_1\mathfrak{A}_m J_m(\gamma\rho) + a_2\mathfrak{M}_m I_m(\beta\rho)]e^{im\varphi} \tag{11.84}$$

Note that both P and Q are slowly varying functions in the vicinity of the rod. In accord with the premises established above, we shall neglect the derivatives $\partial P/\partial d_1$ and $\partial Q/\partial d_1$ at $d_1 = b_*$ and evaluate P at $d_1 = 0$ ($\rho = a$, $\varphi = 0$) instead of at $d_1 = b_*$. Then, the application of condition (1) from (11.74) yields [using (11.84) to replace the series in (11.76)]

$$\mathcal{E}_0 = -\mathcal{F}_0\left(\frac{a_2}{a_1}\right)\frac{K_0(\beta b_*)}{Y_0(\gamma b_*)} - \frac{Q(a,0)}{a_1 Y_0(\gamma b_*)} \tag{11.85}$$

where

$$Q(a,0) = \sum_{m=-\infty}^{\infty} [a_1\mathfrak{A}_m J_m(\gamma a) + a_2\mathfrak{M}_m I_m(\beta a)] \tag{11.86}$$

The application of condition (2) on the fast flux gives

$$\mathcal{E}_0 = -\mathcal{F}_0\frac{\beta K_1(\beta b)}{\gamma Y_1(\gamma b)} \tag{11.87}$$

wherein we have neglected the derivative of P with respect to d_1 as planned.

The criticality condition for this system is obtained by eliminating the constants $\mathcal{E}_0$, $\mathcal{F}_0$, $\mathfrak{A}_m$, and $\mathfrak{M}_m$ between (11.81), (11.82), (11.85), and (11.87). First we use the relations (11.81) and (11.82) in (11.86). The resulting expression for $Q(a,0)$ when substituted into (11.85) yields one equation relating $\mathcal{E}_0$ and $\mathcal{F}_0$. For a second equation we use (11.87). If we divide one by the other we obtain

$$\left(\frac{a_1}{a_2}\right)\frac{\beta K_1(\beta b)}{\gamma Y_1(\gamma b)} = \frac{K_0(\beta b_*) - I_0^2(\beta a)\dfrac{K_0(\beta\tilde{R})}{I_0(\beta\tilde{R})} - 2\displaystyle\sum_{m=1}^{\infty} I_m^2(\beta a)\frac{K_m(\beta\tilde{R})}{I_m(\beta\tilde{R})}}{Y_0(\gamma b_*) - J_0^2(\gamma a)\dfrac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} - 2\displaystyle\sum_{m=1}^{\infty} J_m^2(\gamma a)\frac{Y_m(\gamma\tilde{R})}{J_m(\gamma\tilde{R})}} \tag{11.88}$$

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wherein we have used the relations

$$\begin{aligned} J_{-n} &= (-1)^n J_n & Y_{-n} &= (-1)^n Y_n \\ I_{-n} &= I_n & K_{-n} &= K_n \end{aligned} \tag{11.89}$$

For most practical situations, all the I_m terms in the numerator of (11.88) are negligibly small in comparison to $K_0(\beta b_*)$, and the series in the denominator may be neglected in comparison to the term in J_0^2. Thus a good approximation to the criticality equation for the bare cylindrical reactor with eccentric control rod is

$$\frac{Y_0(\gamma \tilde{R})}{J_0(\gamma \tilde{R})} \simeq \frac{1}{J_0^2(\gamma a)} \left[Y_0(\gamma b_*) - \left(\frac{a_2}{a_1}\right) \frac{\gamma Y_1(\gamma b)}{\beta K_1(\beta b)} K_0(\beta b_*) \right] \tag{11.90}\star$$

Note that a further reduction is possible when $b \to 0$; then, Eq. (11.53) may be applied.

11.4 Ring of Control Rods in Two-group Model

a. Analytical Model. In the present section we develop a general method for computing the effectiveness of a ring of equally spaced solid absorber rods on the basis of the two-group model. Specific criticality calculations are carried out for the case of zero fast current at the rod surface, as in (11.74). The analytical approach used in these calculations follows that of Codd and Rennie.[1] The more general situations of two rings of control rods, and one or two rings with central rod, have been analyzed by Garabedian.[2] These calculations were performed for the bare cylindrical reactor; however, Garabedian considers also the problem of the single ring of rods in a reflected reactor. In all these analyses, he applies the condition that the fast flux remains finite inside the rods [refer to Eqs. (11.25)].

The problem of determining the criticality condition and the flux distribution in the two-group model for a ring of *gray* rods has been considered by Avery.[3] The method used is a generalization of the Nordheim-Scalettar approach and may be applied when either the rods or the reactor, or both, are multiplying media.

b. Application to Solid-absorber Rod. The derivation of the criticality condition for a ring of M solid-absorber control rods in a bare cylindrical reactor follows the procedure used for the eccentric rod. The configuration to be analyzed is shown in Fig. 11.10. As before, we select a pair of functions for each rod which are singular at the center of the rod;

[1] Codd and Rennie, *op. cit.*

[2] H. L. Garabedian, "Control Rod Theory for a Cylindrical Reactor," AECD-3666, Aug. 9, 1950.

[3] R. Avery, "Two-group Diffusion Theory for a Ring of Cylindrical Rods," ANL-5729, June, 1957.

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thus, we take for rod m [cf. (11.75)]

$$\mathcal{E}_0 Y_0(\gamma d_m) + \mathcal{F}_0 K_0(\beta d_m) \tag{11.91}$$

If we choose M such pairs for the M rods, and add also a general solution

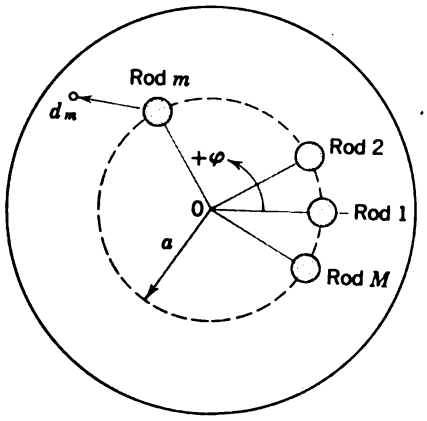

FIG. 11.10 Ring of control rods.

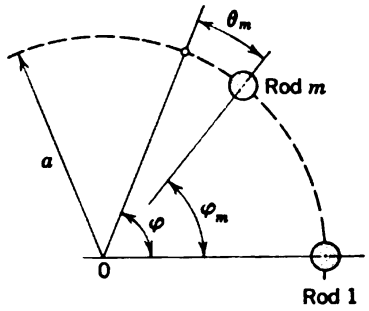

FIG. 11.11 Angular coordinates for ring of M rods.

to help meet all the boundary conditions, then the solutions for the fluxes can be constructed in the form [cf. Eqs. (11.76) and (11.77)]

$$\phi_1(\rho,\varphi,z) \equiv H(z)\left\{\mathcal{E}_0 \sum_{m=1}^{M} Y_0(\gamma d_m) + \mathcal{F}_0 \sum_{m=1}^{M} K_0(\beta d_m) + \sum_{n=-\infty}^{\infty} [\mathfrak{A}_n J_n(\gamma\rho) + \mathfrak{M}_n I_n(\beta\rho)]e^{in\varphi}\right\} \tag{11.92}$$

$$\phi_2(\rho,\varphi,z) \equiv H(z)\left\{a_1\mathcal{E}_0 \sum_{m=1}^{M} Y_0(\gamma d_m) + a_2\mathcal{F}_0 \sum_{m=1}^{M} K_0(\beta d_m) + \sum_{n=-\infty}^{\infty} [a_1\mathfrak{A}_n J_n(\gamma\rho) + a_2\mathfrak{M}_n I_n(\beta\rho)]e^{in\varphi}\right\} \tag{11.93}$$

Again, it is convenient to use a series representation for each of the singular functions. Thus, as in (11.78), we write for rod m (refer to Fig. 11.11)

$$Y_0(\gamma d_m) \equiv \sum_{s=-\infty}^{\infty} J_s(\gamma a)\, Y_s(\gamma\rho) e^{is\theta_m} \tag{11.94}$$

where θ_m is the azimuthal coordinate measured counterclockwise from the radial line passing through the center of rod m. If φ is the angular coordinate measured from rod 1, then

$$\theta_m \equiv \varphi - \frac{2\pi}{M}(m-1) \tag{11.95}$$

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Note that for a ring of M rods the angular interval between rods is $2\pi/M$; the interval between rod m and rod 1 is, therefore, $\varphi_m = 2\pi(m-1)/M$. If we use (11.95) in (11.94) and sum over all M rods as indicated in (11.92), then

$$\sum_{m=1}^{M} Y_0(\gamma d_m) = \sum_{s=-\infty}^{\infty} J_s(\gamma a)\, Y_s(\gamma\rho) e^{is\varphi} \sum_{m=1}^{M} e^{-i2\pi s(m-1)/M} \tag{11.96}$$

Now

$$\begin{aligned} \sum_{m=1}^{M} e^{-i2\pi s(m-1)/M} &= \frac{1-e^{-i2\pi s}}{1-e^{-i2\pi s/M}} = M \qquad \text{if } \frac{s}{M} \text{ is an integer} \\ \sum_{m=1}^{M} e^{-i2\pi s(m-1)/M} &= \frac{1-e^{-i2\pi s}}{1-e^{-i2\pi s/M}} = 0 \qquad \text{if } \frac{s}{M} \text{ is not an integer} \end{aligned} \tag{11.97}$$

Thus all terms in Eq. (11.96) vanish except those in the form J_{Ms}, Y_{Ms}, and $e^{isM\varphi}$, and these appear with factor M. Equation (11.96) therefore reduces to

$$\sum_{m=1}^{M} Y_0(\gamma d_m) = M \sum_{s=-\infty}^{\infty} J_{Ms}(\gamma a)\, Y_{Ms}(\gamma\rho) e^{iMs\varphi} \tag{11.98}$$

Similar results can be obtained for the functions $K_0(\beta d_m)$; the substitution of these, along with (11.98), into the expression for the fast flux yields

$$\begin{aligned} \phi_1(\rho,\varphi,z) = H(z) \Big\{ M \sum_{s=-\infty}^{\infty} &[\mathcal{E}_0 J_{Ms}(\gamma a)\, Y_{Ms}(\gamma\rho) + \mathcal{F}_0 I_{Ms}(\beta a)\, K_{Ms}(\beta\rho)] e^{iMs\varphi} \\ &+ \sum_{n=-\infty}^{\infty} [\mathfrak{A}_n J_n(\gamma\rho) + \mathfrak{M}_n I_n(\beta\rho)] e^{in\varphi} \Big\} \end{aligned} \tag{11.99}$$

The corresponding form for ϕ_2 is obtained by inserting the factor a_1 in each term involving a regular Bessel function of the first kind (J) and the factor a_2 in each term with a modified Bessel function of the first kind (I).

In order to satisfy the boundary conditions at the outer surface of the reactor, we must equate to zero the coefficients of *like* functions $e^{in\varphi}$ (that is, all exponentials with $n = Ms$). If this is done for both ϕ_1 and ϕ_2, the following relations are obtained:

$$\mathfrak{A}_{Ms} = -M\mathcal{E}_0 J_{Ms}(\gamma a) \frac{Y_{Ms}(\gamma\tilde{R})}{J_{Ms}(\gamma\tilde{R})} \tag{11.100}$$

$$\mathfrak{M}_{Ms} = -M\mathcal{F}_0 I_{Ms}(\beta a) \frac{K_{Ms}(\beta\tilde{R})}{I_{Ms}(\beta\tilde{R})} \tag{11.101}$$

All other $\mathfrak{M}_n$ and $\mathfrak{A}_n$ are zero.

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In view of the symmetry of the present system, it will suffice to satisfy the rod-reactor interface condition (11.74) at one rod, say at $m = 1$. For this purpose we write the flux equations (11.92) and (11.93) in a form that separates out the contributions due to rod 1; thus,

$$\phi_1(\rho,\varphi,z) \equiv H(z)[\mathcal{E}_0 Y_0(\gamma d_1) + \mathcal{F}_0 K_0(\beta d_1) + V(\rho,\varphi)] \tag{11.102}$$

$$\phi_2(\rho,\varphi,z) \equiv H(z)[a_1\mathcal{E}_0 Y_0(\gamma d_1) + a_2\mathcal{F}_0 K_0(\beta d_1) + U(\rho,\varphi)] \tag{11.103}$$

Here the functions V and U denote all the remaining terms in (11.92) and (11.93), respectively; these terms are all slowly varying functions in the vicinity of rod 1 [cf. P and Q in (11.83) and (11.84)]. The application of condition (1) from (11.74) to (11.103) yields an equation analogous to (11.85); this is

$$\mathcal{E}_0 = -\mathcal{F}_0\left(\frac{a_2}{a_1}\right)\frac{K_0(\beta b_*)}{Y_0(\gamma b_*)} - \frac{U(a,0)}{a_1 Y_0(\gamma b_*)} \tag{11.104}$$

where

$$U(a,0) \equiv a_1\mathcal{E}_0 \sum_{m=2}^{M} Y_0(\gamma d_{1m}) + a_2\mathcal{F}_0 \sum_{m=2}^{M} K_0(\beta d_{1m}) + \sum_{n=-\infty}^{\infty} [a_1\mathfrak{A}_n J_n(\gamma a) + a_2\mathfrak{M}_n I_n(\beta a)] \tag{11.105}$$

and d_{1m} denotes the distance between the centers of rod 1 and rod m.

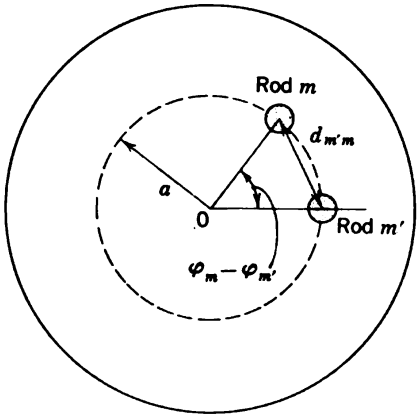

FIG. 11.12 Distance between rod centers.

In the general case, the distance between the centers of rod m' and rod m is given by (refer to Fig. 11.12)

$$d_{m'm} = 2a \sin\left[\frac{\pi}{M}(m' - m)\right] \tag{11.106}$$

The application of condition (2) from (11.74) to (11.102) yields a relation identical to (11.87), if we neglect the derivative of V with respect to d_1. This equation, together with (11.104), provides two rela-

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tions involving $\mathfrak{E}_0$ and $\mathfrak{F}_0$. The coefficients $\mathfrak{A}_n$ and $\mathfrak{M}_n$ in (11.105) may be eliminated with the aid of (11.100) and (11.101). The simultaneous solution of (11.104) and (11.87) then yields the criticality condition for the system. This is found to be

$$\left(\frac{a_1}{a_2}\right)\frac{\beta K_1(\beta b)}{\gamma Y_1(\gamma b)} = \frac{K_0(\beta b_*) + \sum_{m=2}^{M} K_0(\beta d_{1m}) - MI_0^2(\beta a)\dfrac{K_0(\beta\tilde{R})}{I_0(\beta\tilde{R})} - 2M\sum_{s=1}^{\infty} I_{Ms}^2(\beta a)\dfrac{K_{Ms}(\beta\tilde{R})}{I_{Ms}(\beta\tilde{R})}}{Y_0(\gamma b_*) + \sum_{m=2}^{M} Y_0(\gamma d_{1m}) - MJ_0^2(\gamma a)\dfrac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} - 2M\sum_{s=1}^{\infty} J_{Ms}^2(\gamma a)\dfrac{Y_{Ms}(\gamma\tilde{R})}{J_{Ms}(\gamma\tilde{R})}} \tag{11.107}$$

where we have again used the relations (11.89).

As in the case of the eccentric rod [cf. Eq. (11.88)], a good approximation to (11.107) can be obtained by neglecting all the terms in the numerator which involve the modified Bessel functions of the first kind, and the sum over s indicated in the denominator; then, the criticality condition for the ring of M solid-absorber rods in a bare cylindrical reactor may be written

$$\frac{Y_0(\gamma\tilde{R})}{J_0(\gamma\tilde{R})} \simeq \frac{1}{MJ_0^2(\gamma a)}\left\{Y_0(\gamma b_*) + \sum_{m=2}^{M} Y_0(\gamma d_{1m}) - \left(\frac{a_2}{a_1}\right)\frac{\gamma Y_1(\gamma b)}{\beta K_1(\beta b)}\left[K_0(\beta b_*) + \sum_{m=2}^{M} K_0(\beta d_{1m})\right]\right\} \tag{11.108}\star$$

This relation will be adequate for most computations.

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CHAPTER 12

HYDROGENOUS SYSTEMS

12.1 The Special Treatment Required for Hydrogen

a. General Considerations. Most of the discussion of reactor calculations in this work has been based on the age-diffusion model of neutron behavior. This model is very convenient to use and has yielded valuable results in innumerable cases. However, there is an important class of systems to which this model is poorly applicable. If a system includes a large proportion of hydrogen, the approximations involved in the age-diffusion method are really not valid. As indicated in the discussion of Sec. 6.1a, the age-diffusion method would not be expected to hold in situations where the neutrons make appreciable numbers of collisions with light nuclei; nevertheless, in a practical sense, this model yields acceptable results for any nonhydrogenous material. The case of hydrogen is unique since the neutron can lose all its energy in one elastic-scattering collision, and the energy-angle relationship (7.134) takes a very special form.

The practical advantages of the use of water as a moderator and as a coolant make it necessary to consider methods of improving the age-diffusion approximation.

The more accurate treatment required for hydrogenous systems can be separated into two classifications:

1. A complete set of equations covering the slowing down and diffusion process, replacing the age-diffusion set, may be found in two ways: (*a*) An intuitive approach using physical understanding of the neutron-proton elastic scattering process may form the basis for a set of equations. (*b*) The original Boltzmann equation may be examined in an attempt to discover a practical approximation which is an improvement over the age-diffusion approximation. It is this method which is emphasized in the present chapter.

2. The essentials of the age-diffusion method may be retained, but various parameters of the calculation may be chosen to give better results for hydrogenous reactors: (*a*) The group diffusion models discussed in Chap. 8 may be applied with particular choices of the group constants designed to yield improved accuracy. These constants are not necessarily calculated in any prescribed manner but are obtained from meas-

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urements or by trial and error in the calculations. (*b*) An empirical slowing-down kernel may be used with the usual one-velocity diffusion equations for the thermal neutrons.

In general, a particular method of the second class will give useful results only for a limited range of types of reactors; thus, in undertaking a new design analysis, considerable effort must be applied to find the appropriate constants.

b. Utility of the Measured Neutron Age in Water. One very important quantity involved in all the methods indicated above is the neutron age to thermal (or to indium resonance) in water. A simple example of the one approach (*b*) would be the application of the criticality condition (6.80) to a bare water-moderated system, wherein for the fast leakage losses one uses exp $(-B^2\tau_{th})$ with the age to thermal as obtained from experiment. More fundamentally, an experimental determination of the age in water actually gives the spatial distribution of neutrons of various energies as they slow down from a fission source. Thus an obvious requirement for a proposed calculational method is that when it is applied to the geometry of the experiment it should yield the observed distribution. In practice this is most conveniently approached by requiring that the calculational method give accurately a finite number of the lower *moments* of the slowing-down distribution. If, in a spherical geometry, the spatial distribution of neutrons reaching thermal energy at a distance r from a fission source is $K_{th}(r)$, then the nth moment $\overline{r^n}$ is defined as

$$\overline{r^n} \equiv 4\pi \int_0^\infty r^n K_{th}(r) r^2 \, dr \tag{12.1}\star$$

The most important of these is the second, $\overline{r^2}$, and in many cases, to require that the second moment be adequately represented is regarded as sufficient. In the age-diffusion approximation, the second moment is proportional to the age, and it was shown previously that

$$\overline{r^2(u)} = 6\tau(u) \tag{6.52}$$

In analogy to this result, it is convenient to define a generalized age for use in approximations other than the age diffusion; thus, the age to lethargy u is given by

$$\text{Age} \equiv \tfrac{1}{6}\overline{r^2(u)} \tag{12.2}\star$$

Then, the simplest and most direct criterion for evaluating a proposed calculational method for water-moderated systems is that it yield correctly the measured age to thermal in water (i.e., the correct second moment).

In the sections which follow, we examine some of the improvements over the age-diffusion approximation presently in use. The discussion of the calculation of the age from the equations which are developed

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is deferred until the last section of this chapter. The primary purpose of the present chapter is to demonstrate the basic ideas on which the improvements are based and to indicate the general direction for the development of a calculational method. In no case is a complete and detailed method of computation (including various regions, boundary conditions, neutron balance, etc.) given, such as was done previously for the age-diffusion case. Once the basic ideas are understood, the detailed system applied to the age-diffusion case may be easily adapted to the cases presented here.

12.2 Elementary Development of the Goertzel-Selengut Equations

Some improvement over the use of the ordinary age-diffusion equations for the hydrogenous case may be obtained by starting with the age-diffusion model but giving neutron scattering collisions with hydrogen special attention. It has been shown by Goertzel and Selengut[1] that the equations resulting from this approach may be derived directly in a certain approximation to the Boltzmann equation. In developing these relations in the present formulation, we will use a somewhat intuitive approach in order to display more clearly the physical basis of the model.

Consider the balance of neutrons in volume $d\mathbf{r}$ about $\mathbf{r}$ and with lethargies in du about u. At steady state the neutrons lost from the differential volume $d\mathbf{r}\,du$ will be just balanced by the gains, as indicated in Eq. (6.2). In the Goertzel-Selengut approach the leakage term is given by the usual age-diffusion form (6.3), and the absorption losses are given by the form $\Sigma_a(u)\,\phi(\mathbf{r},u)\,d\mathbf{r}\,du$. The volume sources will be represented by $S(\mathbf{r},u)\,d\mathbf{r}\,du$. The slowing-down terms, however, are given special attention. These we divide into two parts:

$$\begin{aligned} q(\mathbf{r},u) &\equiv \text{number of neutrons slowing down past lethargy } u \text{ per unit time per unit volume around } \mathbf{r} \text{ which made their last scattering collision with a nonhydrogen nucleus} \\ \eta(\mathbf{r},u)\,du &\equiv \text{number of neutrons slowing down into lethargy interval } du \text{ about } u \text{ per unit time per unit volume about } \mathbf{r} \text{ which made their last scattering collision with hydrogen} \end{aligned} \tag{12.3}$$

The slowing-down density given here is related to the flux $\phi(\mathbf{r},u)$ in the usual way, viz.,

$$q(\mathbf{r},u) = \xi\Sigma_s^{(0)}(u)\phi(\mathbf{r},u) \tag{12.4}$$

where $\xi\Sigma_s^{(0)}$ is taken as the sum of the individual $(\xi\Sigma_s)_i$ for all materials *except* hydrogen. The new quantity $\eta(\mathbf{r},u)$ may be computed from the

[1] G. Goertzel and D. Selengut (unpublished).

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integral relation (4.97); thus,

$$\eta(\mathbf{r},u)\,du = \int_0^u \Sigma_s^{(H)}(u')\,\phi(\mathbf{r},u')e^{u'-u}\,du'\,du \tag{12.5}$$

where (H) refers to the hydrogen nuclei, and it is implied that there are no sources above $u = 0$. In order to determine the net gain in neutron density in $d\mathbf{r}\,du$ from scattering collisions in and out of this differential volume, we require also

$$\Sigma_s^{(H)}(u)\,\phi(\mathbf{r},u)\,d\mathbf{r}\,du \equiv \begin{array}{l}\text{number of neutrons being scattered out}\\ \text{of } d\mathbf{r} \text{ about } \mathbf{r} \text{ and } du \text{ about } u \text{ because of}\\ \text{collisions with hydrogen nuclei}\end{array} \tag{12.6}$$

and $q(\mathbf{r},\, u + du)$. Then, the net gain may be written

$$[q(\mathbf{r},u) + \eta(\mathbf{r},u)\,du]\,d\mathbf{r} - [q(\mathbf{r},u + du) + \Sigma_s^{(H)}(u)\,\phi(\mathbf{r},u)\,du]\,d\mathbf{r} \tag{12.7}$$

Note that in (12.7) each scattering collision with hydrogen is counted as a loss to be consistent with Eq. (12.5) in which each neutron lost by scattering from a given lethargy interval is counted as a gain to some interval at higher lethargy.

If we use expression (12.7) for the slowing-down term, and the definitions for the other loss and gain terms given above, then in the present case Eq. (6.2) may be written

$$\begin{aligned}-D(u)\,\nabla^2\phi(\mathbf{r},u)\,du + \Sigma_a(u)\,\phi(\mathbf{r},u)\,du + q(\mathbf{r},u + du)\\ + \Sigma_s^{(H)}(u)\,\phi(\mathbf{r},u)\,du = S(\mathbf{r},u)\,du + q(\mathbf{r},u) + \eta(\mathbf{r},u)\,du\end{aligned} \tag{12.8}$$

But, from (12.5)

$$\frac{\partial}{\partial u}\,\eta(\mathbf{r},u) = -\eta(\mathbf{r},u) + \Sigma_s^{(H)}(u)\,\phi(\mathbf{r},u) \tag{12.9}\star$$

Substitution into (12.8) yields

$$-D(u)\,\nabla^2\phi(\mathbf{r},u) + \Sigma_a(u)\,\phi(\mathbf{r},u) = -\frac{\partial}{\partial u}\,q(\mathbf{r},u) - \frac{\partial}{\partial u}\,\eta(\mathbf{r},u) + S(\mathbf{r},u) \tag{12.10}\star$$

with $q(\mathbf{r},u)$ related to $\phi(\mathbf{r},u)$ by (12.4).

In applying these results to a reactor calculation the pair (12.9) and (12.10) is taken for each region containing hydrogen along with, as needed, ordinary age-diffusion equations for nonhydrogenous regions. To this system is applied the usual spatial boundary conditions, thermal-group relations, and coupling equations. A discussion of computational results using this set is given in Sec. 12.4.

The derivation of the Goertzel-Selengut equations given above brings out clearly the fact that, while the distribution of neutrons in energy is properly handled through exact treatment of the slowing-down in hydrogen, the treatment of the space-energy correlation during the slowing-down process is not improved over that of the age-diffusion equations.

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A possibility exists for such improvement in the calculation of the diffusion coefficient, but as will be shown in the next section, a consistent derivation of these equations from the Boltzmann equation indicates that $D(u)$ is to be calculated as in Eq. (7.161), with hydrogen treated on the same basis as all other substances present.

12.3 Solution of Boltzmann Equation for Hydrogenous Systems

a. Computation of Second Moment for Mixtures of Nuclear Species. The present analysis is a more general development of the ideas previously considered in Sec. 7.6a. For this purpose we require a statement of the Boltzmann equation for a system containing a mixture of N nuclear species. The direct extension of the steady-state form of (7.110) is

$$\boldsymbol{\Omega} \cdot \nabla\phi(\mathbf{r},v,\boldsymbol{\Omega}) + \Sigma_t(v)\,\phi(\mathbf{r},v,\boldsymbol{\Omega}) = S(\mathbf{r},v,\boldsymbol{\Omega}) + \frac{1}{2\pi}\sum_{i=1}^{N}\int_{\mathbf{v}'}\Sigma_s^{(i)}(v')\,\phi(\mathbf{r},v',\boldsymbol{\Omega}')\,\mathfrak{q}^{(i)}(v,\mu_0;\mathbf{v}')\,dv'\,d\boldsymbol{\Omega}' \quad (12.11)$$

The conditional frequency $\mathfrak{q}^{(i)}$ is defined according to (7.114), and the superscript (i) means that the scattering collision occurred with a nucleus of type i. In the present application we consider the case of one-dimensional symmetry so that spatial variation occurs only in the x direction, and the only angular variable of interest is μ, the cosine of the angle between the x axis and the direction of motion of the neutron. The appropriate one-dimensional form of (12.11) is [cf. Eq. (7.336)]

$$\mu\frac{\partial}{\partial x}\phi(x,u,\mu) + \Sigma_t(u)\,\phi(x,u,\mu) = S(x,u,\mu) + \frac{1}{2\pi}\sum_{i=1}^{N}\int_{u'}du'\int_{\boldsymbol{\Omega}'}\Sigma_s^{(i)}(u')\,\phi(x,u',\mu')\,\mathfrak{s}^{(i)}(u,\mu_0;u',\boldsymbol{\Omega}')\,d\boldsymbol{\Omega}' \quad (12.12)$$

where we have transformed from the speed to the lethargy variable. The $\mathfrak{s}^{(i)}$ are defined according to Eq. (7.127).

We obtain the solution to Eq. (12.12) by expanding the flux and the frequency functions $\mathfrak{s}^{(i)}$ in infinite series of Legendre functions. For the frequency function we use the general form (7.129), and for the flux [cf. Eq. (7.74)],

$$\phi(x,u,\mu) \equiv \sum_{n=0}^{\infty}\left(\frac{2n+1}{2}\right)\phi_n(x,u)P_n(\mu) \quad (12.13)$$

A similar series is defined also for the source function $S(x,u,\mu)$. If these expansions are used in Eq. (12.12), along with the addition theorem

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(7.21), and the orthogonality properties for the Legendre functions are applied, the result is an infinite set of coupled equations in the coefficients ϕ_n:

$$-\Sigma_t(u)\,\phi_n(x,u) + S_n(x,u) + \sum_{i=1}^{N} \int_{u+\ln \alpha_i}^{u} \Sigma_s^{(i)}(u')\,\phi_n(x,u')\,\mathfrak{s}_n^{(i)}(u;u',\Omega')\,du'$$

$$= \left(\frac{n+1}{2n+1}\right)\frac{\partial}{\partial x}\,\phi_{n+1}(x,u) + \left(\frac{n}{2n+1}\right)\frac{\partial}{\partial x}\,\phi_{n-1}(x,u) \quad (12.14)$$

where

$$\alpha_i \equiv \left(\frac{A_i - 1}{A_i + 1}\right)^2 \quad (12.15)$$

and A_i is the mass number of the ith nuclear species. Note that the range of integration for the lethargy variable extends over the usual collision interval appropriate to the particular species. It is of interest to mention that the set (12.14) may also be derived directly from the general form (7.125) by applying the procedure used in Sec. 7.2f. The computation requires first that Eq. (7.125) be integrated term by term over the range of the azimuthal variable ψ, in order to obtain an expression suitable to the one-dimensional problem. The next step is to multiply through by some $P_m(\mu)$ and integrate over μ. These operations yield an equation in the coefficients ϕ_n^0. The final reduction to Eq. (12.14) is obtained by the application of (7.77).

To proceed further we require a specification of the coefficients $\mathfrak{s}_n^{(i)}$. For this purpose we introduce the assumption that the scattering collisions are isotropic in the center-of-mass system of coordinates. In that case we obtain

$$\mathfrak{s}_n^{(i)}(u;u',\Omega) \equiv \frac{P_n(\zeta_i)e^{u'-u}}{1-\alpha_i} \quad (12.16)$$

with [cf. Eq. (7.134)]

$$\zeta_i(u,u') \equiv \tfrac{1}{2}[(A_i+1)e^{\frac{1}{2}(u'-u)} - (A_i-1)e^{-\frac{1}{2}(u'-u)}] \quad (12.17)$$

as demonstrated in the derivation of Sec. 7.3e. If we use the expression (12.16) in (12.14) and approximate the infinite set by the first three harmonics, we obtain, for a system with isotropic sources,

$$\frac{\partial}{\partial x}\,\phi_1(x,u) + \Sigma_t(u)\,\phi_0(x,u) = S_0(x,u) + \sum_{i=1}^{N} \int_{u+\ln \alpha_i}^{u} \Sigma_s^{(i)}(u')\,\phi_0(x,u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \quad (12.18)$$

$$\frac{1}{3}\frac{\partial}{\partial x}\,\phi_0(x,u) + \frac{2}{3}\frac{\partial}{\partial x}\,\phi_2(x,u) + \Sigma_t(u)\,\phi_1(x,u) = \sum_{i=1}^{N} \int_{u+\ln \alpha_i}^{u} \Sigma_s^{(i)}(u')\,\phi_1(x,u')\,\zeta_i(u,u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \quad (12.19)$$

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$$\frac{2}{5}\frac{\partial}{\partial x}\phi_1(x,u) + \frac{3}{5}\frac{\partial}{\partial x}\phi_3(x,u) + \Sigma_t(u)\,\phi_2(x,u) = \sum_{i=1}^{N}\int_{u+\ln\alpha_i}^{u}\Sigma_s^{(i)}(u')\,\phi_2(x,u')\,\frac{[3\zeta_i^2(u,u')-1]}{2(1-\alpha_i)}\,e^{u'-u}\,du' \quad (12.20)$$

It is convenient for the demonstration of some of the properties of these equations to work with their Fourier transforms; we introduce

$$\Psi_n(s,u) \equiv \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\phi_n(x,u)e^{isx}\,dx$$
$$\mathcal{S}_0(s,u) \equiv \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty}S_0(x,u)e^{isx}\,dx \quad (12.21)$$

The transformation of Eqs. (12.18) through (12.20) yields

$$is\Psi_1(s,u) + \Sigma_t(u)\,\Psi_0(s,u) = \mathcal{S}_0(s,u) + \sum_{i=1}^{N}\int_{u+\ln\alpha_i}^{u}\Sigma_s^{(i)}(u')\,\Psi_0(s,u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \quad (12.22)$$

$$\frac{is}{3}\Psi_0(s,u) + \frac{2is}{3}\Psi_2(s,u) + \Sigma_t(u)\,\Psi_1(s,u) = \sum_{i=1}^{N}\int_{u+\ln\alpha_i}^{u}\Sigma_s^{(i)}(u')\,\Psi_1(s,u')\,\zeta_i(u,u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \quad (12.23)$$

$$\frac{2is}{5}\Psi_1(s,u) + \frac{3is}{5}\Psi_3(s,u) + \Sigma_t(u)\,\Psi_2(s,u) = \sum_{i=1}^{N}\int_{u+\ln\alpha_i}^{u}\Sigma_s^{(i)}(u')\,\Psi_2(s,u')\,\frac{[3\zeta_i^2(u,u')-1]}{2(1-\alpha_i)}\,e^{u'-u}\,du' \quad (12.24)$$

If the medium under consideration and the source distribution are symmetric in x, then the total flux $\phi_0(x,u)$ is symmetric in x, and consequently, $\Psi_0(s,u)$ and $\mathcal{S}_0$ are symmetric in s. Thus, we may represent Ψ_0 and $\mathcal{S}_0$ in the following power series in s:

$$\Psi_0(s,u) \equiv \Psi_{00}(u) + \Psi_{02}(u)\frac{s^2}{2!} + \Psi_{04}(u)\frac{s^4}{4!} + \cdots$$
$$\mathcal{S}_0(s,u) \equiv \mathcal{S}_{00}(u) + \mathcal{S}_{02}(u)\frac{s^2}{2!} + \mathcal{S}_{04}(u)\frac{s^4}{4!} + \cdots \quad (12.25)$$

Then, the form of Eqs. (12.22) through (12.24) shows that it is sufficient to represent the higher-order coefficients in power series beginning with

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the power corresponding to the order of the coefficient; thus,

$$\Psi_1(s,u) \equiv \Psi_{11}(u)s + \Psi_{13}(u)\frac{s^3}{3!} + \cdots \tag{12.26}$$

$$\Psi_2(s,u) \equiv \Psi_{22}(u)\frac{s^2}{2!} + \Psi_{24}(u)\frac{s^4}{4!} + \cdots \tag{12.27}$$

$$\Psi_3(s,u) \equiv \Psi_{33}(u)\frac{s^3}{3!} + \Psi_{35}(u)\frac{s^5}{5!} + \cdots \tag{12.28}$$

and so on. The lethargy-dependent coefficients defined by these equations are determined by substituting these series into Eqs. (12.22) through (12.24) and equating coefficients of like powers of s. The result is a set of relations between the coefficients; for example,

$$\Sigma_t(u)\,\Psi_{00}(u) = \mathcal{S}_{00}(u) + \sum_{i=1}^{N}\int_{u+\ln\alpha_i}^{u}\Sigma_s^{(i)}(u')\,\Psi_{00}(u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \tag{12.29}$$

$$i\Psi_{11}(u) + \frac{1}{2}\Sigma_t(u)\,\Psi_{02}(u) = \frac{1}{2}\mathcal{S}_{02}(u) + \frac{1}{2}\sum_{i=1}^{N}\int_{u+\ln\alpha_i}^{u}\Sigma_s^{(i)}(u')\,\Psi_{02}(u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \tag{12.30}$$

$$\frac{i}{3}\Psi_{00}(u) + \Sigma_t(u)\,\Psi_{11}(u) = \sum_{i=1}^{N}\int_{u+\ln\alpha_i}^{u}\Sigma_s^{(i)}(u')\Psi_{11}(u')\,\zeta_i(u,u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \tag{12.31}$$

If the cross sections and the source distribution are known, these equations may be solved simultaneously; thus, (12.29) yields Ψ_{00}, which when applied to (12.31) yields Ψ_{11}. This in turn gives Ψ_{02} from (12.30), and so on.

The interest in these particular coefficients follows from the discussion given in Sec. 12.1b on the generalized definition of the age (12.2). In order to compute the age, we require the mean squared distance to lethargy u. A suitable expression is given by Eq. (7.343), which in the present case reduces to the form

$$\overline{x^2(u)} = \frac{\int_{-\infty}^{\infty} x^2\phi_0(x,u)\,dx}{\int_{-\infty}^{\infty}\phi_0(x,u)\,dx} \tag{12.32}$$

This relation may also be expressed in terms of the Fourier transforms (12.21). It was shown previously [refer to discussion beginning with Eq. (7.364) and extending through Eq. (7.374)] that these are related

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to the moments of the function, and in the present case we obtain [cf. Eq. (7.374)]

$$\overline{x^2(u)} = -\left[\frac{(\partial^2/\partial s^2)\Psi_0(s,u)}{\Psi_0(s,u)}\right]_{s=0} = -\frac{\Psi_{02}(u)}{\Psi_{00}(u)} \tag{12.33}$$

the last equality following from (12.25).

We find then that Eqs. (12.29) through (12.31) are sufficient for computing the correct second moment of the flux. Equations (12.22) through (12.24) show that, by working with the transformed equations, only the first two terms of the series (12.13) for the flux are required to obtain the second moment and, therefore, the generalized age accurately.

The accurate P_1 set of approximating equations to the Boltzmann equation is obtained from (12.22) through (12.24) by neglecting Ψ_2 and higher terms; thus,

$$is\Psi_1(s,u) + \Sigma_t(u)\,\Psi_0(s,u) = \mathcal{S}_0(s,u) + \sum_{i=1}^{N}\int_{u+\ln\alpha_i}^{u} \Sigma_s^{(i)}(u')\,\Psi_0(s,u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \tag{12.34}$$

$$\frac{is}{3}\Psi_0(s,u) + \Sigma_t(u)\,\Psi_1(s,u) = \sum_{i=1}^{N}\int_{u+\ln\alpha_i}^{u} \Sigma_s^{(i)}(u')\,\Psi_1(s,u')\,\zeta_i(u,u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \tag{12.35}$$

b. Reduction to the Age-diffusion Equations. As in Sec. 7.3g, the basis of the reduction to the age-diffusion equations is the assumption that the collision densities are slowly varying with lethargy over the range of integration required on the right-hand sides of Eqs. (12.34) and (12.35); that is,

$$\Sigma_s^{(i)}(u')\,\Psi_0(s,u') \simeq \Sigma_s^{(i)}(u)\,\Psi_0(s,u) + (u'-u)\,\frac{\partial}{\partial u}\left[\Sigma_s^{(i)}(u)\,\Psi_0(s,u)\right] \tag{12.36}$$

$$\Sigma_s^{(i)}(u')\,\Psi_1(s,u') \simeq \Sigma_s^{(i)}(u)\,\Psi_1(s,u) \qquad \text{for } i = 1, 2, \ldots, N \tag{12.37}$$

These approximations reduce Eqs. (12.34) and (12.35) to

$$is\Psi_1(s,u) + \Sigma_t(u)\,\Psi_0(s,u) = \mathcal{S}_0(s,u) + \Sigma_s(u)\,\Psi_0(s,u) - \frac{\partial}{\partial u}\left[\xi\Sigma_s(u)\,\Psi_0(s,u)\right] \tag{12.38}$$

$$\frac{is}{3}\Psi_0(s,u) + \Sigma_t(u)\,\Psi_1(s,u) = \overline{\mu_0}\Sigma_s(u)\,\Psi_1(s,u) \tag{12.39}$$

where $\xi\Sigma_s(u)$ is defined according to Eq. (4.121), and

$$\overline{\mu_0}\Sigma_s(u) \equiv \sum_{i=1}^{N} \overline{\mu_0^{(i)}}\Sigma_s^{(i)}(u) \tag{12.40}$$

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The quantities ξ_i and $\overline{\mu_0^{(i)}}$ are related to A_i in the usual way. If we define $D(u)$ according to (7.161), then Eq. (12.38) may be reduced to the form

$$s^2 D(u)\,\Psi_0(s,u) + \Sigma_a(u)\,\Psi_0(s,u) = \mathcal{S}_0(s,u) - \frac{\partial}{\partial u}\left[\xi\Sigma_s(u)\,\Psi_0(s,u)\right] \tag{12.41}$$

by the elimination of Ψ_1. The Fourier inversion of Eq. (12.41) yields the familiar age-diffusion equation (7.164).

c. Reduction to the Goertzel-Selengut Equations. The treatment above demonstrates that the age-diffusion equations rest on two principal approximations: (1) The angular distribution of the flux[1] can be represented by the first two harmonics [refer to Eqs. (12.38) and (12.39)]. (2) The collision densities are sufficiently slowly varying over the lethargy range $\ln 1/\alpha_i$ for each species so that they may be represented by one or two terms in a Taylor expansion [as in (12.36) and (12.37)]. These conditions were established previously in Secs. 7.3f and 7.3g. The general method for improving the age-diffusion equations for application to hydrogenous systems is achieved by discarding some or all of approximation (2). In the particular case of scatterings from hydrogen, the range $(u + \ln \alpha_H) \to (u)$ is the entire lethargy scale[2] from $u = 0$ to u; thus, approximation (2) is inapplicable in the case of hydrogen. However, since the approximation is made separately for each nuclear species over different lethargy ranges, there is no inconsistency involved in using approximation (2) for some substances and dropping it for others. In particular, to obtain the Goertzel-Selengut equations: (1) Equation (12.36) is applied for all substances *except* hydrogen; that is, (12.36) holds for all $i = 1, \ldots, N-1$, but not for $i = N$, which we use to denote hydrogen. (2) Equation (12.37) is retained for all i. With these conditions, Eqs. (12.34) and (12.35) become

$$\begin{aligned} is\Psi_1(s,u) + \Sigma_t(u)\,\Psi_0(s,u) &= \mathcal{S}_0(s,u) + \Sigma_s^{(0)}(u)\,\Psi_0(s,u) \\ &\quad - \frac{\partial}{\partial u}\left[\xi_0\Sigma_s^{(0)}(u)\,\Psi_0(s,u)\right] + \int_0^u \Sigma_s^{(H)}(u')\,\Psi_0(s,u')e^{u'-u}\,du' \end{aligned} \tag{12.42}$$

$$\frac{is}{3}\,\Psi_0(s,u) + \Sigma_t(u)\,\Psi_1(s,u) = \overline{\mu_0}\Sigma_s(u)\,\Psi_1(s,u) \tag{12.43}$$

[1] H. Hurwitz and P. Zweifel, *J. Appl. Phys.*, **26**, 923 (1955), have considered the expansion of the scattering function, rather than the flux, in spherical harmonics; by dropping all terms beyond the nth, they obtain the so-called B_n approximation, which has been shown to be more accurate than the corresponding P_n approximation developed here.

[2] Actually, the lethargy scale extends from $-\infty$ to u, but it is assumed here, as usual, that the zero of lethargy is taken to correspond to a sufficiently high energy so that there are a negligible number of neutrons of higher energy in the system.

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Here, the index 0 refers to all substances other than hydrogen, that is,

$$\Sigma_s^{(0)}(u) \equiv \sum_{i=1}^{N-1} \Sigma_s^{(i)}(u) \tag{12.44}$$

$$\bar{\xi}_0 \Sigma_s^{(0)}(u) \equiv \sum_{i=1}^{N-1} \xi_i \Sigma_s^{(i)}(u) \tag{12.45}$$

If we eliminate Ψ_1 from Eqs. (12.42) and (12.43), we obtain for Ψ_0 [using (7.161) for $D(u)$]

$$s^2 D(u)\, \Psi_0(s,u) + \Sigma_t(u)\, \Psi_0(s,u) = \mathcal{S}_0(s,u) + \Sigma_s^{(0)}(u)\, \Psi_0(s,u) - \frac{\partial}{\partial u}[\bar{\xi}_0 \Sigma_s^{(0)}(u) \Psi_0(s,u)] + \int_0^u \Sigma_s^{(H)}(u')\, \Psi_0(s,u') e^{u'-u}\, du' \tag{12.46}$$

The inversion of this equation yields:

$$-D(u) \frac{\partial^2}{\partial x^2} \phi_0(x,u) + \Sigma_a(u)\, \phi_0(x,u) = S_0(x,u) - \Sigma_s^{(H)}(u)\, \phi_0(x,u) - \frac{\partial}{\partial u} q(x,u) + \eta(x,u) \tag{12.47}$$

with the definitions

$$q(x,u) \equiv \bar{\xi}_0 \Sigma_s^{(0)}(u)\, \phi_0(x,u) \tag{12.48}$$

$$\eta(x,u) \equiv \int_0^u \Sigma_s^{(H)}(u')\, \phi_0(x,u') e^{u'-u}\, du' \tag{12.49}$$

and the application of Eq. (12.9). Thus we obtain the one-dimensional form of the Goertzel-Selengut equations (12.9) and (12.10) given previously.

d. Reduction to the Consistent P_1 Approximation. Another improvement on the age-diffusion equations for use in the hydrogenous media which has been sufficiently widely used to merit discussion here is the so-called "consistent P_1 approximation." This model also represents an improvement over the Goertzel-Selengut equations. In the consistent P_1 approximation, neither of the expansions (12.36) or (12.37) is applied to hydrogen, though both are used for all other nuclear species. Equations (12.34) and (12.35) become, for this case,

$$is\Psi_1(s,u) + \Sigma_t(u)\, \Psi_0(s,u) = \mathcal{S}_0(s,u) + \Sigma_s^{(0)}(u)\, \Psi_0(s,u) - \frac{\partial}{\partial u}[\bar{\xi}_0 \Sigma_s^{(0)}(u)\, \Psi_0(s,u)] + \int_0^u \Sigma_s^{(H)}(u')\, \Psi_0(s,u') e^{u'-u}\, du' \tag{12.50}\star$$

$$is\Psi_0(s,u) + \Sigma_t(u)\, \Psi_1(s,u) = \overline{\mu_0^{(0)}} \Sigma_s^{(0)}(u)\, \Psi_1(s,u) + \int_0^u \Sigma_s^{(H)}(u')\, \Psi_1(s,u') e^{\frac{2}{3}(u'-u)}\, du' \tag{12.51}\star$$

where

$$\overline{\mu_0^{(0)}} \Sigma_s^{(0)}(u) \equiv \sum_{i=1}^{N-1} \overline{\mu_0^{(i)}} \Sigma_s^{(i)}(u) \tag{12.52}$$

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These equations are, in general, difficult to handle analytically. Some analytical treatment has been given to the consistent P_1 set for the special case of pure hydrogen by Simon[1] and by Welton and Biedenharn.[2] In the present discussion we shall content ourselves with a comparison of the various models on the basis of numerical calculations.

12.4 Numerical Results

a. Comparison of Methods by Age Calculations. The primary purpose of the present discussion is to compare the accuracy of the various models for hydrogenous systems on the basis of their prediction of the neutron age in water. Thus the information given here will supplement the detailed discussion of age computations given in Sec. 7.6 which includes a development for water-metal systems based on the Fermi-Marshak method. The comparison of the methods is drawn on the basis of the computed age-to-indium resonance. Besides the relative ease of measurement (as contrasted with the age to thermal) it has the advantage of easier interpretation because the thermal motion and molecular effects are not generally important above the indium resonance (1.44 ev).

According to the general definition of the age, the quantity of interest is the second moment of the neutron flux at the energy of the indium resonance.[3] It is also possible to calculate the second moment of the slowing-down density. This quantity will be slightly smaller than the second moment of the flux because of the "last flight" of the neutrons after reaching the specified energy (refer to Sec. 7.6c). It is clear also that the second moment of the flux corresponds more closely to the physically measured quantity. Some calculated results are given in Table 12.1.

An acceptable *experimental* value at the present time is (see Table 6.1)

$$\tau_{1n} = 26.7 \text{ cm}^2$$

The agreement between the more accurately calculated values, as represented by the Monte Carlo and consistent P_1 approximations in Table 12.1, and the experimental result is seen to be quite good.

b. Reactor Calculations. The improvements discussed in this chapter may be incorporated into the multigroup method of calculating reactors which has been developed for use on fast computing machines (refer to Sec. 8.8). Examples of such programs are discussed by Alexander and

[1] A. Simon, "Neutron Slowing Down by Hydrogen in the Consistent P_1 Approximation," Oak Ridge National Laboratory, ORNL-2098, July 19, 1956.

[2] L. C. Biedenharn and T. A. Welton, "Some Remarks on the Slowing Down of Neutrons in Hydrogen," Oak Ridge National Laboratory, ORNL-2107, Aug. 21, 1956.

[3] That is, τ_N, in the notation of Hurwitz and Zweifel, *op. cit.*

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Given[1] and by Kinney et al.[2] The "Eyewash" code described by Alexander and Given was developed for the UNIVAC and incorporates the Goertzel-Selengut method in a 30 lethargy-group program designed for a multiregion spherical reactor. A similar 30-group code called "Medusa" is also available for the UNIVAC, and this is based entirely on the age-diffusion model. When applied to nonhydrogenous reactors these two methods give essentially identical results.[3]

TABLE 12.1 COMPARISON OF MODELS ON BASIS OF AGE OF FISSION NEUTRONS IN WATER AT STANDARD CONDITIONS

Model for calculation	Age (cm^2) to indium resonance	Comments	Reference
Age-diffusion.....	17.5	Oxygen mass assumed infinite	
Monte Carlo.....	25.6	Second moment of slowing-down density	R. R. Coveyou, ORNL-2081, p. 144.
Consistent P_1.....	25.3	.....	H. Hurwitz and P. Zweifel, *J. Appl. Phys.*, **26**, 923 (1955)
Goertzel-Selengut...	30.9	.....	H. Hurwitz and P. Zweifel, *J. Appl. Phys.*, **26**, 923 (1955)

As an example of the results which may be obtained by use of the different approximations consider the following reactor system:

Core: Sphere of radius 12.06 cm containing
5.94 kg of H_2O
8.88 kg of UO_2F_2 (containing 6.84 kg of 93 per cent enriched uranium)
Reflector: Spherical shell of pure water, 10.56 cm thick

For the analysis of this system the lethargy scale was divide into 29 fast groups plus a thermal group corresponding to 183°F. The spatial integration was carried out using intervals of 0.754 cm which yielded 16 space points in the core and 14 in the reflector. The results of this computation are given in Table 12.2.

[1] J. H. Alexander and N. D. Given, "A Machine Multigroup Calculation: The Eye Wash Program for UNIVAC," Oak Ridge National Laboratory, ORNL-1925, Sept. 15, 1955.

[2] W. E. Kinney, R. R. Coveyou, J. G. Sullivan, and B. J. Garrick, "Multigroup Calculations for Multiregion Reactor," Applied Nuclear Physics Division Annual Report for Period Ending Sept. 10, 1956, ORNL-2081, pp. 133–139.

[3] Actually there are some slight differences because of different assumptions in the averaging process for obtaining group constants from cross-section data.

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This somewhat extreme case was chosen to exhibit the difference between the age-diffusion and the Goertzel-Selengut models. The assembly is very small, and therefore the neutron leakage plays an important role in determining the neutron economy. The effect of the different approximations in the treatment of the energy distribution of neutrons after scattering collisions with hydrogen will be magnified because of the resultant effect on the neutron leakage in the fast range. An examination of the results given in Table 12.2 shows that the age-diffusion approximation gives a higher estimate of the multiplication

TABLE 12.2 MULTIGROUP CALCULATIONS FOR WATER-MODERATED REACTOR

Model	k	Fraction of neutrons absorbed in reactor	Fraction of absorptions in core	Total neutrons escaping	Fraction of neutrons escaping as fast neutrons	Fraction of fissions due to thermal neutrons
Age-diffusion (Medusa).....	1.17	0.789	0.763	0.211	0.107	0.375
Goertzel-Selengut (Eye Wash)....	1.00	0.663	0.770	0.337	0.239	0.430

constant than does the Goertzel-Selengut. Evidently this is due to the greater fast leakage in the Goertzel-Selengut case, being over twice as large as that predicted by the age model. This is entirely consistent with the larger computed age for water as obtained from the G-S model (refer to Table 12.1).

In contrast to this discrepancy in the multiplication constant, the spatial distribution of the group fluxes does not differ appreciably. This point is illustrated in Fig. 12.1, which shows that the computed power distributions for the two methods are nearly identical.

The multigroup calculational method based on the Goertzel-Selengut model has also been tested in the calculation of critical assemblies. E. G. Silver[1] has compared the computed critical masses from the Eye Wash code with the experimentally determined values for the case of enriched uranium assemblies in water.[2] Unfortunately the evaluation of the analytical model was not clear cut since it was necessary to apply a transformation from the actual completely reflected parallelepiped geometry to a three-region spherical model. Some aspects of this problem were discussed previously in Sec. 8.6. These considerations are discussed in

[1] E. G. Silver, "Comparison of Calculated Critical Mass for BSR with Experiment," Applied Nuclear Physics Division Annual Report for Period Ending Sept. 10, 1956, Oak Ridge National Laboratory, ORNL-2081, pp. 47–58.

[2] The ORNL Bulk Shielding Reactor.

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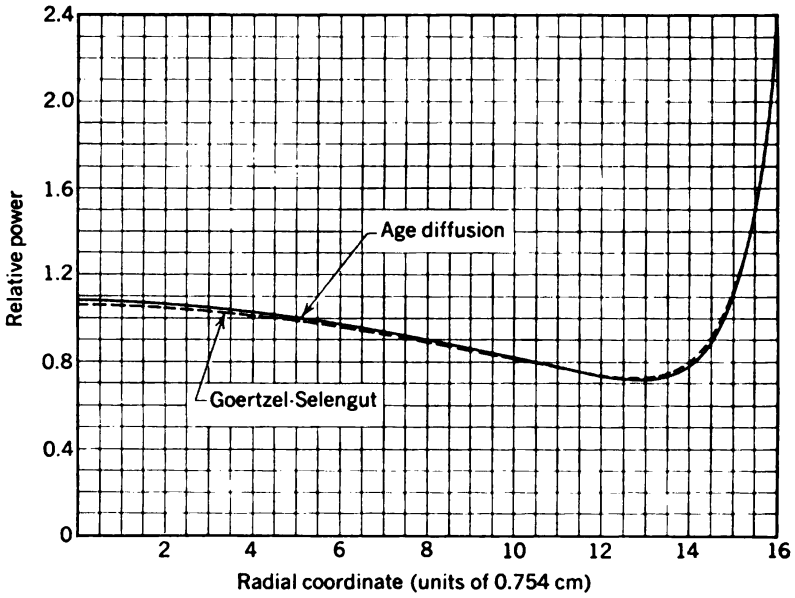

FIG. 12.1 Comparison of the power distributions obtained from age-diffusion and Goertzel-Selengut models.

detail by Silver. A partial set of the results obtained from the comparison are listed in Table 12.3. As may be seen, the comparison is entirely satisfactory. Since the results of Table 12.2 indicate that the age-diffusion model would have given considerably different results, it may be concluded that the G-S model represents a worthwhile improvement.

TABLE 12.3 COMPARISON WITH EXPERIMENT OF CRITICAL MASSES COMPUTED BY GOERTZEL-SELENGUT METHOD

Core dimensions, cm	Critical mass of U^{235}, kg	
	Computed	Experimental
61.3 × 45 × 31	3.242	3.26
61.3 × 38.5 × 40.5	3.246	3.27

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CHAPTER 13

PERTURBATION THEORY

13.1 The Scope and Methods of Perturbation Theory

The methods[1] described in this chapter are designed to meet the general class of problems involving small changes which affect the nuclear features of a reactor system and which require, in general, a recalculation of the principal reactor parameters. The methods developed for treating situations of this sort utilize the fact that the changes are sufficiently small that the characteristics of the perturbed system may be constructed from the analysis of the original. This approach is widely used in the problems of physics and engineering science because it frequently leads to substantial simplifications in the calculational procedure.

As an illustration of such problems, consider the situation wherein a small lump of absorbing material is introduced at some point within a critical assembly. Since the precise nuclear balance of the system has been upset, it can no longer sustain itself in steady-state operation, and it is to be expected that the over-all flux level in the reactor would fall in time, eventually to zero. In order to restore the steady-state condition, it would be necessary to add a certain amount of positive reactivity to the system (presumably by means of the control system). If it were possible to calculate the required additional reactivity, the time-dependent behavior (at least after long time) of the uncompensated case would be predicted as an exponential decay of the flux level with a time constant given by the ratio of the neutron lifetime in the system to the required extra multiplication [see, for example, Eq. (9.23)]. (It would be supposed that the small amount of absorber added would have a negligible effect on the neutron lifetime.) Thus the central problem is the calculation of the reactivity decrease resulting from the introduction of the absorber. It may be seen at once that the result is dependent upon the position in the system at which the absorber is placed. Clearly, if the lump were placed near the outer boundary of the assembly it would absorb much fewer neutrons than if it were placed near the center. The effect

[1] The presentation in this chapter represents a departure from the remainder of the book in that the mathematical background required is considerably higher. No attempt will be made to supply the necessary basic information. The material presented should be of value as a reference, in any event.

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of the lump on the reactivity of the system is therefore much smaller when positioned near the outer edge. Further, the energy dependence of the absorption cross section and the neutron-energy spectrum at the location of the lump also have important influences on its effectiveness. For example, in a highly thermal reactor it would be expected that a material with a high thermal absorption cross section would produce a greater reactivity change than would a material with high absorption resonances in the epithermal range and low absorption cross section in the thermal.

The calculational method for approaching such problems as mentioned here involves the solution of the neutron-balance equations for the system, with small changes allowed in the original parameters. By the application of the well-known mathematical techniques of perturbation theory, one can obtain the corresponding variations in the multiplication constant and in the neutron flux. This general procedure is applied throughout this chapter. Before undertaking detailed calculations, it will be instructive first to examine the over-all problem from a general viewpoint.

a. General Methods. In this section we present the formalism of the perturbation method, leaving the general discussion of the physical interpretation to the next section. The presentation given here is in no sense a rigorous mathematical derivation. It is intended merely as an outline of a procedure which can be applied rigorously to many particular cases.

Suppose that the reactor equations for the critical assembly can be written in the general form

$$\boldsymbol{O}\,\phi_0(\mathbf{r},u) = 0 \tag{13.1}$$

Here $\phi_0(\mathbf{r},u)$ denotes the neutron flux as a function of space and lethargy, and $\boldsymbol{O}$ symbolizes an operator chosen appropriately for the system at hand. For example, in the *one-velocity* model the operator $\boldsymbol{O}$ would have the form

$$\boldsymbol{O} \equiv -\nabla \cdot D\nabla + \Sigma_a - \nu\Sigma_f \tag{13.2}$$

However, much more complicated operators are also encompassed in the general treatment; e.g., in a later section the operator $\boldsymbol{O}$ will be taken as a matrix. It will be convenient for the analysis which follows to define operators $\boldsymbol{O}_1$ and $\boldsymbol{O}_2$ such that

$$\boldsymbol{O} \equiv \boldsymbol{O}_1 - \nu\boldsymbol{O}_2 \tag{13.3}$$

Here the intent is to bring the quantity ν explicitly into the calculation. In the one-velocity case, for example, we have from (13.2)

$$\boldsymbol{O}_1 = -\nabla \cdot D\nabla + \Sigma_a \qquad \boldsymbol{O}_2 = \Sigma_f \tag{13.4}$$

The average number of neutrons per fission ν is very useful for the present

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type of problems, since we may imagine that nonuniform changes in the neutron parameters of the reactor system are compensated by a uniform change in ν, so that the system remains critical. Then, as discussed previously [cf. Eqs. (4.167) and (5.178)], the change in the effective multiplication constant brought about by the nonuniform changes is given by

$$\frac{\delta k}{k} = -\frac{\delta \nu}{\nu} \tag{13.5}$$

If we allow alterations, then, in the reactor system, changes in the operator $\boldsymbol{O}$ are implied; therefore, the flux in the perturbed system will also be different from the original. By allowing hypothetical changes in ν, the perturbed system may also be critical, so that (denoting quantities associated with the perturbed system by a prime)

$$\boldsymbol{O}'\phi_0'(\mathbf{r},u) = (\boldsymbol{O}_1' - \nu' \boldsymbol{O}_2')\,\phi_0'(\mathbf{r},u) = 0 \tag{13.6}$$

The form of this equation implies a recalculation of the entire reactor problem, the determination of the value of ν' for criticality being a primary objective. The complex reanalysis required here is not always necessary, however. If the changes introduced into the system are small, it is possible to make use of the knowledge already available about the unperturbed system from the solution of Eq. (13.1). Suppose for each of the primed quantities Q of Eq. (13.6) we take the form

$$Q' \equiv Q + \delta Q \tag{13.7}$$

where the unprimed quantity refers to the unperturbed system, and the change δQ in passing from the unperturbed to the perturbed system is regarded as small. If we use the form (13.7) everywhere in (13.6), and apply also (13.1) and (13.3), and neglect all terms containing the product of more than one small quantity, we obtain

$$\delta\boldsymbol{O}\,\phi_0 + \boldsymbol{O}\,\delta\phi_0 = (\boldsymbol{O}_1 - \nu\boldsymbol{O}_2)\,\delta\phi_0 + (\delta\boldsymbol{O}_1 - \nu\,\delta\boldsymbol{O}_2)\phi_0 - \delta\nu\,\boldsymbol{O}_2\phi_0 = 0 \tag{13.8}$$

Equation (13.8) is only intended to illustrate the basis of the approach in this section and will not be of direct further use. However, it may be noted that Eq. (13.8) can be formally obtained from (13.1) by application of a differentiation operation through the equation.

Equation (13.8) is still not directly useful, being a very complicated differential equation involving the new operators and the new fluxes. In order to obtain a more tractable form for calculation, we must introduce another function. Let $\psi_0(\mathbf{r},u)$ be an as yet unspecified function of space and lethargy which will later be carefully defined. We multiply Eq. (13.1) through by the function ψ_0 and integrate over the reactor volume V and over all lethargies. If we use Eq. (13.3) and collect terms so as to

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solve for ν, we obtain

$$\nu = \frac{\int_V d\mathbf{r} \int_u \psi_0(\mathbf{r},u)\, O_1\phi_0(\mathbf{r},u)\, du}{\int_V d\mathbf{r} \int_u \psi_0(\mathbf{r},u)\, O_2\phi_0(\mathbf{r},u)\, du} \tag{13.9}$$

Changes are now allowed in the operators O_1 and O_2 and in ν to retain criticality; there are then resultant changes in ϕ_0 and ψ_0, in general (the latter function is not as yet associated with the reactor system, but will be so associated later). The altered quantities may then be formally obtained to a first approximation, as shown in Eq. (13.8), by applying a differentiation through Eq. (13.9). The result is

$$\delta\nu = \frac{\int_V d\mathbf{r} \int_u [\delta\psi_0(O_1 - \nu O_2)\phi_0 + \psi_0(\delta O_1 - \nu\,\delta O_2)\phi_0 + \psi_0(O_1 - \nu O_2)\delta\phi_0]\, du}{\int_V d\mathbf{r} \int_u \psi_0\, O_2\phi_0\, du} \tag{13.10}$$

Because of Eq. (13.1) the first of the three terms under the integral in the numerator of Eq. (13.10) vanishes. We consider now the last of the three terms. In order to evaluate this term we introduce two new operators $O_1^\dagger$ and $O_2^\dagger$, which are related to O_1 and O_2 by the equations

$$\begin{aligned} \int_V d\mathbf{r} \int_u \psi_0 O_1 F\, du &= \int_V d\mathbf{r} \int_u F O_1^\dagger \psi_0\, du \qquad (a) \\ \int_V d\mathbf{r} \int_u \psi_0 O_2 F\, du &= \int_V d\mathbf{r} \int_u F O_2^\dagger \psi_0\, du \qquad (b) \end{aligned} \tag{13.11}$$

where $F(\mathbf{r},u)$ denotes some arbitrary function. With this notation the last integral in (13.10) becomes

$$\int_V d\mathbf{r} \int_u \psi_0(O_1 - \nu O_2)\delta\phi_0\, du = \int_V d\mathbf{r} \int_u \delta\phi_0(O_1^\dagger - \nu O_2^\dagger)\psi_0\, du \tag{13.12}$$

We are now prepared to specify the function $\psi_0(\mathbf{r},u)$, and we require that it satisfy the equation

$$O^\dagger \psi_0(\mathbf{r},u) \equiv (O_1^\dagger - \nu O_2^\dagger)\, \psi_0(\mathbf{r},u) = 0 \tag{13.13}$$

With this choice the last integral of Eq. (13.10) vanishes, and the change in the multiplication constant may be expressed, finally, as

$$-\frac{\delta k}{k} = \frac{\delta\nu}{\nu} = \frac{\int_V d\mathbf{r} \int_u \psi_0(\delta O_1 - \nu\,\delta O_2)\phi_0\, du}{\nu \int_V d\mathbf{r} \int_u \psi_0 O_2\phi_0\, du} \tag{13.14}$$

Equation (13.14) is of the general form in which the principal results of this chapter will be cast. The fractional change in the multiplication constant is calculated, to first order, as an integral over the changes in the reactor operators, with a pair of weighting functions ψ_0 and ϕ_0 associ-

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ated with the unperturbed system. [The integral in the denominator of Eq. (13.14) may be regarded as a normalization factor.] The symbol ϕ_0 denotes the neutron flux in the original system; ψ_0 is closely related to the neutron flux and is calculated from Eq. (13.13), which involves only parameters of the original reactor. Note that the changes in the flux are not calculated in this approximation. It is, of course, possible by an extension of this method to obtain approximations to the perturbed flux; however, such a calculation is generally quite involved and the results are seldom of direct interest for reactor work.

b. The Statistical Weight and the Importance Function. Let us turn to a consideration of the physical interpretation of the result given in Eq. (13.14). The situation discussed above in which the neutron absorption cross section is locally increased will serve as an illustrative example. Suppose we concentrate on a thermal reactor and use the one-velocity approximation [thus eliminating the energy dependence in Eq. (13.14)] so that the extra absorber is for thermal neutrons. Let the additional absorption, which is introduced nonuniformly, be denoted by $\delta\Sigma_a(\mathbf{r})$; further, choose the normalization of ψ_0 and ϕ_0 so that

$$\nu \int_V \psi_0(\mathbf{r})\, O_2\, \phi_0(\mathbf{r})\, d\mathbf{r} = \frac{1}{A} \tag{13.15}$$

Then, from (13.14)

$$-\frac{\delta k}{k} = A \int_V \delta\Sigma_a(\mathbf{r})[\psi_0(\mathbf{r})\ \phi_0(\mathbf{r})]\, d\mathbf{r} \tag{13.16}$$

As a further illustration of this point, suppose the absorption is changed only at point $\mathbf{r}_0$, so that $\delta\Sigma_a(\mathbf{r}) = a\,\delta(\mathbf{r} - \mathbf{r}_0)$; then,

$$\psi_0(\mathbf{r}_0) = \frac{-\delta k/k}{Aa\phi_0(\mathbf{r}_0)}$$

This shows that $\psi_0(\mathbf{r}_0)$, with appropriate normalization, is just the fractional change in the multiplication constant per neutron absorbed per unit time.

As Eq. (13.16) shows, changes in the absorption cross section at various points in the reactor are weighted by a function

$$W(\mathbf{r}) = \psi_0(\mathbf{r})\ \phi_0(\mathbf{r}) \tag{13.17}$$

which is called the *statistical weight.* Another interesting interpretation of Eq. (13.16) is also possible. If $\phi_0(\mathbf{r})$ is a good approximation to the flux in the perturbed system (as is essentially assumed in this treatment), the product $\delta\Sigma_a(\mathbf{r})\ \phi_0(\mathbf{r})$ is a good estimate of the *additional* rate of removal of neutrons at the point $\mathbf{r}$ in the perturbed system over the original system. Then from (13.16) it may be seen that this extra removal of neutrons is weighted by the function $\psi_0(\mathbf{r})$, which is, in this sense, called the *importance function.* For the simple one-velocity case

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(as will be shown later), the function $\psi_0(\mathbf{r})$ is just equal to the unperturbed flux $\phi_0(\mathbf{r})$. (Here we are not concerned with normalization of either the statistical weight or the importance function, but only with the relative values of either from space point to space point.) Thus for the one-velocity case, the importance (to the multiplication constant of the system) of the introduction of extra neutrons (or the removal of neutrons) from the system varies as the original neutron flux, that is,

$$\begin{matrix}\text{Importance of neutrons at } \mathbf{r} \\ \text{(in one-velocity model)}\end{matrix} \sim \psi_0(\mathbf{r}) = \phi_0(\mathbf{r}) \tag{13.18}$$

Further, the statistical weight varies as the square of the flux, that is,

$$W(\mathbf{r}) \sim \phi_0^2(\mathbf{r}) \qquad \text{in one-velocity model} \tag{13.19}$$

Thus the influence of extra thermal neutron absorption placed in a thermal reactor is very strong at the center, where the flux is high. In a slab reactor, for example, the statistical weight for absorption would fall off as $\cos^2 x$. (The implications of this behavior are discussed further in subsequent analysis when the one-velocity model is considered in detail.)

Of course, such results as expressed in general terms by Eq. (13.14) encompass much more complicated behaviors than the simple one-velocity absorption. Even in the one-velocity case changes in the diffusion coefficient bring about somewhat different considerations than those presented above; the energy dependence of the changes has not been discussed here at all. Still it is a broadly applicable conclusion that the importance function tends to behave space-wise as the neutron flux, and thus the statistical weight, in so far as changes in the absorption and fission cross sections are concerned, tends to vary spatially as the square of the flux.

c. The Adjoint Function and Operators. We return now to the formal outline of the perturbation method as started in Sec. 13.1a. In the usual mathematical terminology the operators $O_1^\dagger$ and $O_2^\dagger$ as defined by Eqs. (13.11*a*) and (13.11*b*) are called the "adjoint operators" to O_1 and O_2; similarly ψ_0, which satisfies Eq. (13.13) involving the adjoint operators to those of Eq. (13.1) for ϕ_0, is said to be the "adjoint function" to ϕ_0. It is clear that the emphasis of the succeeding discussion must be placed on the calculation of the adjoint function ψ_0 for the various models considered. It is also clear, from Eq. (13.13), that the calculation of the adjoint function is very much like the calculation of the neutron flux itself. With appropriately chosen boundary conditions [not specified in order to reach the result (13.14), and which will be selected in the examples below for computational convenience] the machinery used for the calculation of the flux may be applied directly to the construction of the adjoint function. The advantage of this procedure is seen from Eq. (13.14); the two calculations one for ϕ_0 and one for ψ_0, on the original

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system, allow an easy evaluation of the effects of many types and distributions of small changes from the original. In fact, in some cases, as for the one-velocity model mentioned above, only one calculation is required since the adjoint function is just equal to the flux itself. This situation arises when the operators in the reactor equation are *self-adjoint*, i.e., when $O^\dagger$ and O are identical. Before turning to the calculation of adjoint functions for special models, however, we shall examine one general property of the adjoint function.

Consider the following problem. A burst of neutrons of lethargy u is introduced into a just-critical reactor at space point $\mathbf{r}'$, and it is required that the resultant neutron flux be found as a function of space and lethargy for all subsequent time. Since the system is critical, a neutron introduced into the system will *on the average* just reproduce itself indefinitely, giving rise to a persistent steady-state flux corresponding to a neutron density of just one neutron in the reactor volume. Now the phrase "on the average" here implies an average taken over neutrons introduced into the system in the fundamental mode. It is clear that there will arise a persistent, ultimate flux (in the fundamental mode) from the introduction of a neutron at any point in the system, but the magnitude of this ultimate flux will vary with the point of introduction. For instance, it would be expected that a neutron introduced near the outside would give rise to a much smaller ultimate flux than that from the introduction of a neutron near the center of the system, since the fraction of the neutron which would "leak out" before the flux settled down into the fundamental mode would be higher in the former case. We now attempt to solve this problem in general terms and find the form and magnitude of the ultimate flux.

The flux in the reactor for all time $t > 0$ is taken to satisfy the equation

$$O\phi(\mathbf{r},u,t;\mathbf{r}',u') = -\frac{1}{v}\frac{\partial}{\partial t}\phi(\mathbf{r},u,t;\mathbf{r}',u') + \delta(t)\,\delta(\mathbf{r}')\,\delta(u') \quad (13.20)$$

Here $\phi(\mathbf{r},u,t;\mathbf{r}',u')$ is the neutron flux of lethargy u, at space point $\mathbf{r}$, at time t after the introduction of one neutron of lethargy u' at position $\mathbf{r}'$. The symbol O denotes the reactor operator of Eq. (13.1). The approach to the solution of this problem is through expansion of the flux in space modes as was done for a more specialized case in Sec. 5.4. Suppose that it is possible to find functions $\phi_i(\mathbf{r},u)$ which satisfy the following equation:

$$O\phi_i(\mathbf{r},u) = \lambda_i\phi_i(\mathbf{r},u) \qquad i = 0, 1, 2, \ldots \quad (13.21)$$

The eigenvalues λ_i are assumed to be arranged in order of increasing magnitude with increasing index, and λ_0 is taken to be zero so that $\phi_0(\mathbf{r},u)$ is just the reactor flux of Eq. (13.1). The basic assumption here is that the eigenvalues of Eq. (13.21) can be arranged in order of increasing magnitude of the *real* part. The eigenvalues and eigenfunctions of

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(13.21) may, of course, be complex; indeed, little is known of them except for the simplest cases, e.g., the one-velocity case discussed later in this chapter. However, it does seem reasonable to expect, from the results obtained below for the time-dependent behavior, that when the operator O corresponds to a physically operating system the assumption used here must hold. In the case of complex eigenfunctions, more convenient definitions of the adjoint operators might be chosen. It is important to point out again that the present discussion is merely an outline of a procedure which might be carried out in detail for particular cases.

We now expand the time-dependent flux of Eq. (13.20) in the functions ϕ_i; thus,

$$\phi(\mathbf{r},u,t;\mathbf{r}',u') = \sum_{i=0}^{\infty} a_i(t;\mathbf{r}',u')\ \phi_i(\mathbf{r},u) \tag{13.22}$$

Here a departure from the procedure of Sec. 5.4 must be made because the members of the set ϕ_i are not necessarily orthogonal. However, we can construct a set of functions $\psi_j(\mathbf{r},u)$ which are orthogonal to the set $\phi_i(\mathbf{r},u)$ in the sense

$$\int_V d\mathbf{r} \int_u \psi_j(\mathbf{r},u)\ \phi_i(\mathbf{r},u)\ du = \delta_{ij} \tag{13.23}$$

Further, the set $\psi_j(\mathbf{r},u)$ is unique (except for normalization, which is unimportant here; we are not concerned here even with the dimensionality of the adjoint function which can be specified; but this also is not important). Now multiply Eq. (13.21) through by the function $\psi_j(\mathbf{r},u)$ and integrate over space and lethargy.

$$\int_V d\mathbf{r} \int_u \psi_j O\phi_i\, du = \lambda_i \int_V d\mathbf{r} \int_u \psi_j\phi_i\, du = \lambda_i\delta_{ij} \tag{13.24}$$

If we introduce the adjoint operator $O^\dagger$ of Eq. (13.13),

$$\int_V d\mathbf{r} \int_u \psi_j O\phi_i\, du = \int_V d\mathbf{r} \int_u \phi_i[O^\dagger\psi_j]\, du = \lambda_i\delta_{ij} \tag{13.25}$$

The form of Eq. (13.25), along with the uniqueness of the functions ψ_j, shows that the ψ_j must satisfy the equation

$$O^\dagger\psi_j(\mathbf{r},u) = \lambda_j\psi_j(\mathbf{r},u) \tag{13.26}$$

Thus the orthogonal functions are just the adjoint functions, and $\psi_0(\mathbf{r},u)$ is just the flux adjoint of Eq. (13.13).

We now introduce the expansion (13.22) into Eq. (13.20), multiply through by $\psi_j(\mathbf{r},u)$, and integrate over space and lethargy. The resultant equations for the coefficients in the expansion are

$$\lambda_j a_j(t;\mathbf{r}',u') = -\frac{1}{v}\,\dot{a}_j(t;\mathbf{r}',u') + \delta(t)\ \psi_j(\mathbf{r}',u') \tag{13.27}$$

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This equation shows the expected result that the coefficients of all the higher modes decay exponentially in time ($t > 0$) and that the coefficient of the fundamental mode is a constant after the burst. Thus the ultimate form of the flux ϕ_∞ long after the burst is

$$\phi_\infty(\mathbf{r},u;\mathbf{r}'u') = v\,\psi_0(\mathbf{r}',u')\,\phi_0(\mathbf{r},u) \tag{13.28}$$

We find, then, that the relative values of the adjoint function to the neutron flux from one space point to another gives directly the ratio of the magnitudes of the ultimate fluxes from bursts of neutrons introduced at these points. Again it may be seen that the adjoint function measures the importance of the introduction (or removal) of neutrons at a given point in a reactor assembly.

It should be pointed out that this rather interesting approach to the adjoint function is not of direct importance for developing useful results which will be obtained in this chapter. In fact, this line of thinking can be overextended if, for example, it is taken to imply that an extra *source* of neutrons can be compensated by an appropriate change in the multiplication constant.

In the sections which follow we develop with the aid of various reactor models the application of the techniques discussed above (as well as others) to the calculation of the effects of parameter changes on the reactor system.

In many cases it is possible to use Eqs. (13.11) to find explicit forms for the adjoint operators so that (13.13) may be written as an explicit set of equations for $\psi_0(\mathbf{r},u)$ (along with appropriate boundary conditions). In the general case this particular approach leads to the consideration of the adjoint to the Boltzmann equation in which all the generality might be preserved, including angular dependence of the functions. On the other hand, the procedure represented by (13.20) might correspond to the time-dependent Boltzmann equation, and the eventual flux resulting from the introduction of one neutron might be found in the most general case. However, such ambitious programs must meet with exactly the same difficulties as are encountered in applying the Boltzmann equation directly to reactor-flux calculations. Further, it is hardly desirable to know the importance function to a better approximation than the flux itself is known, since the basic result, as typified by Eq. (13.14), uses the two functions symmetrically. Consequently, it is customary and proper to base perturbation theory and the calculation of the adjoint function on the model used for the calculation of the flux itself.

13.2 Perturbation in the Infinite Multiplying Medium

a. The Unperturbed System. The discussion of the case of the infinite multiplying medium, in which the disturbances are changes in the energy-

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dependent cross sections only, is of value as an introduction to methods which will be of use in subsequent analyses. The full power of the perturbation techniques is not needed, since all the important results may be easily obtained from the explicit form of the criticality condition (4.162); however, this very same fact makes this case an interesting one for the introduction to the methods since the results obtained by somewhat elaborate methods may be easily checked from considerations of the criticality condition.

For this treatment we consider the lethargy range $0 \leq u \leq u_0$, where the two limiting values of the lethargy are chosen so that practically no neutrons are found in the system with lethargies beyond these limits. Thus, lethargy zero should correspond to an energy which is a practical upper limit for the fission spectrum, and lethargy u_0 should correspond to an energy just below thermal. All functions are treated as continuously varying over this lethargy range, and the special conditions in the region of thermal energy are imagined to be handled by allowing the quantity ξ (the average logarithmic changes in the energy per collision) to take on very small values in the thermal range. All physically interesting results can be obtained as limiting cases of the lethargy-dependent functions obtained in this way.

The results for the unperturbed system are summarized from the derivations of Sec. 4.6. The appropriate form of the neutron-balance relation (4.145) for the case of the normalized fission spectrum $\chi(u)$ is seen to be

$$\frac{d}{du} q(u) + \beta(u)\, q(u) = \nu\chi(u) \int_0^{u_0} \gamma(u')\, q(u')\, du' \tag{13.29}$$

where $\beta(u)$ and $\gamma(u)$ are defined by Eqs. (4.161). The solution to this equation is

$$q(u) = \int_0^u \chi(u') \exp\left[- \int_{u'}^u \beta(u'')\, du'' \right] du' \tag{13.30}$$

and for the criticality condition we previously obtained the form [cf. Eq. (4.162)]

$$1 = \nu \int_0^{u_0} \gamma(u)\, du \int_0^u \chi(u') \exp\left[- \int_{u'}^u \beta(u'')\, du'' \right] du' \tag{13.31}$$

b. The Adjoint Equations. The first step in the derivation of the importance function by the technique of adjoint functions is to cast Eq. (13.29) into the form of Eqs. (13.1) and (13.3) and to multiply through by a function $q^\dagger(u)$, as yet unspecified. The next step is to integrate over lethargy and then solve for ν:

$$\nu = \frac{\displaystyle\int_0^{u_0} q^\dagger(u) \frac{d}{du} q(u)\, du + \int_0^{u_0} q^\dagger(u)\, \beta(u)\, q(u)\, du}{\displaystyle\int_0^{u_0} q^\dagger(u)\, \chi(u)\, du \int_0^{u_0} \gamma(u')\, q(u')\, du'} \tag{13.32}$$

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Now allow changes in the cross sections, implied in β and γ, and consequent changes in q and $q^\dagger$. (Other changes are also imaginable: the fission spectrum χ might be altered by, for example, the introduction of new species of fissionable nuclei; this is a rather unimportant case, and the treatment presented here may be easily generalized to include it. The limit u_0 might change, if it is imagined that u_0 corresponds roughly to thermal energy, if the temperature were changed. However, as chosen, u_0 corresponds to an energy so far below thermal that practically no neutrons are found there, and the changes consequent upon the temperature change may be regarded as changes in β and γ.) The application of (13.29) eliminates one integral, and the result may be written

$$\delta\nu = \frac{\int_0^{u_0} q^\dagger(u)\ \delta\beta(u)\ q(u)\ du - \nu \int_0^{u_0} \chi(u)\ du \int_0^{u_0} q^\dagger(u')\ \delta\gamma(u')\ q(u')\ du'}{\int_0^{u_0} q^\dagger(u)\ \chi(u)\ du \int_0^{u_0} \gamma(u')\ q(u')\ du'}$$

$$- \nu \frac{\int_0^{u_0} \left[q^\dagger(u) \frac{d}{du} \delta q(u) + q^\dagger(u)\, \beta(u)\, \delta q(u) - \nu\chi(u) \int_0^{u_0} \gamma(u')\, \delta q(u')\, du' \right] du}{\int_0^{u_0} q^\dagger(u)\ \chi(u)\ du \int_0^{u_0} \gamma(u')\ q(u')\ du'} \tag{13.33}$$

The first term on the right-hand side of (13.33), involving the cross-section changes $\delta\beta$ and $\delta\gamma$, is in the desired form; the last term on the right corresponds to the last term on the right of Eq. (13.10) and will be treated analogously. We first look for the operators adjoint to the operators of the differential equation (13.29). The only one that presents any difficulty is the differential operator d/du; following the requirement of Eq. (13.11a), consider

$$\int_0^{u_0} q^\dagger\ (u) \frac{d}{du} \delta q(u)\ du = \int_0^{u_0} \frac{d}{du} [q^\dagger(u)\ \delta q(u)]\ du - \int_0^{u_0} \delta q(u) \frac{d}{du} q^\dagger(u)\ du \tag{13.34}$$

But,

$$\int_0^{u_0} \frac{d}{du} [q^\dagger(u)\ \delta q(u)]\ du = q^\dagger(u_0)\ \delta q(u_0) - q^\dagger(0)\ \delta q(0) = 0 \tag{13.35}$$

The last step follows since the slowing-down density, originally and after the change in constants, is zero at both $u = 0$ and $u = u_0$; consequently, no specification of $q^\dagger$ at either limit is required. For this case, then, the operator adjoint to d/du is $-d/du$. The numerator of the

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last term on the right-hand side of (13.33) may be written

$$\int_0^{u_0} \left[q^\dagger(u) \frac{d}{du} \delta q(u) + q^\dagger(u)\, \beta(u)\, \delta q(u) - \nu \chi(u) \int_0^{u_0} \gamma(u')\, \delta q(u')\, du' \right] du$$
$$= \int_0^{u_0} \delta q(u) \left[- \frac{d}{du} q^\dagger(u) + \beta(u)\, q^\dagger(u) - \nu\gamma(u) \int_0^{u_0} \chi(u')\, q^\dagger(u')\, du' \right] du \tag{13.36}$$

Thus, by a proper choice of the function $q^\dagger(u)$, the entire last term on the right-hand side of Eq. (13.33) may be made to vanish. From (13.36) the required specification of $q^\dagger(u)$ is seen to be

$$- \frac{d}{du} q^\dagger(u) + \beta(u)\, q^\dagger(u) = \nu\gamma(u) \int_0^{u_0} \chi(u')\, q^\dagger(u')\, du' \tag{13.37}$$

By comparison with (13.29), it may be seen that besides the sign change on the derivative term the fission spectrum and the fission cross section have exchanged roles for the calculation of the adjoint slowing-down density. The calculation of the adjoint slowing-down density is very much like a "backward" calculation of the slowing-down density itself; the adjoint function is started in the thermal region with a high value, the fission cross section acting as a source. The "adjoint neutrons" are then followed *up* in energy (because of the negative derivative) with losses occurring by absorption just as in the case of the slowing-down density. Finally, at high energies the value of the fission spectrum multiplied by the adjoint function determines the number of "adjoint neutrons" started up the scale again from thermal. This feature of the lethargy adjoint functions has lead to the use of the expression "forward" calculation when referring to the calculation of the slowing-down density or the flux from the slowing-down equation (13.29), as contrasted with the adjoint or "backward" calculation of Eq. (13.37).

The normalization of the adjoint functions is still to be chosen; we take

$$\nu \int_0^{u_0} \chi(u)\, q^\dagger(u)\, du \equiv 1 \tag{13.38}$$

which is analogous to (13.31), if we use (13.30) for $q(u)$ and note that γ and χ have exchanged roles. With this requirement we find the solution to Eq. (13.37) for the adjoint function to be

$$q^\dagger(u) = \int_u^{u_0} \gamma(u') \exp\left[- \int_u^{u'} \beta(u'')\, du'' \right] du' \tag{13.39}$$

which may be compared with (13.30) for the "forward" function. Finally, from (13.33) we obtain for the change in the eigenvalue, or

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multiplication constant,

$$\frac{\delta\nu}{\nu} = -\frac{\delta k}{k} = \nu\left[\int_0^{u_0} q^\dagger(u)\,\delta\beta(u)\,q(u)\,du - \int_0^{u_0}\delta\gamma(u)\,q(u)\,du\right] \quad (13.40)\star$$

In this result, $q^\dagger(u)$ indeed plays the role of an importance function for the introduction of neutrons, as discussed in Sec. 13.1. In the first integral on the right-hand side of Eq. (13.40), $\delta\beta(u)q(u)$ are the extra neutrons removed per unit volume per unit time by the additional absorption cross section, assuming that the change in the slowing-down density itself is small. Then $q^\dagger(u)$ is a measure of the (relative) importance of removing a neutron at lethargy u. Note that an importance function is not needed in the second integral; $\delta\gamma(u)q(u)$ gives the extra fissions, and all fission neutrons appear with the same distribution in lethargy regardless of the energy of the neutron causing the fission (at least by assumption in this model).

c. The Importance Function. By substituting the explicit forms for $q(u)$ and $q^\dagger(u)$, from (13.30) and (13.39), respectively, into Eq. (13.40), a complete expression for the fractional change in multiplication constant may be obtained. This is not of particular interest, however, and is not recorded here. In fact, as mentioned previously, the entire technique used here is unnecessary for this simple case, and the value of the discussion is the ease with which the basic ideas, to be used later, can be exhibited. For example, the result (13.40), with the explicit forms of the functions used, may be obtained directly by a differential technique applied to the criticality condition, Eq. (13.31). Even the importance function Eq. (13.39) may be readily constructed from a physical understanding of the model; the relative importance of a neutron introduced at lethargy u is just proportional to the number of fissions it would eventually cause in slowing down from u toward u_0. Equation (13.39) is the mathematical statement of this fact. In the very simple case in which fissions are caused only by thermal neutrons, it is evident that the relative importance of a neutron introduced at lethargy u is just the probability that the neutron escape absorption in slowing down to thermal; thus in this case, wherein fissions are caused only by thermal neutrons, both physical considerations and Eq. (13.39) lead to the result

$$\text{Importance function} \sim q^\dagger(u) \sim p(u;u_{\text{th}}) = e^{-\int_u^{u_{\text{th}}}\beta(u')\,du'} \quad (13.41)\star$$

The behavior of the slowing-down density and its adjoint are presented in Fig. 13.1. The thermal range is represented as a small interval in lethargy just below $u = u_0$; throughout this interval the functions are rapidly varying. Changes in the absorption cross section, as seen from Eq. (13.40), are weighted by the function $q^\dagger(u)q(u)$; from Fig. 13.1 this is seen to be nearly a constant over the lethargy interval below the fission

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spectrum and above thermal. In fact this may be easily seen to be true in the limiting case in which fissions occur only at thermal and the fission

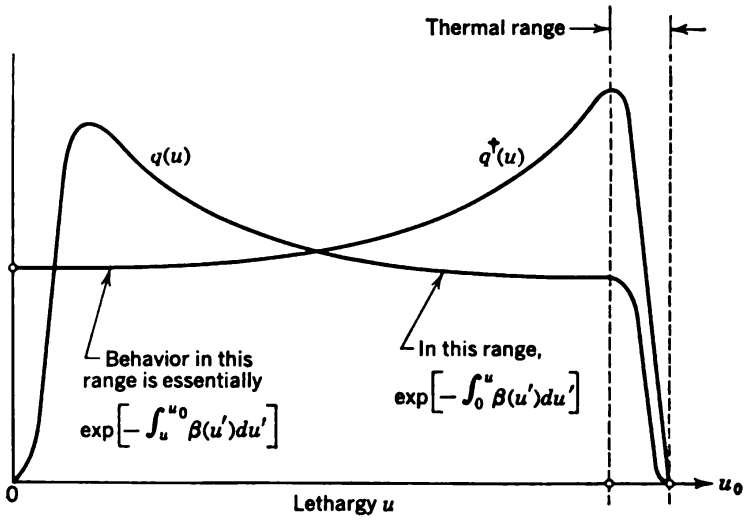

FIG. 13.1 The slowing-down density $q(u)$ and the importance function $q^\dagger(u)$ for the infinite multiplying medium.

spectrum is represented as a delta function at the zero of lethargy; then, in the intermediate lethargy range

$$q(u) \sim e^{-\int_0^u \beta(u')\,du'} \tag{13.42}$$

$$q^\dagger(u) \sim e^{-\int_u^{u_{th}} \beta(u')\,du'} \tag{13.43}$$

and

$$q^\dagger(u)\, q(u) = \text{constant in lethargy} = \text{"statistical weight"} \tag{13.44}\star$$

[Actually, this result can be somewhat misleading unless it is noted that the statistical weight of Eq. (13.44) really applies to changes in $\beta \equiv \Sigma_a/\xi\Sigma_t$, as seen from Eq. (13.40).]

13.3 Perturbation in the One-velocity Model of the Bare Reactor

a. Eigenfunctions of the Bare Reactor. The method of adjoint functions introduced in Sec. 13.1, and applied to the multiplying medium in Sec. 13.2, may also be applied directly to the one-velocity diffusion equation. The general development for these systems is given in Sec. 13.4. In the particular case of the bare reactor, we discard the method of adjoint functions in favor of an alternate approach which yields more information.

Consider a bare reactor in the one-velocity approximation in which the flux satisfies the equation

$$-\nabla \cdot D\nabla\phi + \Sigma_a\phi = \nu\Sigma_f p_{fl}\phi \tag{13.45}$$

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In this equation the parameters D, Σ_a, Σ_f, and p_{fl} are, in general, functions of position; the parameter p_{fl} is taken to represent the losses in the fast range, so that in the application of any of the results of this section p_{fl} may be replaced by whatever estimate of the fast losses, leakage and absorption, is regarded as satisfactory in a particular case. In the undisturbed homogeneous system, the unperturbed flux $\phi_0(\mathbf{r})$ satisfies the equation

$$-D_0\nabla^2\phi_0(\mathbf{r}) + \Sigma_a^{(0)}\phi_0(\mathbf{r}) = \nu\Sigma_f^{(0)}p_{fl}\phi_0(\mathbf{r}) \tag{13.46}$$

where D_0, $\Sigma_a^{(0)}$, and $\Sigma_f^{(0)}$ are independent of position. The solution to this equation may be written in the form

$$\phi_0(\mathbf{r}) = cF_0(\mathbf{r}) \tag{13.47}$$

where, in the usual way, $F_0(\mathbf{r})$ satisfies

$$\nabla^2F_0(\mathbf{r}) + B_0^2F_0(\mathbf{r}) = 0 \tag{13.48}$$

and B_0 is determined from the requirement

$$F_0(\tilde{\mathbf{R}}) = 0 \tag{13.49}$$

The vector $\tilde{\mathbf{R}}$ denotes the outer extrapolated boundary of the reactor. The critical equation for this system is

$$B_0^2(\tilde{\mathbf{R}}) = \frac{\nu\Sigma_f^{(0)}p_{fl} - \Sigma_a^{(0)}}{D_0} \tag{13.50}$$

It will be of interest in connection with the perturbations to be discussed here to consider the complete set of eigenfunctions $F_n(\mathbf{r})$ of the bare reactor, of which $F_0(\mathbf{r})$ is the lowest mode. These functions are solutions to the equation [cf. Eq. (8.260)]

$$\nabla^2F_n(\mathbf{r}) + B_n^2F_n(\mathbf{r}) = 0 \qquad \text{with } F_n(\tilde{\mathbf{R}}) = 0 \tag{13.51}$$

It is convenient to construct this set so that the $F_n(\mathbf{r})$ are orthonormal; thus [cf. Eq. (8.249)]

$$\int_V F_n(\mathbf{r})\,F_m(\mathbf{r})\,d\mathbf{r} = \delta_{mn} \tag{13.52}$$

where V denotes the volume enclosed within the surface represented by $\tilde{\mathbf{R}}$. The set $F_n(\mathbf{r})$ may be ordered to correspond to the eigenvalues B_n such that $B_n^2 < B_{n+1}^2$. It should be noted that in the present formulation this set of functions must be constructed in complete generality in order to meet physical situations involving nonsymmetric perturbations.[1] Appropriate functions for the four elementary reactor geometries are listed in Table 13.1.

[1] For example, the block-reactor eigenfunctions must include the sine functions as well as the cosine, and in the spherical and cylindrical cases, the angular harmonics are required.

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b. Change in Multiplication Constant. Now suppose that in the bare system under consideration a change is made in the cross sections; thus,

$$\Sigma_a(\mathbf{r}) \equiv \Sigma_a^{(0)} + \delta\Sigma_a(\mathbf{r}) \qquad \Sigma_f(\mathbf{r}) \equiv \Sigma_f^{(0)} + \delta\Sigma_f(\mathbf{r}) \tag{13.53}$$

(Problems involving changes in the diffusion coefficient are deferred until the next section.) The flux in this perturbed system may be defined, then,

$$\phi(\mathbf{r}) \equiv \phi_0(\mathbf{r}) + \delta\phi(\mathbf{r}) \tag{13.54}$$

In order to maintain steady-state operation, we must allow a hypothetical change in ν:

$$\nu \longrightarrow \nu + \delta\nu \tag{13.55}$$

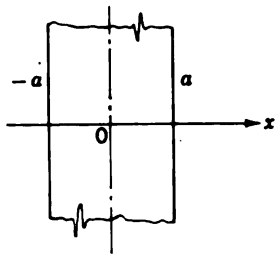
The substitution of these forms into Eq. (13.45), along with the application of (13.46), yields

$$\begin{aligned} -D_0\nabla^2\delta\phi(\mathbf{r}) + \delta\Sigma_a(\mathbf{r})\,\phi_0(\mathbf{r}) + \Sigma_a^{(0)}\delta\phi(\mathbf{r}) \\ = \nu\Sigma_f^{(0)}p_{fl}\delta\phi(\mathbf{r}) + \nu\delta\Sigma_f(\mathbf{r})\,p_{fl}\phi_0(\mathbf{r}) + \delta\nu\Sigma_f^{(0)}p_{fl}\phi_0(\mathbf{r}) \end{aligned} \tag{13.56}$$

where we have neglected products of the various differential quantities. This equation may be solved for the fractional change in k by expanding

TABLE 13.1 ORTHONORMAL SET OF EIGENFUNCTIONS FOR THE FOUR ELEMENTARY REACTOR GEOMETRIES

1. Infinite slab reactor



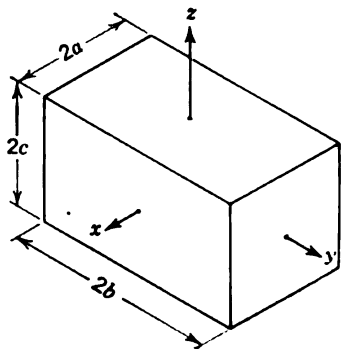
$$F_n(x) \equiv \begin{cases} A_n \cos B_n x & n \text{ even} \\ A_n \sin B_n x & n \text{ odd} \end{cases}$$

$$B_n \equiv \frac{\pi}{2\tilde{a}}(n+1) \qquad A_n \equiv \frac{1}{\tilde{a}^{\frac{1}{2}}} \qquad n = 0, 1, 2, \ldots$$

2. Rectangular block reactor

$$F_{ijk}(x,y,z) \equiv A_{ijk}f(l_i x)g(m_j y)h(n_k z)$$
$$B_{ijk}^2 \equiv l_i^2 + m_j^2 + n_k^2$$

with $i, j, k = 0, 1, 2, \ldots$, and



$$l_i \equiv \frac{\pi}{2\tilde{a}}(i+1) \qquad m_j \equiv \frac{\pi}{2\tilde{b}}(j+1)$$

$$n_k \equiv \frac{\pi}{2\tilde{c}}(k+1)$$

$$A_{ijk} \equiv (\tilde{a}\tilde{b}\tilde{c})^{-\frac{1}{2}}$$

and

$$f(l_i x) \equiv \begin{cases} \cos l_i x \\ \sin l_i x \end{cases} \qquad g(m_j y) \equiv \begin{cases} \cos m_j y \\ \sin m_j y \end{cases} \qquad h(n_k z) \equiv \begin{cases} \cos n_k z & i,j,k \text{ even} \\ \sin n_k z & i,j,k \text{ odd} \end{cases}$$

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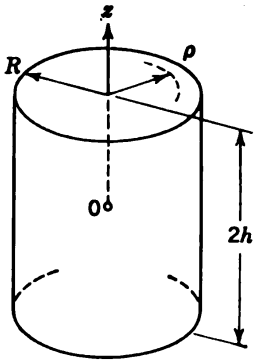
TABLE 13.1 ORTHONORMAL SET OF EIGENFUNCTIONS FOR THE FOUR ELEMENTARY REACTOR GEOMETRIES (*Continued*)

3. Cylindrical reactor



$$F_{ijk}(\rho,\theta,z) \equiv A_{ijk} f_{jk}(\rho)\, g_j(\theta)\, h_i(z)$$

$$B_{ijk}^2 \equiv l_i^2 + \left(\frac{\nu_{jk}}{\tilde{R}}\right)^2$$

with $i, j = 0, 1, 2, \ldots$, and $k = 1, 2, 3, \ldots$

$l_i \equiv \dfrac{\pi}{2\tilde{h}}(i+1)$ $\qquad$ $\nu_{jk} \equiv k$th zero of Bessel function J_j

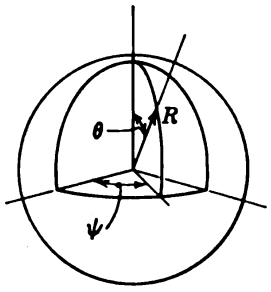
and

$$A_{ijk} \equiv \{\pi\tilde{h}[-\tfrac{1}{2}\tilde{R}^2 J_{j-1}(\nu_{jk}) J_{j+1}(\nu_{jk})]\}^{-\frac{1}{2}}$$

$$f_{jk}(\rho) \equiv J_j(\nu_{jk}\rho/\tilde{R}) \qquad g_j(\theta) \equiv \begin{cases}\cos j\theta \\ \sin j\theta\end{cases} \qquad h_i(z) \equiv \begin{cases}\cos l_i z & i, j \text{ even} \\ \sin l_i z & i, j \text{ odd}\end{cases}$$

For each value of j there are two functions F_{ijk}, one with $\cos j\theta$, the other with $\sin j\theta$.

4. Spherical reactor



$$F_{ijk}(r,\theta,\psi) \equiv A_{ijk} f_{ik}(r)\, g_{ij}(\theta)\, h_j(\psi)$$

$$B_{ik}^2 \equiv \left(\frac{\mu_{ik}}{\tilde{R}}\right)^2$$

with $\mu_{ik} \equiv k$th zero of the spherical Bessel function[a] j_i, and the indices $i, j = 0, 1, \ldots$, and $k = 1, 2, \ldots$.

$$A_{ijk} \equiv \left\{\pi\left[-\frac{1}{2}\tilde{R}^3 j_{i-1}(\mu_{ik}) j_{i+1}(\mu_{ik})\right]\frac{2(i+j)!}{(2i+1)(i-j)!}\right\}^{-\frac{1}{2}}$$

$f_{ik}(r) \equiv j_i(\mu_{ik} r/\tilde{R})$ spherical Bessel function

$g_{ij}(\theta) \equiv P_i^j(\cos\theta)$ associated Legendre function

$h_j(\psi) \equiv \begin{cases}\cos j\psi \\ \sin j\psi\end{cases}$ for each value of j there are two functions F_{ijk}, one with $\cos j\psi$, the other with $\sin j\psi$

[a] P. M. Morse and H. Feshbach," Methods of Theoretical Physics," McGraw-Hill Book Company, Inc., New York, 1953.

the perturbed flux (13.54) in a series using the complete set of eigenfunctions given by (13.51). Therefore set

$$\phi(\mathbf{r}) \equiv \sum_{n=0}^{\infty} a_n F_n(\mathbf{r}) \equiv cF_0(\mathbf{r}) + \sum_{n=1}^{\infty} a_n F_n(\mathbf{r}) \tag{13.57}$$

Here the contribution from the original flux has been separated out so that the last sum on the right of Eq. (13.57) represents $\delta\phi(\mathbf{r})$. Note also that we have taken $a_0 \equiv c$ for convenience since the magnitude of

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$\delta\phi$ relative to ϕ_0 is not of interest here; only the spatial variation of $\delta\phi$ will be considered in the present calculation. The analysis of the perturbed flux, taking into account its actual magnitude and buildup in time, is an entirely different problem and is not examined here.

The substitution of the expression (13.57) into (13.56), followed by multiplication through by $\phi_0(\mathbf{r})$ and integration over the volume enclosed by the extrapolated outer surface of the reactor, gives the following result:

$$-\frac{\delta\nu}{\nu} = \frac{\delta k}{k} = \frac{\int_V \phi_0(\mathbf{r})[\nu p_{fl}\delta\Sigma_f(\mathbf{r}) - \delta\Sigma_a(\mathbf{r})]\phi_0(\mathbf{r})\,d\mathbf{r}}{\nu p_{fl}\Sigma_f^{(0)}\int_V \phi_0^2(\mathbf{r})\,d\mathbf{r}} \qquad (13.58)\star$$

In this expression the change in the multiplication constant resulting

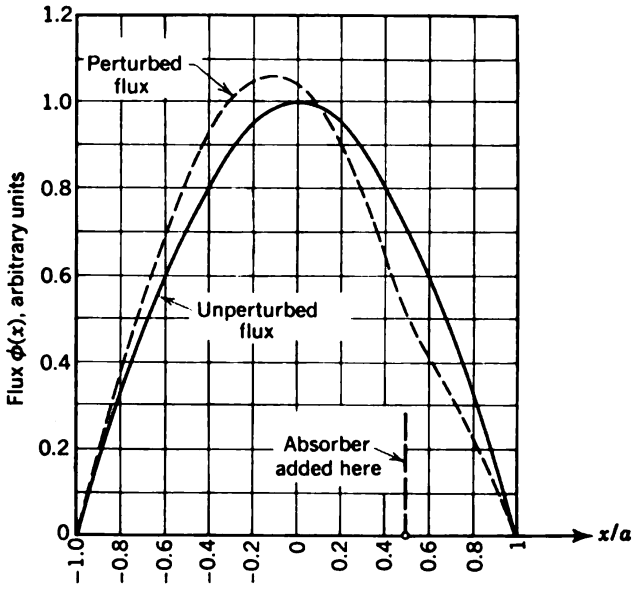

FIG. 13.2 Perturbed one-velocity flux in a bare slab reactor due to the addition of small absorber at $x = a/2$.

from a change in the parameters is calculated in the usual form, from which it is evident that for this case

$$\begin{aligned} \text{Importance function} &\sim \phi_0(\mathbf{r}) \\ \text{Statistical weight} &\sim \phi_0^2(\mathbf{r}) \end{aligned} \qquad (13.59)\star$$

The same result will be obtained in Sec. 13.4, somewhat more generally, and will be discussed in more detail there.

For the present section, we shall concentrate on the extra information to be obtained by the eigenfunction expansion. If Eq. (13.57) is multiplied through by $F_m(\mathbf{r})$, where $m \neq 0$, and integrated over the volume V, an expression may be obtained for the coefficients a_n of the expansion for $\delta\phi(\mathbf{r})$. With the application of Eqs. (13.50) and (13.52) the result may

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be written

$$a_n = \frac{\int_V F_n(\mathbf{r})[\nu p_{fl}\delta\Sigma_f(\mathbf{r}) - \delta\Sigma_a(\mathbf{r})]\phi_0(\mathbf{r})\,d\mathbf{r}}{D_0(B_n^2 - B_0^2)} \qquad n \neq 0 \qquad (13.60)$$

Since the $F_n(\mathbf{r})$ are known, the specification of the perturbing functions $\delta\Sigma_f$ and $\delta\Sigma_a$ will allow explicit evaluation of the coefficients a_n from the above relation and, therefore, also an infinite series representation for $\delta\phi(\mathbf{r})$. The perturbed flux $\phi(\mathbf{r})$ may be conveniently given the same normalization as $\phi_0(\mathbf{r})$ by requiring that

$$\int_V \delta\phi(\mathbf{r})\,d\mathbf{r} = 0 \qquad (13.61)$$

By so doing $\phi(\mathbf{r})$ may be directly compared with $\phi_0(\mathbf{r})$. Figure 13.2 shows an example of the results of a calculation using Eqs. (13.57) and (13.60) for the special case of a bare slab with locally added absorber.

13.4 Perturbation in the One-velocity Model of a General System

a. General Formulation. The treatment of the general reactor problem in the one-velocity model using the method of adjoint functions is based on the neutron balance relation (13.45). It is understood that in the present application all the quantities appearing in that equation are, in general, functions of the spatial coordinates so that this equation may be used to describe any and all regions of the reactor. For convenience in writing, we combine the two factors Σ_f and p_{fl} into a single function Σ_{fl}, where

$$\Sigma_{fl}(\mathbf{r}) \equiv \Sigma_f(\mathbf{r})\,p_{fl}(\mathbf{r}) \qquad (13.62)$$

Then (13.45) becomes

$$-\nabla \cdot D(\mathbf{r})\,\nabla\phi(\mathbf{r}) + \Sigma_a(\mathbf{r})\,\phi(\mathbf{r}) = \nu\Sigma_{fl}(\mathbf{r})\,\phi(\mathbf{r}) \qquad (13.63)$$

In order to determine the required modification in ν to compensate for a general change in any of the nuclear parameters, we first multiply through Eq. (13.63) by some, as yet unspecified, function $\phi^\dagger(\mathbf{r})$ and integrate over the volume V enclosed within the outer extrapolated surface of the reactor. The resulting equation when solved for ν yields

$$\nu = \frac{\int_V \phi^\dagger(\mathbf{r})[-\nabla \cdot D(\mathbf{r})\,\nabla + \Sigma_a(\mathbf{r})]\,\phi(\mathbf{r})\,d\mathbf{r}}{\int_V \phi^\dagger(\mathbf{r})\,\Sigma_{fl}(\mathbf{r})\,\phi(\mathbf{r})\,d\mathbf{r}} \qquad (13.64)$$

As before, we next allow variations in the cross sections, which in turn cause variations in ϕ and $\phi^\dagger$. Formally, this is accomplished by applying a differential operator through Eq. (13.64). Again the original differential equation (13.63) may be used to eliminate some terms. The

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resulting expression is

$$\delta\nu = \frac{\int_V \phi^\dagger(\mathbf{r})[-\nabla\cdot\delta D(\mathbf{r})\,\nabla + \delta\Sigma_a(\mathbf{r}) - \nu\,\delta\Sigma_{fl}(\mathbf{r})]\,\phi(\mathbf{r})\,d\mathbf{r}}{\int_V \phi^\dagger(\mathbf{r})\,\Sigma_{fl}(\mathbf{r})\,\phi(\mathbf{r})\,d\mathbf{r}} + \frac{\int_V \phi^\dagger(\mathbf{r})[-\nabla\cdot D(\mathbf{r})\,\nabla + \Sigma_a(\mathbf{r}) - \nu\Sigma_{fl}(\mathbf{r})]\delta\phi(\mathbf{r})\,d\mathbf{r}}{\int_V \phi^\dagger(\mathbf{r})\,\Sigma_{fl}(\mathbf{r})\,\phi(\mathbf{r})\,d\mathbf{r}} \tag{13.65}$$

The first term on the right side of this equation is of the desired form. The immediate problem is to choose $\phi^\dagger(\mathbf{r})$ according to the method outlined in Sec. 13.1 so that the second term vanishes.

b. The Adjoint Function. The first step in the analysis is the determination of the operators adjoint to those within the brackets in the integrand of the last term of Eq. (13.65). The only one of these which presents any difficulty is the operator $\nabla\cdot D(\mathbf{r})\nabla$. In order to treat this term we introduce the following identities:

$$\int_V \phi^\dagger\nabla\cdot D\,\nabla\delta\phi\,d\mathbf{r} = \int_V \nabla\cdot(\phi^\dagger D\,\nabla\delta\phi)\,d\mathbf{r} - \int_V (\nabla\phi^\dagger)\cdot(D\,\nabla\delta\phi)\,d\mathbf{r} \tag{13.66}$$

$$\int_V \delta\phi\nabla\cdot D\,\nabla\phi^\dagger\,d\mathbf{r} = \int_V \nabla\cdot(\delta\phi\,D\,\nabla\phi^\dagger)\,d\mathbf{r} - \int_V (\nabla\delta\phi)\cdot(D\,\nabla\phi^\dagger)\,d\mathbf{r} \tag{13.67}$$

If the functions whose divergence is required in the first integral on the right-hand side of each of these equations are nonsingular and continuous, Gauss' theorem may be applied; then,

$$\int_V \nabla\cdot(\phi^\dagger D\,\nabla\delta\phi)\,d\mathbf{r} = \int_S (\phi^\dagger D\,\nabla\delta\phi)\cdot d\mathbf{S} \tag{13.68}$$

$$\int_V \nabla\cdot(\delta\phi\,D\,\nabla\phi^\dagger)\,d\mathbf{r} = \int_S (\delta\phi\,D\,\nabla\phi^\dagger)\cdot d\mathbf{S} \tag{13.69}$$

where S denotes the extrapolated surface of the reactor and $d\mathbf{S}$ a differential area on that surface. Since both the perturbed and unperturbed fluxes (and thus $\delta\phi$) must vanish on this surface, the integrals of (13.69) vanish. If it now be required that $\phi^\dagger$ vanish on the surface S, the integrals of (13.68) also vanish. Thus the first integrals on the right-hand side of Eqs. (13.66) and (13.67) vanish, and inasmuch as the order of scalar multiplication is immaterial, the second integrals are equal, and we conclude that

$$\int_V \phi^\dagger(\mathbf{r})\,\nabla\cdot D(\mathbf{r})\,\nabla\delta\phi(\mathbf{r})\,d\mathbf{r} = \int_V \delta\phi(\mathbf{r})\,\nabla\cdot D(\mathbf{r})\,\nabla\phi^\dagger(\mathbf{r})\,d\mathbf{r} \tag{13.70}$$

if $\phi^\dagger(\mathbf{r})$ and $D(\mathbf{r})\nabla\phi^\dagger(\mathbf{r})$ are continuous and nonsingular throughout the volume V, and $\phi^\dagger(\tilde{\mathbf{R}}) = 0$, where $\tilde{\mathbf{R}}$ denotes the surface S. We find, then, that under these conditions, the operator $\nabla\cdot D\nabla$ is *self-adjoint*. Since $\Sigma_a(\mathbf{r})$ and $\nu\Sigma_{fl}(\mathbf{r})$ are obviously self-adjoint, the numerator of the second term in (13.65) may be made to vanish if the function $\phi^\dagger(\mathbf{r})$ is

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selected so as to satisfy the equation

$$-\nabla \cdot D(\mathbf{r})\,\nabla\phi^{\dagger}(\mathbf{r}) + \Sigma_a(\mathbf{r})\,\phi^{\dagger}(\mathbf{r}) = \nu\Sigma_{fl}(\mathbf{r})\,\phi^{\dagger}(\mathbf{r}) \qquad (13.71)\star$$

But this is precisely the equation satisfied by the flux (13.63). Since the boundary conditions specified above for $\phi^{\dagger}$ are identical to those for ϕ, it must be concluded that

$$\phi^{\dagger}(\mathbf{r}) = \phi(\mathbf{r})$$

With the above result the change in the multiplication constant may be calculated from (13.65) as

$$-\frac{\delta\nu}{\nu} = \frac{\delta k}{k} = \frac{\int_V \phi(\mathbf{r})[\nu\,\delta\Sigma_{fl}(\mathbf{r}) - \delta\Sigma_a(\mathbf{r}) + \nabla\cdot\delta D(\mathbf{r})\nabla]\phi(\mathbf{r})\,d\mathbf{r}}{\nu\int_V \Sigma_{fl}(\mathbf{r})\phi^2(\mathbf{r})\,d\mathbf{r}} \qquad (13.72)\star$$

In so far as the changes in the absorption and fission cross sections are concerned, this result is seen to be formally identical to that of Eq. (13.58); however, Eq. (13.72) applies more generally to a multiregion reactor. It must be concluded, then, that (for absorption and fission changes) in the one-velocity diffusion approximation [cf. Eqs. (13.59)],

$$\text{Importance function} \sim \phi(\mathbf{r}) \qquad (13.73)$$

$$\text{Statistical weight} \sim \phi^2(\mathbf{r}) \qquad (13.74)$$

In Eq. (13.74) we have directly the result mentioned in the discussion following Eq. (13.19). In conjunction with this it is important to point out the great effectiveness of changes in fission and absorption cross sections at the center of the reactor where the neutron flux is high. For example, in a bare cubic homogeneous one-velocity reactor, the effectiveness in changing the multiplication constant of adding a given amount of absorber material at the center of the cube is eight times as great as if the same amount of absorber were spread uniformly over the reactor.

c. Changes Involving the Diffusion Coefficient. Equation (13.72) contains information not previously considered regarding the effect of changes in the diffusion coefficient. In order to gain insight into the nature of the weighting functions for changes in the diffusion coefficient, consider, for the moment, cases in which $\delta D(\mathbf{r})\nabla\phi(\mathbf{r})$ is a continuous function. Following the technique of Eqs. (13.66) through (13.69) it can be shown that the portion of (13.72) involving the diffusion coefficient may be written in the form

$$\frac{\delta k}{k} = -\frac{\int_V \delta D(\mathbf{r})[\nabla\phi(\mathbf{r})]^2\,d\mathbf{r}}{\nu\int_V \Sigma_{fl}(\mathbf{r})\,\phi^2(\mathbf{r})\,d\mathbf{r}} \qquad (13.75)\star$$

We find, then, that for the case of changes in the diffusion coefficient,

$$\text{Statistical weight} \sim (\nabla\phi)^2 \qquad (13.76)\star$$

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In the case of a bare slab reactor, the statistical weight for absorption and fission cross-section changes would vary across the slab as $\cos^2 Bx$ [from Eq. (13.74)], whereas the statistical weight for changes in the diffusion coefficient would go as $\sin^2 Bx$ [from Eq. (13.76)]. These forms are shown in Fig. 13.3. This figure reveals that changes in the absorption and fission cross sections are most important if made near the center of the reactor; changes made near the edge are not as important since extra neutrons there are likely to leak out and therefore unlikely to contribute to the next generation. Very nearly the opposite behavior is observed

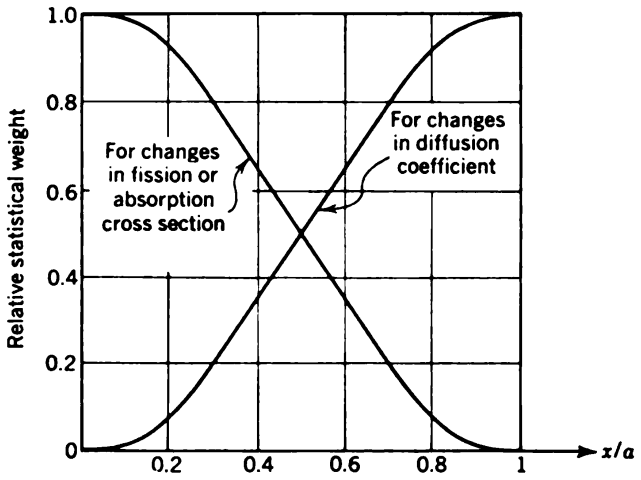

FIG. 13.3 Statistical weighting functions in a bare slab reactor in one-velocity model.

for changes in the diffusion coefficient; changes near the center are unimportant while changes near the edge contribute heavily. We may approach the understanding of this result by considering the introduction of a small void at various points in the reactor (neglecting the effects of the displaced absorbing and fissionable material). The presence of the void increases the effective diffusion coefficient in the nearby region, and this, according to (13.75), will result in a *reduction* in the reactivity of the system. This is seen to depend on the fact that neutrons are streaming from regions of higher importance to regions of lower importance, and an enhanced diffusion coefficient will aid this streaming. In regions where the gradient of the flux is large, the increased diffusion coefficient will be most effective. At the center, where the gradient vanishes, no finite increase in D will make the net current anything but zero, and thus will not affect the reactivity. This increased net current due to an increased diffusion coefficient accounts for one factor of $\nabla\phi$ in Eq. (13.75); the other arises from the fact that the change in importance function due to streaming is proportional to the change in the flux itself since the flux is self-adjoint in this case.

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13.5 Perturbation in the Two-group Model

For the purposes of the present treatment we write the two-group equations in the form [cf. Eqs. (8.147)]

$$\begin{aligned} -\nabla \cdot D_1(\mathbf{r})\, \nabla\phi_1(\mathbf{r}) + \Sigma_R^{(1)}(\mathbf{r})\, \phi_1(\mathbf{r}) &= \nu\Sigma_f(\mathbf{r})\, \phi_2(\mathbf{r}) \\ -\nabla \cdot D_2(\mathbf{r})\, \nabla\phi_2(\mathbf{r}) + \Sigma_a^{(2)}(\mathbf{r})\, \phi_2(\mathbf{r}) &= p_C(\mathbf{r})\, \Sigma_R^{(1)}(\mathbf{r})\, \phi_1(\mathbf{r}) \end{aligned} \tag{13.77}$$

where the various symbols and subscripts have the previous definitions. In this analysis we require that the functions ϕ_1, ϕ_2, $D_1\nabla\phi_1$, and $D_2\nabla\phi_2$ be continuous and nonsingular throughout the volume V (as previously specified) and that $\phi_1(\tilde{\mathbf{R}}) = \phi_2(\tilde{\mathbf{R}}) = 0$, where $\tilde{\mathbf{R}}$ denotes the outer surface of V. In the present formulation all parameters appearing in (13.77) are taken generally as functions of spatial coordinates so as to allow for all variations, including those involving regional differences in the system.

It is convenient and customary in analyzing this system to use matrix notation. Thus we define the fluxes as a column matrix and the operators as square matrices. In this construction, Eqs. (13.77) may be written

$$\begin{bmatrix} -\nabla \cdot D_1\nabla + \Sigma_R^{(1)} & 0 \\ -p_C\Sigma_R^{(1)} & -\nabla \cdot D_2\nabla + \Sigma_a^{(2)} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \nu \begin{bmatrix} 0 & \Sigma_f \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{13.78}$$

If the square matrices are represented by the operator symbols $\boldsymbol{O}_1$ and $\boldsymbol{O}_2$ [cf. Eqs. (13.2) and (13.3)] and the column matrix of the fluxes by $\boldsymbol{\Phi}$, then (13.78) reduces to the expression

$$\boldsymbol{O}_1\boldsymbol{\Phi} = \nu\boldsymbol{O}_2\boldsymbol{\Phi} \tag{13.79}$$

This equation is in the form of the general relation (13.1), and may therefore be treated by the method outlined in Sec. 13.1.

The analysis begins by introducing a yet unspecified row matrix $\boldsymbol{\Phi}^\dagger$:

$$\boldsymbol{\Phi}^\dagger \equiv [\phi_1^\dagger\ \phi_2^\dagger] \tag{13.80}$$

As in the earlier calculations, this quantity is multiplied into Eqs. (13.79), the result integrated over volume V and solved for ν:

$$\nu = \frac{\int_V \boldsymbol{\Phi}^\dagger \boldsymbol{O}_1 \boldsymbol{\Phi}\, d\mathbf{r}}{\int_V \boldsymbol{\Phi}^\dagger \boldsymbol{O}_2 \boldsymbol{\Phi}\, d\mathbf{r}} \tag{13.81}$$

Next, we allow general changes in the parameters of the system which cause changes in $\boldsymbol{\Phi}$ and $\boldsymbol{\Phi}^\dagger$. In order to maintain the system at steady state, we introduce a change also in ν. This operation is performed by differentiating through Eq. (13.81); the differentiated quantities are

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preceded by the symbol δ.

$$\delta\nu = \frac{\int_V \delta\Phi^\dagger(O_1 - \nu O_2)\Phi\, d\mathbf{r} + \int_V \Phi^\dagger(\delta O_1 - \nu\,\delta O_2)\Phi\, d\mathbf{r} + \int_V \Phi^\dagger(O_1 - \nu O_2)\delta\Phi\, d\mathbf{r}}{\int_V \Phi^\dagger O_2 \Phi\, d\mathbf{r}} \tag{13.82}$$

The first integral in the numerator vanishes by virtue of (13.79). The second integral is of the desired form. As previously, we begin with some manipulations of the third integral. The procedure is the same as outlined in Sec. 13.1, and used in subsequent sections. New operators $O_1^\dagger$ and $O_2^\dagger$ are defined such that

$$\int_V \Phi^\dagger(O_1 - \nu O_2)\delta\Phi\, d\mathbf{r} = \int_V \delta\Phi_\star(O_1^\dagger - \nu O_2^\dagger)\Phi_\star^\dagger\, d\mathbf{r} \tag{13.83}$$

Thus we require that $\Phi^\dagger$ satisfy the equation

$$O_1^\dagger \Phi_\star^\dagger = \nu O_2^\dagger \Phi_\star^\dagger \tag{13.84}$$

where the subscript $\star$ denotes the *transposed* row or column matrix. The change in the multiplication constant is finally calculated as

$$-\frac{\delta\nu}{\nu} = \frac{\delta k}{k} = -\frac{\int_V \Phi^\dagger(\delta O_1 - \nu\,\delta O_2)\Phi\, d\mathbf{r}}{\int_V \Phi^\dagger O_2 \Phi\, d\mathbf{r}} \tag{13.85}\star$$

In order to determine the adjoint operators it is necessary to examine in detail the operator terms of O_1 and O_2 of Eq. (13.78); these are all multiplicative, and thus self-adjoint except for $\nabla \cdot D_1\nabla$ and $\nabla \cdot D_2\nabla$, which may be treated exactly as in Sec. 13.4. The result is that all the individual operators which make up the matrix operators O_1 and O_2 may be regarded as self-adjoint if we require that $\phi_1^\dagger$, $\phi_2^\dagger$, $D_1\nabla\phi_1^\dagger$, and $D_2\nabla\phi_2^\dagger$ be nonsingular and continuous and that $\phi_1^\dagger(\bar{\mathbf{R}}) = \phi_2^\dagger(\bar{\mathbf{R}}) = 0$. Then the adjoint operators $O_1^\dagger$ and $O_2^\dagger$ are the transposes of the operators O_1 and O_2, respectively (note that all quantities are real here), so that Eq. (13.84) becomes

$$\begin{bmatrix} -\nabla \cdot D_1\nabla + \Sigma_R^{(1)} & -p_C\Sigma_R^{(1)} \\ 0 & -\nabla \cdot D_2\nabla + \Sigma_a^{(2)} \end{bmatrix} \begin{bmatrix} \phi_1^\dagger \\ \phi_2^\dagger \end{bmatrix} = \nu \begin{bmatrix} 0 & 0 \\ \Sigma_f & 0 \end{bmatrix} \begin{bmatrix} \phi_1^\dagger \\ \phi_2^\dagger \end{bmatrix} \tag{13.86}$$

For direct comparison with the "forward" equations (13.77) this may be expanded to

$$\begin{aligned} -\nabla \cdot D_1(\mathbf{r})\,\nabla\phi_1^\dagger(\mathbf{r}) + \Sigma_R^{(1)}(\mathbf{r})\,\phi_1^\dagger(\mathbf{r}) &= p_C(\mathbf{r})\,\Sigma_R^{(1)}(\mathbf{r})\,\phi_2^\dagger(\mathbf{r}) \\ -\nabla \cdot D_2(\mathbf{r})\,\nabla\phi_2^\dagger(\mathbf{r}) + \Sigma_a^{(2)}(\mathbf{r})\,\phi_2^\dagger(\mathbf{r}) &= \nu\Sigma_f(\mathbf{r})\,\phi_1^\dagger(\mathbf{r}) \end{aligned} \tag{13.87}\star$$

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Finally the change in the multiplication constant is calculated from (13.85):

$$\frac{\delta k}{k} = \frac{1}{\nu \int_V \phi_1^\dagger \Sigma_f \phi_2 \, d\mathbf{r}} \left\{ - \int_V \delta D_1 \nabla \phi_1^\dagger \cdot \nabla \phi_1 \, d\mathbf{r} - \int_V \phi_1^\dagger \delta \Sigma_R^{(1)} \phi_1 \, d\mathbf{r} \right.$$
$$+ \int_V \phi_2^\dagger \delta [p_C \Sigma_R^{(1)}] \phi_1 \, d\mathbf{r} - \int_V \delta D_2 \nabla \phi_2^\dagger \cdot \nabla \phi_2 \, d\mathbf{r}$$
$$\left. - \int_V \phi_2^\dagger \delta \Sigma_a^{(2)} \phi_2 \, d\mathbf{r} + \nu \int_V \phi_1^\dagger \delta \Sigma_f \phi_2 \, d\mathbf{r} \right\} \quad (13.88)\star$$

It must be remembered that Eqs. (13.87) apply throughout all regions of the reactor and that the integrals of (13.88) extend over the "extrapolated volume."

It follows from this development that the method employed here is directly applicable to any multigroup model.

13.6 Perturbation in the Age-diffusion Model

a. Adjoint Equations for the Age-diffusion Theory. The methods of the preceding sections may be combined to determine the equations adjoint to the age-diffusion relations (6.54). The solution to the adjoint equations yields the functions adjoint to the slowing-down density and the thermal flux; these will act as importance functions in the treatment of small changes in the parameters of the system. Since all the essential features of the method have already been discussed in considerable detail, only an outline of the procedure is given here, along with a summary of the results.[1]

For the present application we use the following form for the age-diffusion equations:

$$-\nabla \cdot D(\mathbf{r},u) \nabla \left[\frac{q(\mathbf{r},u)}{\xi \Sigma_t(\mathbf{r},u)} \right] + \beta(\mathbf{r},u) \, q(\mathbf{r},u) + \frac{\partial}{\partial u} q(\mathbf{r},u)$$
$$= \nu_\delta(u) \int_0^\infty [\gamma(\mathbf{r},u') \, q(\mathbf{r},u') + \Sigma_f^{\text{th}} \phi(\mathbf{r}) \, \delta(u' - u_{\text{th}})] \, du' \quad (13.89)$$

$$-\nabla \cdot D_{\text{th}}(\mathbf{r}) \nabla \phi(\mathbf{r}) + \Sigma_a^{\text{th}}(\mathbf{r}) \, \phi(\mathbf{r}) = \int_0^\infty \delta(u' - u_{\text{th}}) \, q(\mathbf{r},u') \, du' \quad (13.90)$$

where $\phi(\mathbf{r})$ denotes the thermal flux, and

$$\beta(\mathbf{r},u) \equiv \frac{\Sigma_a(\mathbf{r},u)}{\xi \Sigma_t(\mathbf{r},u)} \qquad \gamma(\mathbf{r},u) \equiv \frac{\Sigma_f(\mathbf{r},u)}{\xi \Sigma_t(\mathbf{r},u)} \quad (13.91)$$

All remaining symbols and notation, including the "th," have the usual definition. Since all quantities (with the exception of the fission spec-

[1] This treatment follows that of M. J. Nielsen given in "The Adjoint Equations and Perturbation Theory for a Reflected Reactor," Oak Ridge National Laboratory, ORNL Y-F10-31, Jan. 8, 1951.

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trum χ) have been specified as general functions of the spatial coordinates, Eqs. (13.89) and (13.90) apply throughout a multiregion system.

The analysis proceeds by putting the age-diffusion equations into operational form:

$$\boldsymbol{O}_1\boldsymbol{\chi} = \nu\boldsymbol{O}_2\boldsymbol{\chi} \tag{13.92}$$

wherein we have introduced the matrix operators

$$\boldsymbol{O}_1 \equiv \begin{bmatrix} -\nabla\cdot D\nabla\frac{1}{\xi\Sigma_t} + \beta + \frac{\partial}{\partial u} & 0 \\ -\int_0^\infty \delta(u-u_{th})\{\cdots\}\,du & -\int_0^\infty (\nabla\cdot D_{th}\nabla + \Sigma_a^{th})\{\cdots\}\,du \end{bmatrix} \tag{13.93}\star$$

$$\boldsymbol{O}_2 \equiv \begin{bmatrix} \chi(u)\int_0^\infty \gamma(\mathbf{r},u)\{\cdots\}\,du & \chi(u)\int_0^\infty \Sigma_f^{th}(\mathbf{r})\{\cdots\}\,du \\ 0 & 0 \end{bmatrix} \tag{13.94}\star$$

and

$$\boldsymbol{\chi} \equiv \begin{bmatrix} q(\mathbf{r},u) \\ \phi(\mathbf{r})\ \delta(u-u_{th}) \end{bmatrix} \tag{13.95}$$

The notation $\int_0^\infty g(u)\{\cdots\}\,du$ signifies that the quantity operated upon is to be put inside the braces and integrated over lethargy, with $g(u)$ as a weighting function.

Next we multiply through Eq. (13.92) by an as yet unspecified matrix $\boldsymbol{\chi}^\dagger$, integrate over the extrapolated volume V and over the entire lethargy range, and solve for ν.

$$\nu = \frac{\int_0^\infty du\int_V \boldsymbol{\chi}^\dagger\boldsymbol{O}_1\boldsymbol{\chi}\,d\mathbf{r}}{\int_0^\infty du\int_V \boldsymbol{\chi}^\dagger\boldsymbol{O}_2\boldsymbol{\chi}\,d\mathbf{r}} \tag{13.96}$$

Now perform the general variation of parameters and functions in Eq. (13.96) by applying the delta operator, as used previously in this chapter; and, again, choose operators adjoint to $\boldsymbol{O}_1$ and $\boldsymbol{O}_2$ such that

$$\int_0^\infty du\int_V \boldsymbol{\chi}^\dagger(\boldsymbol{O}_1 - \nu\boldsymbol{O}_2)\delta\boldsymbol{\chi}\,d\mathbf{r} = \int_0^\infty du\int_V \delta\boldsymbol{\chi}_\star(\boldsymbol{O}_1^\dagger - \nu\boldsymbol{O}_2^\dagger)\boldsymbol{\chi}_\star^\dagger\,d\mathbf{r} \tag{13.97}$$

where $\boldsymbol{\chi}^\dagger$ is now defined by

$$\boldsymbol{O}_1^\dagger\boldsymbol{\chi}_\star^\dagger = \nu\boldsymbol{O}_2\boldsymbol{\chi}_\star^\dagger \tag{13.98}$$

Then,

$$-\frac{\delta\nu}{\nu} = \frac{\delta k}{k} = -\frac{\int_0^\infty du\int_V \boldsymbol{\chi}^\dagger(\delta\boldsymbol{O}_1 - \nu\,\delta\boldsymbol{O}_2)\boldsymbol{\chi}\,d\mathbf{r}}{\nu\int_0^\infty du\int_V \boldsymbol{\chi}^\dagger\boldsymbol{O}_2\boldsymbol{\chi}\,d\mathbf{r}} \tag{13.99}\star$$

as usual. If $\boldsymbol{\chi}^\dagger$ is taken as the row matrix

$$\boldsymbol{\chi}^\dagger \equiv [q^\dagger(\mathbf{r},u) \qquad \phi^\dagger(\mathbf{r})\delta(u-u_{th})] \tag{13.100}$$

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then the equations for the adjoint functions $q^\dagger$ and $\phi^\dagger$ may be shown to be [using conditions (13.97) and (13.98)]

$$-\frac{1}{\xi\Sigma_t(\mathbf{r},u)}\nabla\cdot D(\mathbf{r},u)\,\nabla q^\dagger(\mathbf{r},u)+\beta(\mathbf{r},u)\,q^\dagger(\mathbf{r},u)-\frac{\partial}{\partial u}q^\dagger(\mathbf{r},u)$$
$$=\delta(u-u_{\text{th}})\phi^\dagger(\mathbf{r})+\nu\gamma(\mathbf{r},u)\int_0^\infty \chi(u')\,q^\dagger(\mathbf{r},u')\,du' \quad (13.101)\star$$

$$-\nabla\cdot D_{\text{th}}\nabla\phi^\dagger(\mathbf{r})+\Sigma_a^{\text{th}}\phi^\dagger(\mathbf{r})=\nu\Sigma_f^{\text{th}}(\mathbf{r})\int_0^\infty \chi(u')\,q^\dagger(\mathbf{r},u')\,du' \quad (13.102)\star$$

and $q^\dagger$, $\phi^\dagger$, $D\nabla q^\dagger$, and $D_{\text{th}}\nabla\phi^\dagger$ must be everywhere nonsingular and continuous, and $q^\dagger(\tilde{\mathbf{R}})=\phi^\dagger(\tilde{\mathbf{R}})=0$. With the functions $q^\dagger$ and $\phi^\dagger$ calculated as functions of the spatial coordinates and lethargy, Eq. (13.99) may be used to calculate the effect of any (small) change in parameters.

Note that the pair (13.101) and (13.102) are "backward" equations in just the sense discussed for the infinite medium Eq. (13.37). The general procedure would be to choose a normalized space form for the integral $\int_0^\infty \chi(u')\,q^\dagger(\mathbf{r},u')\,du'$ and proceed to obtain $\phi^\dagger(\mathbf{r})$ from Eq. (13.102). This "adjoint thermal flux" $\phi^\dagger(\mathbf{r})$ acts as a source for beginning the calculation of the "adjoint slowing-down density"; the appropriate starting point in lethargy is $u=u_{\text{th}}$, and the integration is carried "up" (in energy) to $u=0$ [note that as in Eq. (13.37) there is a change in sign of the lethargy derivative in (13.101) as compared with the "forward" Eq. (13.89)].

b. Application to Multigroup Method. The entire scheme described above is readily adapted to the multigroup technique discussed in Sec. 8.8. The adjoint set (13.101) and (13.102) is very like the forward set (13.89) and (13.90). In a similar manner to that used in Sec. 8.8, the adjoint equations may be converted into multigroup equations convenient for numerical integration. Further, once the forward calculation is set up, the adjoint equations are easily solved since the constants involved are the same, and the same network in lethargy and space variables may be used. Thus the calculation of the importance functions $q^\dagger$ and $\phi^\dagger$, in this case, add very little to the time and effort already invested in obtaining the slowing-down density and the flux. In fact, the second calculation may serve as a numerical check on the forward calculation since the same eigenvalue ν should be obtained in both cases.

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